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# RANK PROCEDURES FOR TESTING SUBHYPOTHESES IN LINEAR REGRESSION

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## Summary

In the linear regression model $X_i=\alpha+\boldsymbol{\beta}\boldsymbol{c}_i+Z_i$, we consider the problem of testing the subhypothesis that some (but not all) components of $\boldsymbol{\beta}$ are equal to 0. A class of asymptotically distribution-free tests based on a quadratic form in aligned rank statistic is studied and the asymptotic relative efficiencies of the proposed tests with respect to the general likelihood ratio test and the test based on least-squares estimates of regression parameters are derived. Asymptotic optimality (à la Wald) is also discussed.

## 1. Introduction

Consider a linear regression model

$$X_i=\alpha+\boldsymbol{\beta}\boldsymbol{c}_i+Z_i\ , \qquad (i=1,\cdots,n) \tag{1.1}$$

where the intercept $\alpha$ and the regression parameters $\boldsymbol{\beta}=(\beta_1,\cdots,\beta_q)$, $(q>1)$ are unknown, each $\boldsymbol{c}_i=(c_{1i},\cdots,c_{qi})'$, $(i=1,\cdots,n)$ is a vector of known regression constants, and $Z_1,\cdots,Z_n$ are independent and identically distributed random variables with (unkown) symmetric distribution function $F(x)$. Let $\boldsymbol{\beta}=(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2)$, where $\boldsymbol{\beta}_1=(\beta_1,\cdots,\beta_r)$ and $\boldsymbol{\beta}_2=(\beta_{r+1},\cdots,\beta_q)$, $1\leqq r<q$ are fixed. A problem of interest is that of testing the subhypothesis

$$H_0\colon \boldsymbol{\beta}_2=\boldsymbol{0} \text{ vs. } H\colon \boldsymbol{\beta}_2\neq\boldsymbol{0} \qquad ((\alpha,\boldsymbol{\beta}_1) \text{ nuisance})\,. \tag{1.2}$$

Different versions of this problem (under the non-intercept linear model) have been treated in detail in the context of the classical normal theory of linear regression (see, e.g. Williams [14] and Graybill [4], p. 194). Recent years have seen much interest in rank methods for regression.

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Thus McKean and Hettmansperger [9] have proposed a class of tests for (1.2) based on the drop (or reduction) in Jaeckel's [6] dispersion measure, which, however, is not a pure rank statistic, but rather a mixed linear combination of the $X_i$'s and rank scores. More recently Adichie [1] has studied two classes of tests for a different version of (1.2) based on the difference of two quadratic forms in aligned rank statistics.

In the present paper we study a distribution-free class of rank procedures for testing (1.2). Basic assumptions and notations are given in Section 2. In Section 3 we study a class of tests based on a quadratic form in aligned rank statistics. The approach is similar to that of Sen and Puri [12]. We derive the asymptotic distribution of the test statistics, which is central chi-square under $H_0$ and non-central chi-square under a sequence of local alternatives. In Section 4 we compare the proposed rank procedures with two classical procedures for the same problem: the general likelihood ratio test and the test based on least-squares estimates of $\boldsymbol{\beta}_2$. Asymptotic relative efficiencies are derived. Finally, asymptotic optimality in the sense of Wald [13] is discussed in Section 5.

## 2. Notations and basic assumptions

Let $\boldsymbol{X}_n=(X_1,\cdots,X_n)$, $\boldsymbol{Z}_n=(Z_1,\cdots,Z_n)$, $\boldsymbol{\theta}=(\alpha,\boldsymbol{\beta})$, $\mathbf{1}_n=(1,\cdots,1)\in R^n$, $C_n=(\boldsymbol{c}_1,\cdots,\boldsymbol{c}_n)$ and $C_n^*=(\mathbf{1}_n', C_n')'$. Then (1.1) can be expressed as

$$\boldsymbol{X}_n=\alpha\mathbf{1}_n+\boldsymbol{\beta}C_n+\boldsymbol{Z}_n=\boldsymbol{\theta}C_n^*+\boldsymbol{Z}_n\,. \tag{2.1}$$

We consider only $n>q+1$ and make the usual assumptions of full rank, namely that the $(q+1)\times n$ matrix $C_n^*$ has rank $q+1$. Let

$$\bar{\boldsymbol{c}}_n=n^{-1}\sum_{i=1}^{n}\boldsymbol{c}_i(\bar{c}_{1n},\cdots,\bar{c}_{qn})'\,,\quad \bar{c}_{mn}=n^{-1}\sum_{i=1}^{n}c_{mi}\,,\qquad (m=1,\cdots,q) \tag{2.2}$$

$$D_n=\sum_{i=1}^{n}\boldsymbol{c}_i\boldsymbol{c}_i'\,. \tag{2.3}$$

Then the $(q+1)\times(q+1)$ symmetric matrix

$$A_n=C_n^*C_n^{*\prime}=\begin{bmatrix} n & n\bar{\boldsymbol{c}}_n' \\ n\bar{\boldsymbol{c}}_n & D_n\end{bmatrix} \tag{2.4}$$

has rank $q+1$ and is positive definite. Consider the $q\times q$ matrix

$$\begin{aligned} M_n&=\sum_{i=1}^{n}(\boldsymbol{c}_i-\bar{\boldsymbol{c}}_n)(\boldsymbol{c}_i-\bar{\boldsymbol{c}}_n)'=D_n-n\bar{\boldsymbol{c}}_n\bar{\boldsymbol{c}}_n' \\ &=(\boldsymbol{c}_1-\bar{\boldsymbol{c}}_n,\cdots,\boldsymbol{c}_n-\bar{\boldsymbol{c}}_n)(\boldsymbol{c}_1-\bar{\boldsymbol{c}}_n,\cdots,\boldsymbol{c}_n-\bar{\boldsymbol{c}}_n)'\,. \end{aligned} \tag{2.5}$$

Then it is easy to check that $A_n$ is equivalent to $\begin{bmatrix} n & n\bar{c}'_n \\ 0 & M_n \end{bmatrix}$ which therefore has rank $q+1$. It follows that $M_n$ has rank $q$ and hence is positive definite. We also assume that the limits

$$A=\lim_{n\to\infty} n^{-1}A_n \quad \text{and} \quad M=\lim_{n\to\infty} n^{-1}M_n \tag{2.6}$$

exist and are positive definite.

*Remark.* We have incorporated the assumption of full rank usually made in the least-squares and maximal likelihood procedures. In this respect we have a unified treatment of rank procedures and the classical procedures. The present approach does not have a shortcoming of Adichie's treatment (see Adichie [1], p. 1016, Remarks 1 and 2). Indeed, Assumption $B_1$ (ii) of Adichie [1] does not hold in the present model (2.1).

We also simplify some of Jurečková's [8] conditions on the regression constants by assuming that each $\boldsymbol{c}_i$ can be expressed as a difference

$$\boldsymbol{c}_i=\boldsymbol{c}_{i(1)}-\boldsymbol{c}_{i(2)}\ , \qquad \boldsymbol{c}_{i(j)}=(c_{1i(j)},\cdots,c_{qi(j)})'\ , \qquad (i=1,\cdots,n;\ j=1,2) \tag{2.7}$$

where, for each $m=1,\cdots,q$ and each $j=1,2$, $c_{mi(j)}$ is nondecreasing in $i$, and the $c_{mi(j)}$'s satisfy

$$\lim_{n\to\infty} n^{-1}\max_{1\leq i\leq n}[c_{mi(j)}-\bar{c}_{mn(j)}]^2=0\ , \qquad \bar{c}_{mn(j)}=n^{-1}\sum_{i=1}^{n}c_{mi(j)} \tag{2.8}$$

and

$$\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n}[c_{mi(j)}-\bar{c}_{mn(j)}]^2\in(0,\infty)\ , \qquad (m=1,\cdots,q;\ j=1,2)\ . \tag{2.9}$$

Thus the $c_{mi(j)}$'s satisfy the Noether condition

$$\lim_{n\to\infty}\left\{\max_{1\leq i\leq n}[c_{mi(j)}-\bar{c}_{mn(j)}]^2\Big/\sum_{i=1}^{n}[c_{mi(j)}-\bar{c}_{mn(j)}]^2\right\}=0\ , \qquad (m=1,\cdots,q;\ j=1,2)\ .$$

Now let each $\boldsymbol{c}_i$ be partitioned as

$$\boldsymbol{c}_i=(\boldsymbol{c}'_{i,1},\boldsymbol{c}'_{i,2})'\ , \quad \boldsymbol{c}_{i,1}=(c_{1i},\cdots,c_{ri})'\ , \quad \boldsymbol{c}_{i,2}=(c_{r+1,i},\cdots,c_{qi})'\ , \qquad (i=1,\cdots,n)\ . \tag{2.10}$$

Then $H_0$ can be expressed as

$$H_0\colon P_{(\alpha,\boldsymbol{\beta}_1,0)}(X_i\leqq x)=F(x-\alpha-\boldsymbol{\beta}_1\boldsymbol{c}_{i,1})\ , \qquad (i=1,\cdots,n) \tag{2.11}$$

where $P_{(\alpha,\boldsymbol{\beta}_1,0)}$ denotes the probability distribution of $\boldsymbol{X}_n$ under $H_0\colon \boldsymbol{\beta}_2$

$=\mathbf{0}$. More generally $P_{\boldsymbol{\theta}}$ will denote the probability distribution of $\boldsymbol{X}_n$ when $\boldsymbol{\theta}$ is the true value of $(\alpha, \boldsymbol{\beta})=(\alpha, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$.

For $\boldsymbol{b}=(b_1, \cdots, b_q) \in R^q$, let

$$(2.12) \quad R_{ni}(\boldsymbol{b}) = \text{the rank of } X_i-\boldsymbol{b}\boldsymbol{c}_i \text{ among } X_1-\boldsymbol{b}\boldsymbol{c}_1, \cdots, X_n-\boldsymbol{b}\boldsymbol{c}_n \text{ in the ascending order, } (i=1, \cdots, n),$$

$$(2.13) \quad S_{nm}(\boldsymbol{b})=\sum_{i=1}^{n}(c_{mi}-\bar{c}_{mn})a_n[R_{ni}(\boldsymbol{b})], \qquad (m=1, \cdots, q)$$

and

$$(2.14) \quad \boldsymbol{S}_n(\boldsymbol{b})=(S_{n1}(\boldsymbol{b}), \cdots, S_{nq}(\boldsymbol{b})),$$

where the scores $a_n(1), \cdots, a_n(n)$ are generated by a non-constant and square-integrable function $\psi$ defined on $(0, 1)$, in one of the following two ways:

$$(2.15) \qquad a_n(i)=\psi(i/(n+1)) \quad \text{or} \quad a_n(i)=\mathrm{E}\,[\psi(U_{ni})], \qquad (i=1, \cdots n)$$

where $U_{n1} \leqq \cdots \leqq U_{nn}$ are the order statistics of a random sample of size $n$ from the uniform distribution over $(0, 1)$. We assume that $\psi$ is the difference $\psi=\psi_1-\psi_2$ of two non-decreasing and absolutely continuous functions $\psi_1$ and $\psi_2$ on $(0, 1)$. Let

$$(2.16) \qquad \lambda(\psi)=\left\{\int_0^1 [\psi(u)-\bar{\psi}]^2 du\right\}^{1/2}, \qquad \bar{\psi}=\int_0^1 \psi(u)du.$$

Then $0<\lambda(\psi)<\infty$. A class of score-generating functions of particular interest are of the form

$$(2.17) \qquad \psi(u)=\phi_g(u)=-g'[G^{-1}(u)]/g[G^{-1}(u)], \qquad u \in (0, 1)$$

where $G$ is a distribution function whose density $g=G'$ is absolutely continuous and has finite and positive Fisher's information

$$(2.18) \qquad 0<I(g)=\int_{-\infty}^{\infty} [g'(x)/g(x)]^2 dG(x)<\infty.$$

For such a score-generating function $\psi=\phi_g$ we have

$$(2.19) \qquad \bar{\phi}_g=0, \qquad \lambda(\phi_g)=\left\{\int_0^1 [\phi_g(u)]^2 du\right\}^{1/2}=[I(g)]^{1/2}.$$

(For details and examples, see Hájek and Šidák [5], Chapter I, Section 2.)

As for the underlying distribution function $F$, we assume that it has an absolutely continuous density $f=F'$ with $0<I(f)<\infty$. We make no assumptions about the specific form of $F$.

To compare the rank procedures with the classical procedures, we will consider a sequence of alternatives

$$H_n: \boldsymbol{\beta}_2 = \boldsymbol{\beta}_{2n} = n^{-1/2}\boldsymbol{b}_2 \tag{2.20}$$

where $0 \neq \boldsymbol{b}_2 \in R^{q-r}$ is arbitrarily fixed.

## 3. Aligned rank order tests

Since the rank statistics (2.14) do not depend on $\alpha$, and since $X_1 - \boldsymbol{\beta}_1\boldsymbol{c}_{1,1}, \cdots, X_n - \boldsymbol{\beta}_1\boldsymbol{c}_{n,1}$ are independently and identically distributed under $H_0$, we need to estimate $\boldsymbol{\beta}_1$ under $H_0$. To this end, we use Jurečková's [8] method. Let

$$B_{1(n)} = \left\{\boldsymbol{b}_1 \in R^r : \sum_{i=1}^{r} |S_{nm}(\boldsymbol{b}_1, \boldsymbol{0})| = \text{minimum}\right\}, \tag{3.1}$$

and choose an element $\bar{\boldsymbol{\beta}}_{1n} \in B_{1(n)}$ as an estimate of $\boldsymbol{\beta}_1$ under $H_0$. (If $B_{1(n)}$ is a convex set, then a natural choice for $\bar{\boldsymbol{\beta}}_{1n}$ is the centre of gravity of $B_{1(n)}$).

*Remark.* As will be seen later (in the proof of Lemma 3.1), the only essential property required of $\bar{\boldsymbol{\beta}}_{1n}$ is that

$$n^{1/2}(\bar{\boldsymbol{\beta}}_{1n} - \boldsymbol{\beta}_1) \text{ is bounded in } P_{(\alpha, \boldsymbol{\beta}_1, \boldsymbol{0})}\text{-probability}. \tag{3.2}$$

While the classical least-squares and maximal likelihood estimates of $\boldsymbol{\beta}_1$ also satisfy (3.2) under suitable conditions, here rank-order estimates of $\boldsymbol{\beta}_1$ are most appropriate for aligned rank-order tests.

For $\boldsymbol{b} \in R^q$ we partition $\boldsymbol{S}_n(\boldsymbol{b})$ (defined by (2.14)) as

$$\boldsymbol{S}(\boldsymbol{b}) = (\boldsymbol{S}_{n(1)}(\boldsymbol{b}), \boldsymbol{S}_{n(2)}(\boldsymbol{b})), \tag{3.3}$$

where

$$\boldsymbol{S}_{n(1)}(\boldsymbol{b}) = (S_{n1}(\boldsymbol{b}), \cdots, S_{nr}(\boldsymbol{b})), \qquad \boldsymbol{S}_{n(2)}(\boldsymbol{b}) = (S_{n,r+1}(\boldsymbol{b}), \cdots, S_{nq}(\boldsymbol{b})).$$

Define the $(q-r)$-dimensional vector of aligned rank statistics

$$\hat{\boldsymbol{S}}_{n(2)} = \boldsymbol{S}_{n(2)}(\bar{\boldsymbol{\beta}}_{1n}, \boldsymbol{0}) = (\hat{S}_{n,r+1}, \cdots, \hat{S}_{nq}) \tag{3.4}$$

where

$$\hat{S}_{nm} = S_{nm}(\bar{\boldsymbol{\beta}}_{1n}, \boldsymbol{0}) = \sum_{i=1}^{n} (c_{mi} - \bar{c}_{mn}) a_n(\hat{R}_{ni}), \qquad (m = r+1, \cdots, q)$$

$\hat{R}_{ni}$ being the rank of $X_i - \bar{\boldsymbol{\beta}}_{1n}\boldsymbol{c}_{i,1}$ among $X_i - \bar{\boldsymbol{\beta}}_{1n}\boldsymbol{c}_{1,1}, \cdots, X_n - \bar{\boldsymbol{\beta}}_{1n}\boldsymbol{c}_{n,1}$, $(i = 1, \cdots, n)$.

Let the matrix $M_n$ (given by (2.5)) be partitioned as

$$M_n = \begin{bmatrix} M_{n11} & M_{n12} \\ M_{n21} & M_{n22} \end{bmatrix} \tag{3.5}$$

where $M_{n11}$ is $r\times r$, and define the $(q-r)\times(q-r)$ matrix

(3.6) $$\bar{M}_n = M_{n22} - M_{n21}M_{n11}^{-1}M_{n12} ,$$

which is symmetric and positive definite (because $M_n$ is). Let

(3.7) $$\lambda_n = \left\{ n^{-1}\sum_{i=1}^{n} [a_n(i) - \bar{a}_n]^2 \right\}^{1/2} , \qquad \bar{a}_n = n^{-1}\sum_{i=1}^{n} a_n(i) .$$

Then aligned rank-order tests for (1.2) can be based on the quadratic form

(3.8) $$Q_n = (\lambda_n)^{-2}\hat{S}_{n(2)}(\bar{M}_n)^{-1}\hat{S}'_{n(2)} ,$$

whose asymptotic distribution under $H_0$ is given by the following theorem.

THEOREM 3.1. *Under $H_0$, $Q_n$ has asymptotically the (central) chi-square distribution $\chi^2_{q-r}$ with $q-r$ degrees of freedom.*

For $0<\varepsilon<1$ let $\chi^2_{q-r,\varepsilon}$ be the upper $100\varepsilon\%$ point of the $\chi^2_{q-r}$ distribution. Then for large $n$ we have the following asymptotically distribution-free test of approximately size $\varepsilon$:

(3.9) Reject $H_0$ (in favor of $H$) if and only if $Q_n \geqq \chi^2_{q-r,\varepsilon}$ .

To prove Theorem 3.1, we need the following lemmas.

LEMMA 3.1. *Under $H_0$,*

(3.10) $n^{-1/2}[\hat{S}_{n(2)} - S_{n(2)}(\beta_1, 0) + \gamma(\psi, f)(\bar{\beta}_{1n} - \beta_1)M_{n12}]$ *converges to* $0$ *in* $P_{(\alpha,\beta_1,0)}$*-probability, where*

(3.11) $$\gamma(\psi,f) = \int_0^1 \psi(u)\phi_f(u)du , \quad \text{and} \quad \phi_f(u) = -f'[F^{-1}(u)]/f[F^{-1}(u)] , \quad u \in (0,1) .$$

PROOF. Let

(3.12) $$\bar{c}_{n,j} = n^{-1}\sum_{i=1}^{n} c_{i,j} , \qquad (j=1, 2) .$$

Then, by (2.2), (2.5), (2.10) and (3.5), we have

(3.13) $$M_{njk} = \sum_{i=1}^{n} (c_{i,j} - \bar{c}_{n,j})(c_{i,k} - \bar{c}_{n,k})' , \qquad (j, k=1, 2) .$$

Let the matrix $M$ (defined in (2.6)) be partitioned as

(3.14) $$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

where $M_{11}$ is $r\times r$. Then $M_{11}$ is symmetric and positive definite (since

$M$ is). By (3.5), we have

(3.15) $$\lim_{n\to\infty} n^{-1}M_{njk}=M_{jk}\ , \qquad (j, k=1, 2)\ .$$

Under $H_0$, since (2.11) holds, by Theorem 4.1 of Jurečková [8], $n^{1/2}(\bar{\boldsymbol{\beta}}_{1n}-\boldsymbol{\beta}_1)$ is asymptotically normal and so (3.2) is satisfied. It follows from a multi-dimensional extension of Theorem 3.1 of Jurečková [7] (cf. Theorem 3.1 of Jurečková [8]), that under $H_0$, $n^{-1/2}\boldsymbol{S}_{n(2)}(\boldsymbol{\beta}_{1n}, \mathbf{0})-n^{-1/2}\boldsymbol{S}_{n(2)}(\boldsymbol{\beta}_1, \mathbf{0})+\gamma(\psi, f)n^{1/2}(\boldsymbol{\beta}_{1n}-\boldsymbol{\beta}_1)n^{-1}M_{n12}$ converges to $\mathbf{0}$ in $P_{(\alpha, \beta_1, 0)}$-probability, which is the same as (3.10).

LEMMA 3.2. *Under* $H_0$,

(3.16) $n^{-1/2}\hat{\boldsymbol{S}}_{n(2)}-\boldsymbol{V}_n$ *converges to* $\mathbf{0}$ *in* $P_{(\alpha, \beta_1, 0)}$*-probability, where*

(3.17) $\boldsymbol{V}_n=n^{-1/2}[\boldsymbol{S}_{n(2)}(\boldsymbol{\beta}_1, \mathbf{0})-\boldsymbol{S}_{n(1)}(\boldsymbol{\beta}_1, \mathbf{0})M_{n11}^{-1}M_{n12}]$ .

PROOF. Under $H_0$, by (2.11), (3.15) (with $j=k=1$) and by Theorems 3.1 and 4.1 and Lemmas 4.1 and 4.5 of Jurečková [8], both

$$n^{-1/2}\boldsymbol{S}_{n(1)}(\bar{\boldsymbol{\beta}}_{1n}, \mathbf{0})-n^{-1/2}\boldsymbol{S}_{n(1)}(\boldsymbol{\beta}_1, \mathbf{0})+\gamma(\psi, f)n^{1/2}(\bar{\boldsymbol{\beta}}_{1n}-\boldsymbol{\beta}_1)n^{-1}M_{n11}$$

and $n^{-1/2}\boldsymbol{S}_{n(1)}(\bar{\boldsymbol{\beta}}_{1n}, \mathbf{0})$ converge to $\mathbf{0}$ in $P_{(\alpha, \beta_1, 0)}$-probability, It follows that

(3.18) $n^{-1/2}\boldsymbol{S}_{n(1)}(\boldsymbol{\beta}_1, \mathbf{0})-n^{-1/2}\gamma(\psi, f)(\boldsymbol{\beta}_{1n}-\boldsymbol{\beta}_1)M_{n11}$ converges to $\mathbf{0}$ in $P_{(\alpha, \beta_1, 0)}$-probability .

Multiplying (3.18) by $M_{n11}^{-1}M_{n12}$ from the right, we see that

(3.19) $n^{-1/2}\boldsymbol{S}_{n(1)}(\boldsymbol{\beta}_1, \mathbf{0})M_{n11}^{-1}M_{n12}-n^{-1/2}\gamma(\psi, f)(\boldsymbol{\beta}_{1n}-\boldsymbol{\beta}_1)M_{n12}$ converges to $\mathbf{0}$ in $P_{(\alpha, \beta_1, 0)}$-probability .

Finally, adding (3.19) to (3.10) and using (3.17), we obtain (3.16).

LEMMA 3.3. *Under* $H_0$, $\boldsymbol{V}_n$ *has asymptotically the* $(q-r)$*-variate normal distribution* $N_{q-r}(\mathbf{0}, \lambda^2(\psi)\bar{M})$, *where*

(3.20) $$\bar{M}=M_{22}-M_{21}M_{11}^{-1}M_{12}\ .$$

PROOF. The distribution of

(3.21) $$\boldsymbol{T}_n=n^{-1/2}\boldsymbol{S}_n(\boldsymbol{\beta}_1, \mathbf{0})$$

under $H_0$: $\boldsymbol{\beta}_2=\mathbf{0}$ is the same as the distribution of $n^{-1/2}\boldsymbol{S}_n(\boldsymbol{\beta})=n^{-1/2}\boldsymbol{S}_n(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ when $\boldsymbol{\beta}=(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ is the true parameter value. So

(3.22) $$\mathscr{D}(\boldsymbol{T}_n\,|\,H_0)\to N_q(\mathbf{0}, \lambda^2(\psi)M)$$

where $\mathscr{D}$ denotes distribution (see (4.1) of Jurečková [8]). Let $\boldsymbol{T}_n=(\boldsymbol{T}_{n(1)}, \boldsymbol{T}_{n(2)})$, $\boldsymbol{T}_{n(j)}=n^{-1/2}\boldsymbol{S}_{n(j)}(\boldsymbol{\beta}_1, \mathbf{0})=\boldsymbol{T}_nI_j^*$, $(j=1, 2)$, where $I_1^*=\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ is $q\times r$

and $I_2^* = \begin{bmatrix} 0 \\ I_{q-r} \end{bmatrix}$ is $q\times(q-r)$, $I$ (with subscript indicating dimension) and 0 being the identety matrix and the zero matrix of the appropriate dimensions. So (3.17) can be rewritten as

$$V_n = T_{n(2)} - T_{n(1)} M_{n11}^{-1} M_{n12} = T_n(I_2^* - I_1^* M_{n11}^{-1} M_{n12}) . \tag{3.23}$$

By (3.15), we have

$$\lim_{n\to\infty} M_{n11}^{-1} M_{n12} = M_{11}^{-1} M_{12} . \tag{3.24}$$

So, by (3.22), $V_n$ under $H_0$ is asymptotically $(q-r)$-variate normal with mean $\mathbf{0}$ and covariance matrix

$$\begin{aligned} \lambda^2(\psi)(I_2^* - I_1^* M_{11}^{-1} M_{12})' M (I_2^* - I_1^* M_{11}^{-1} M_{12}) \\ = \lambda^2(\psi)(M_{22} - M_{21} M_{11}^{-1} M_{12}) = \lambda^2(\psi)\bar{M} \end{aligned} \tag{3.25}$$

where the first equality in (3.25) follows by routine computation from the symmetry and partitioning (3.14) of $M$, which implies that $I_j^* M I_k^* = M_{jk}$ $(j, k=1, 2)$.

PROOF OF THEOREM 3.1. Lemmas 3.2 and 3.3 together imply

$$\mathcal{D}(n^{-1/2} S_{n(2)} \mid H_0) \to N_{q-r}(\mathbf{0}, \lambda^2(\psi)\bar{M}) .$$

By (2.15)-(2.16) and (3.7) we have

$$\lim_{n\to\infty} \lambda_n = \lambda(\psi) . \tag{3.26}$$

It follows that $\mathcal{D}[(n\lambda_n^2)^{-1/2} \hat{S}_{n(2)} \mid H_0] \to N_{q-r}(\mathbf{0}, \bar{M})$ and hence that $\mathcal{D}[(n\lambda_n^2)^{-1} \cdot \hat{S}_{n(2)}(\bar{M})^{-1} \hat{S}'_{n(2)} \mid H_0] \to \chi^2_{q-r}$. By (3.6), (3.15), (3.20) and (3.24), we have

$$\lim_{n\to\infty} n^{-1} \bar{M}_n = \bar{M} \tag{3.27}$$

and hence

$$\lim_{n\to\infty} n(\bar{M}_n)^{-1} = (\bar{M})^{-1} . \tag{3.28}$$

So, by (3.8), $Q_n$ under $H_0$ is asymptotically $\chi^2_{q-r}$.

The following theorem gives the asymptotic distribution of $Q_n$ under $H_n$ (see (2.20)).

THEOREM 3.2. *Under $H_n$, $Q_n$ has asymptotically the non-central chi-square distribution $\chi^2_{q-r}(\Delta_Q)$ with $q-r$ degrees of freedom and non-centrality parameter.*

$$\Delta_Q = [\gamma(\psi, f)/\lambda(\psi)]^2 \boldsymbol{b}_2 \bar{M} \boldsymbol{b}_2' . \tag{3.29}$$

We prove Theorem 3.2 through the following lemmas.

LEMMA 3.4. *Under $H_n$: $\boldsymbol{\beta}_2 = \boldsymbol{\beta}_{2n} = n^{-1/2} \boldsymbol{b}_2$,* (3.30) $n^{-1/2}S_{n(2)}-V_n$ *converges to* $\mathbf{0}$ *in* $P_{(\alpha,\beta_1,\beta_{2n})}$*-probability* .

*Remark.* Thus Lemma 3.4 states that (3.16) still holds with $P_{(\alpha,\beta_1,0)}$ replaced by $P_{(\alpha,\beta_1,\beta_{2n})}$.

PROOF. We first observe the following facts.

( I ) Assuming $\beta_1=0$ has the same effect as replacing each $X_i$ by $X_i^*=X_i-\beta_1 c_{i,1}$, $(i=1,\cdots,n)$.

( II ) So the distribution of $S_{n(j)}(\beta_1,0)$ under $P_{(\alpha,\beta_1,\beta_2)}$ is the same as the distribution of $S_{n(j)}=S_{n(j)}(0)$ under $P_{(\alpha,0,\beta_2)}$, $(j=1,2)$, whatever the value of $\beta_2$ is.

(III) If we write $\bar{\beta}_{1n}=\bar{\beta}_1(X_n)$ to indicate that the estimate is based on the observations $X_n$, then $\bar{\beta}_1$ has the invariance property

$$(3.31)\quad \bar{\beta}_1(X_n+a1_n+b_1C_{n(1)})=\bar{\beta}_1(X_n)+b_1 \qquad \text{for all } a\in R \text{ and } b_1\in R^r,$$

where $C_{n(1)}=(c_{1,1},\cdots,c_{n,1})$ (see Jurečková [8], Section 5).

(IV) $\hat{S}_{n(2)}$ remains unchanged if each $X_i$ is replaced by $X_i^*$, $(i=1,\cdots,n)$.

To see this last fact, we rewrite $S_{n(j)}(0)$ as $S_{n(j)}(X_n)$, $(j=1,2)$ to indicate the dependence on $X_n$. Then $\hat{S}_{n(2)}$ can be rewritten as

$$(3.32)\qquad \hat{S}_{n(2)}(X_n)=S_{n(2)}(X_n-\bar{\beta}_1(X_n)C_{n(1)})\ .$$

Replacing $X_n$ by $X_n^*=(X_1^*,\cdots,X_n^*)=X_n-\beta_1C_{n(1)}$ in (3.32) and using (3.31), we have $\bar{\beta}_1(X_n^*)=\bar{\beta}_1(X_n)-\beta_1$ and hence $\hat{S}_{n(2)}(X_n^*)=\hat{S}_{n(2)}(X_n)$.

By (I)-(IV) and (3.17), we can restate (3.16) and (3.30) respectively as

(3.33) $n^{-1/2}\hat{S}_{n(2)}-n^{-1/2}[S_{n(2)}-S_{n(1)}M_{11}^{-1}M_{12}]$ converges to $\mathbf{0}$ in $P_{(\alpha,0)}$-probability

and

(3.34) $n^{-1/2}S_{n(2)}-n^{-1/2}[S_{n(2)}-S_{n(1)}M_{n11}^{-1}M_{n12}]$ converges to $\mathbf{0}$ in $P_{(\alpha,0,\beta_{2n})}$)-probability .

Thus it suffices to establish (3.34).

The joint probability densities of $X_n$ corresponding to $P_{(\alpha,0)}$ and $P_{(\alpha,0,\beta_{2n})}$ are respectively

$$p_n(x)=\prod_{i=1}^n f(x_i-\alpha)$$

and

$$q_n(x)=\prod_{i=1}^n f(x_i-\alpha-\beta_{(n)}c_i)=\prod_{i=1}^n f(x_i-d_{ni}) \qquad \text{for } x=(x_1,\cdots,x_n)\in R^n,$$

where $\beta_{(n)}=(0,\beta_{2n})=(0,n^{-1/2}b_2)$ and $d_{ni}=\alpha+\beta_{(n)}c_i$, $(i=1,\cdots,n)$. It is easy to check following Hájek and Šidák ([5] p. 211 and Theorem VI.2.1) that the densities $\{q_n\}$ are contiguous to the densities $\{\bar{q}_n\}$ defined by

$$\bar{q}_n(\boldsymbol{x})=\prod_{i=1}^{n} f(x_i-\bar{d}_n)=\prod_{i=1}^{n} f(x_i-\alpha-\boldsymbol{\beta}_{(n)}\bar{\boldsymbol{c}}_n)\ , \qquad \boldsymbol{x}=(x_1\cdots, x_n)\in R^n$$

where $\bar{d}_n=n^{-1}\sum_{i=1}^{n} d_{ni}=\alpha+\boldsymbol{\beta}_{(n)}\bar{\boldsymbol{c}}_n$.

Note that by (1.2), (2.2) and (3.38) we have

$$n^{-1}M_{ni}=[n^{-1/2}(c_{mi}-\bar{c}_{mn})\cdot n^{-1/2}(c_{m'i}-\bar{c}_{m'n})]_{m,m'=1,\cdots,q}\ , \quad (i=1,\cdots,n)\ . \tag{3.35}$$

Now by (3.33) and with the notation introduced in (III) and (IV) above, we have

$$n^{-1/2}\boldsymbol{S}_{n(2)}(\boldsymbol{X}_n-\bar{\boldsymbol{\beta}}_1(\boldsymbol{X}_n)C_{n(1)})-n^{-1/2}[\boldsymbol{S}_{n(2)}(\boldsymbol{X}_n)-\boldsymbol{S}_{n(1)}(\boldsymbol{X}_n)M_{n11}^{-1}M_{n12}]\xrightarrow{p_n} 0\ .$$

Since $\boldsymbol{\beta}_{(n)}\bar{\boldsymbol{c}}_n$ is a scalar, we have $\boldsymbol{S}_{n(j)}(\boldsymbol{X}_n+\boldsymbol{\beta}_{(n)}\bar{\boldsymbol{c}}_n\boldsymbol{1}_n)=\boldsymbol{S}_{n(j)}(\boldsymbol{X}_n)$, $(j=1, 2)$, and by (3.31) we have $\bar{\boldsymbol{\beta}}_1(\boldsymbol{X}_n+\boldsymbol{\beta}_{(n)}\bar{\boldsymbol{c}}_n\boldsymbol{1}_n)=\bar{\boldsymbol{\beta}}_1(\boldsymbol{X}_n)$. Thus $S_{n(j)}$, $(j=1, 2)$ and $\bar{\boldsymbol{\beta}}_{1n}$ remain unchanged under the simultaneous transformation $X_i\rightarrow X_i+\boldsymbol{\beta}_{(n)}\bar{\boldsymbol{c}}_n$ for all $i=1,\cdots,n$, which is the same as replacing $p_n$ by $\bar{q}_n$. So we have

$$n^{-1/2}\hat{\boldsymbol{S}}_{n(2)}-n^{-1/2}[\boldsymbol{S}_{n(2)}-\boldsymbol{S}_{n(1)}M_{n11}^{-1}M_{n12}]\xrightarrow{\bar{q}_n}\boldsymbol{0}\ . \tag{3.36}$$

Hence, by contiguity, (3.36) still holds with $\bar{q}_n$ replaced by $q_n$, which is the same as (3.34), as was to be proved.

LEMMA 3.5. *Under $H_n$, we have*

$$\mathcal{D}(\boldsymbol{V}_n\,|\,H_n)\rightarrow N_{q-r}(\gamma(\psi, f)\boldsymbol{b}_2\bar{M}, \lambda^2(\psi)\bar{M})\ .$$

PROOF. Let $\boldsymbol{S}_n=\boldsymbol{S}_n(\boldsymbol{0})$. Then $n^{-1/2}\boldsymbol{S}_n$ is asymptotically normal $N_q(\gamma(\psi, f)n^{1/2}\boldsymbol{\beta}M, \lambda^2(\psi)M)$ (see Theorem 3.1 and (4.1) of Jurečková [8]). So, under $H_n^*$: $\boldsymbol{\beta}=\boldsymbol{\beta}_{(n)}=(0, n^{-1/2}\boldsymbol{b}_2)$, by (3.14) we have

$$\mathcal{D}(n^{-1/2}\boldsymbol{S}_n\,|\,H_n^*)\rightarrow N_q(\gamma(\psi, f)\boldsymbol{b}_2(M_{21}, M_{22}), \lambda^2(\psi)M)\ .$$

The distribution of $\boldsymbol{T}_n$ (see (3.21)) under $H_n$ is the same as the distribution of $n^{-1/2}\boldsymbol{S}_n$ under $H_n^*$. Hence

$$\mathcal{D}(\boldsymbol{T}_n\,|\,H_n)\rightarrow N_q(\gamma(\psi, f)\boldsymbol{b}_2(M_{21}, M_{22}), \lambda^2(\psi)M)\ .$$

So, by (3.23)-(3.25), $\boldsymbol{V}_n$ under $H_n$ is asymptotically normal with covariance matrix $\lambda^2(\psi)\bar{M}$ and mean

$$\gamma(\psi, f)\boldsymbol{b}_2(M_{21}, M_{22})(I_2^*-I_1^*M_{11}^{-1}M_{12}) \\ =\gamma(\psi, f)\boldsymbol{b}_2(M_{22}-M_{21}M_{11}^{-1}M_{21})=\gamma(\psi, f)\boldsymbol{b}_2\bar{M}\ .$$

PROOF OF THEOREM 3.2. By Lemmas 3.4 and 3.5 and by (3.26), under $H_n$, the statistic $(n\lambda_n^2)^{-1}\hat{\boldsymbol{S}}_{n(2)}(\bar{M})^{-1}\hat{\boldsymbol{S}}'_{n(2)}$ is asymptotically non-central chi-square with $q-r$ degrees of freedom and noncentrality parameter

$$\{[\gamma(\psi, f)/\lambda(\psi)]\boldsymbol{b}_2\bar{M}\}(\bar{M})^{-1}\{[\gamma(\psi, f)/\lambda(\psi)]\boldsymbol{b}_2\bar{M}\}' \\ =[\gamma(\psi, f)/\lambda(\psi)]^2\boldsymbol{b}_2\bar{M}\boldsymbol{b}_2'=\Delta_Q .$$

Hence unsing (3.28) we have $\mathcal{D}(Q_n|H_n)\rightarrow\chi^2_{q-r}(\Delta_Q)$.

## 4. Asymptotic efficiency

We now compare the rank procedures for testing (1.2) based on $Q_n$ with the general likelihood ratio test and the test of the same hypothesis based on the least-squares estimates of $\boldsymbol{\beta}_2$. For the likelihood procedure, we make Assumptions I–V and VII of Wald [13]. And for the least-squares procedure we make the following usual assumptions:

$$\text{(4.1)} \qquad \mathrm{E}(Z_i)=0$$

$$\text{(4.2)} \qquad 0<\mathrm{Var}(Z_i)=\sigma^2<\infty , \qquad (i=1,\cdots, n) .$$

*Remarks.* 1. A dual form of Wald's [13] Assumption VI is trivially satisfied by the present problem. Assumption III(c) is also redundant, since here it reduces to the assumption that the determinant of the information matrix $I(f)A_n$ is positive, which is equivalent to the finiteness of $I(f)$ and the positive definiteness of $A_n$ (see (2.4)).

2. (4.1) also follows from the symmetry of $F$. In the special case that $F$ is normal, we have $\sigma^2=1/I(f)$, and (4.2) is equivalent to the finiteness of $I(f)$.

The likelihood ratio test rejects $H_0$ (in favor of $H$) when the likelihood ratio statistic

$$\Lambda_n=\sup\left\{\prod_{i=1}^n f(X_i-a-\boldsymbol{b}_1\boldsymbol{c}_{i,1}):\ a\in R,\ \boldsymbol{b}_1\in R^r\right\}\Big/ \\ \sup\left\{\prod_{i=1}^n f(X_i-a-\boldsymbol{b}\boldsymbol{c}_i):\ a\in R,\ \boldsymbol{b}\in R^q\right\}$$

is small, or equivalently when

$$\text{(4.3)} \qquad L_n=-2\log\Lambda_n$$

is large. Here $f$ (or equivalently $F$) is assumed to be known but not necessarily normal.

Let the least-squares estimate $\tilde{\boldsymbol{\beta}}_n=(\tilde{\beta}_{n1},\cdots,\tilde{\beta}_{nq})$ of $\boldsymbol{\beta}$ (based on $\boldsymbol{X}_n$) be partitioned as

$$\text{(4.4)} \qquad \tilde{\boldsymbol{\beta}}_n=(\tilde{\boldsymbol{\beta}}_{1n}, \tilde{\boldsymbol{\beta}}_{2n}) \quad \text{where} \quad \tilde{\boldsymbol{\beta}}_{2n}=(\tilde{\beta}_{n,r+1},\cdots,\tilde{\beta}_{nq})$$

and let $s_n^2$ be the corresponding unbiased estimate of $\sigma^2$. We also partition $A_n$ as

$$A_n = \begin{bmatrix} A_{n11} & A_{n12} \\ A_{n21} & A_{n22} \end{bmatrix} \tag{4.5}$$

where $A_{n11}$ is $(r+1)\times(r+1)$, and define

$$\bar{A}_n = A_{n22} - A_{n21}A_{n11}^{-1}A_{n12} . \tag{4.6}$$

Then the classical normal-theory test of (1.2) is based on the statistic

$$\mathcal{F}_n = \tilde{\boldsymbol{\beta}}_{2n}\bar{A}_n\tilde{\boldsymbol{\beta}}'_{2n}/[(q-r)s_n^2] \tag{4.7}$$

(see, e.g., Anderson [2], Section 2.2), or equivalently on the statistic

$$L_n^* = (q-r)\mathcal{F}_n = \tilde{\boldsymbol{\beta}}_{2n}\bar{A}_n\tilde{\boldsymbol{\beta}}'_{2n}/s_n^2 . \tag{4.8}$$

It is well known that if $F$ is normal, then $\mathcal{F}_n = (\Lambda_n^{-2/n}-1)(n-q-1)/(q-r)$ (see, e.g., Scheffé [11], p. 36), and $\mathcal{F}_n$ under $H_0$ has the $F$-distribution with $q-r$ and $n-q-1$ degrees of freedom. It will be shown later (in the proof of Theorem 4.1) that

$$\bar{A}_n = \bar{M}_n . \tag{4.9}$$

Thus $L_n^*$ can also be expressed as

$$L_n^* = \tilde{\boldsymbol{\beta}}_{2n}\bar{M}_n\tilde{\boldsymbol{\beta}}'_{2n}/s_n^2 . \tag{4.10}$$

The following two theorems give the asymptotic distribution of $L_n$ and $L_n^*$ under $H_n$, but under no assumptions about the specific form of $F$.

THEOREM 4.1. *Under $H_n$, $L_n$ is asymptotically $\chi^2_{q-r}(\Delta_L)$, where*

$$\Delta_L = I(f)\boldsymbol{b}_2\bar{M}\boldsymbol{b}'_2 . \tag{4.11}$$

THEOREM 4.2. *Under $H_n$, $L_n^*$ is asymptotically $\chi^2_{q-r}(\Delta_L^*)$, where*

$$\Delta_L^* = \sigma^{-2}\boldsymbol{b}_2\bar{M}\boldsymbol{b}'_2 . \tag{4.12}$$

*Remark* 3. In the special case that $F$ is normal, we have $\Delta_L = \Delta_L^*$, which comes as no surprise because in this case the likelihood procedure is equivalent to the least-squares procedure.

It follows from Theorems 3.2, 4.1 and 4.2 that the asymptotic relative efficiencies of the rank procedures based on $Q_n$ with respect to the likelihood procedure (based on $L_n$) and the least-squares procedure (based on $L_n^*$) are given by

$$\begin{aligned} e_{Q,L}(F) &= \gamma^2(\phi, f)/I(f)\lambda^2(\phi) \\ &= \left[\int_0^1 \phi(u)\phi_f(u)du\right]^2 \Big/ \left\{I(f)\int_0^1 [\phi(u)-\bar{\phi}]^2 du\right\} \end{aligned} \tag{4.13}$$

and

$$e_{Q,L^*}(F)=\sigma^2\gamma^2(\psi, f)/\lambda^2(\psi) \tag{4.14}$$
$$=\sigma^2\left[\int_0^1 \psi(u)\phi_f(u)du\right]^2\Big/\left[\int_0^1 \psi(u)-\bar{\psi}\right]^2 du .$$

*Remarks.* 4. If the score-generating function for $Q_n$ is $\psi=\phi_f$ (see (3.11)) then by (2.16), (2.19) and (4.13) we have $e_{Q,L}(F)=1$, i.e., the $Q_n$-test is asymptotically power-equivalent to the likelihood ratio test.

5. The quantity on the right-hand side of (4.13) or (4.14) has been extensively studied. Thus with $\psi=\Phi^{-1}$, $e_{Q,L^*}(F)$ is not less than 1, and is equal to 1 if and only if $F$ is normal (see, e.g., Puri and Sen [10], p. 118, Theorem 3.8.2). Thus the $Q_n$-test using normal scores is asymptotically at least as powerful as the classical least-squares procedure, and more powerful than the latter unless the underlying distribution $F$ is actually normal.

PROOF OF THEOREM 4.1. We first prove (4.9). By (2.2)–(2.4), (2.10) and (4.5) we have

$$A_{n11}=\begin{bmatrix} n & n\bar{\boldsymbol{c}}'_{n,1} \\ n\bar{\boldsymbol{c}}_{n,1} & \sum_{i=1}^{n} \boldsymbol{c}_{i,1}\boldsymbol{c}'_{i,1} \end{bmatrix}, \tag{4.15}$$

$$A_{n22}=\sum_{i=1}^{n} \boldsymbol{c}_{i,2}\boldsymbol{c}'_{i,2} , \tag{4.16}$$

$$A_{n21}=\left(\sum_{i=1}^{n} \boldsymbol{c}_{i,2}, \sum_{i=1}^{n} \boldsymbol{c}_{i,2}\boldsymbol{c}'_{i,1}\right) \tag{4.17}$$

and

$$A_{n12}=A'_{n21} , \tag{4.18}$$

where $\bar{\boldsymbol{c}}_{n,j}$, $(j=1, 2)$ is defined by (3.12). By using the generalized Gauss algorithm (see, e.g., Gantmacher [3], Vol. I, p. 45) and the obvious identity

$$M_{njk}=\sum_{i=1}^{n} \boldsymbol{c}_{i,j}\boldsymbol{c}'_{i,k}-n\bar{\boldsymbol{c}}_{n,j}\bar{\boldsymbol{c}}'_{n,k} , \qquad (j, k=1, 2) \tag{4.19}$$

(which follows immediately from (3.13)) we have

$$A_{n11}^{-1}=\begin{bmatrix} \dfrac{1}{n}+\bar{\boldsymbol{c}}'_{n,1}M_{n11}^{-1}\bar{\boldsymbol{c}}_{n,1} & -\boldsymbol{c}'_{n,1}M_{n11}^{-1} \\ -M_{n11}^{-1}\bar{\boldsymbol{c}}_{n,1} & M_{n11}^{-1} \end{bmatrix}. \tag{4.20}$$

By (4.6), (3.6) and routine computation we have $A_{n21}A_{n11}^{-1}A_{n12}=n\bar{\boldsymbol{c}}_{n,2}\bar{\boldsymbol{c}}'_{n,2}+M_{n21}M_{n11}^{-1}M_{n12}$ and hence $\bar{A}_n=\bar{M}_n$. Thus (4.9) is established. It follows from Theorem IX of Wald ([13], p. 480) that $L_n$ under $H_n$: $\boldsymbol{\beta}_2 = n^{-1/2}\boldsymbol{b}_2$ is asymptotically non-central chi-square with $q-r$ degrees of freedom and noncentrality parameter

$$\begin{aligned}\Delta_{L_n} &= I(f)(n^{-1/2}\boldsymbol{b}_2)\bar{A}_n(n^{-1/2}\boldsymbol{b}_2)' \\ &= I(f)\boldsymbol{b}_2(n^{-1}\bar{M}_n)\boldsymbol{b}_2' \to I(f)\boldsymbol{b}_2\bar{M}\boldsymbol{b}_2' = \Delta_L\ , \qquad \text{as } n\to\infty\ ,\end{aligned}$$

where the convergence follows from (3.27). The proof is completed.

PROOF OF THEOREM 4.2. Let $\tilde{\alpha}_n$ be the least-squares estimate of $\alpha$ (based on $\boldsymbol{X}_n$). Then $n^{1/2}[(\tilde{\alpha}_n, \tilde{\boldsymbol{\beta}}_n)-(\alpha, \boldsymbol{\beta})] = (n^{1/2}[(\tilde{\alpha}_n, \tilde{\boldsymbol{\beta}}_{1n})-(\alpha, \boldsymbol{\beta}_1)],\ n^{1/2}(\tilde{\boldsymbol{\beta}}_{2n}-\boldsymbol{\beta}_2))$ is asymptotically $N_{q+1}(\boldsymbol{0}, \sigma^2 A^{-1})$ (where $A$ is given by (2.6)), and $s_n^2$ is a consistent estimate of $\sigma^2$ (see Anderson [2], p. 25, Corollary 2.6.1 and Theorem 2.6.2). Let $A$ be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix} \tag{4.21}$$

where $A_{11}$ is $(r+1)\times(r+1)$, and define

$$\bar{A} = A_{22} - A_{21}A_{11}^{-1}A_{12}\ . \tag{4.22}$$

Then by (4.4) we have

$$\mathscr{D}[n^{1/2}(\tilde{\boldsymbol{\beta}}_{2n}-\boldsymbol{\beta}_2)] \to N_{q-r}(\boldsymbol{0},\ \sigma^2(\bar{A})^{-1}) \tag{4.23}$$

(see, e.g., Theorem 1.3.1 of Graybill [4] for inverses of partitioned matrices). By (2.6), (4.5) and (4.21) we have $\lim_{n\to\infty} n^{-1}A_{njk} = A_{jk}$ ($j, k=1, 2$). So by (4.6) we have $\lim_{n\to\infty} n^{-1}\bar{A}_n = \bar{A}$. It follows from (3.27) and (4.9) that $\bar{A} = \bar{M}$. Now, by (4.23), under $H_n$: $\boldsymbol{\beta}_2 = n^{-1/2}\boldsymbol{b}_2$ we have

$$\mathscr{D}(n^{1/2}\tilde{\boldsymbol{\beta}}_{2n}\,|\,H_n) \to N_{q-r}(\boldsymbol{b}_2,\ \sigma^2(\bar{A})^{-1})\ .$$

Hence, by (4.8) and consistent estimation of $\sigma^2$ by $s_n^2$, $L_n^*$ under $H_n$ is asymptotically non-central chi-square with $q-r$ degrees of freedom and noncentrality parameter.

$$\sigma^{-2}\boldsymbol{b}_2\bar{A}\boldsymbol{b}_2' = \sigma^{-2}\boldsymbol{b}_2\bar{M}\boldsymbol{b}_2' = \Delta_L^*\ .$$

## 5. Asymptotic optimality

Let $B_n$ be the $(q-r)\times(q-r)$ non-singular matrix such that $B_n'B_n = \bar{A}_n = \bar{M}_n$, let $\Gamma_{n1}$ be the $(r+1)\times(r+1)$ non-singular matrix satisfying $\Gamma_{n1}'\Gamma_{n1} = A_{n11}$, and define the $(r+1)\times(q-r)$ matrix $\Gamma_{n2} = (\Gamma_{n1}')^{-1}A_{n12}$ (see (4.15) and (4.17)–(4.18)). Then the $(q+1)\times(q+1)$ matrix

$$K_n=\begin{bmatrix}\Gamma_{n1} & \Gamma_{n2}\\ 0 & B_n\end{bmatrix}$$

is nonsingular and satisfies $K_nA_n^{-1}K_n'=I_{q+1}$.

Let $\Omega=R^{q+1}=\{(a,\boldsymbol{b}):\ a\in R,\ \boldsymbol{b}\in R^q\}$ and $\Omega_0=\{(a,\boldsymbol{b}_1,0)\in\Omega:\ a\in R,\ \boldsymbol{b}_1\in R^r\}$. For $\boldsymbol{\omega}=(a,b_1,0)\in\Omega_0$ and $c>0$ define the surface

$$S(\boldsymbol{\omega},c)=\{(\alpha,\boldsymbol{\beta}_1,\boldsymbol{\beta}_2)\in\Omega:\ I(f)\boldsymbol{\beta}_2\bar{A}_n\boldsymbol{\beta}_2'=c,\ (\alpha,\boldsymbol{\beta}_1,\boldsymbol{\beta}_2)\Gamma_n'=(a,b_1)\Gamma_{n1}'\}$$

where $\Gamma_n=(\Gamma_{n1},\Gamma_{n2})$ is an $(r+1)\times(q+1)$ matrix. Consider the transformation of $\Omega$

$$\boldsymbol{\theta}=(\alpha,\boldsymbol{\beta}_1,\boldsymbol{\beta}_2)\to\boldsymbol{\theta}^*=(\alpha^*,\boldsymbol{\beta}_1^*,\boldsymbol{\beta}_2^*)=[I(f)]^{1/2}\boldsymbol{\theta}K_n'\,, \tag{5.1}$$

which is also given by

$$(\alpha^*,\boldsymbol{\beta}_1^*)=[I(f)]^{1/2}(\alpha,\boldsymbol{\beta}_1,\boldsymbol{\beta}_2)\Gamma_n'\,,\qquad \boldsymbol{\beta}_2^*=[I(f)]^{1/2}\boldsymbol{\beta}_2B_n' \tag{5.2}$$

and maps $S(\boldsymbol{\omega},c)$ into

$$S^*(\boldsymbol{\omega},c)=\{(\alpha^*,\boldsymbol{\beta}_1^*,\boldsymbol{\beta}_2^*)\in\Omega:\ (\alpha^*,\boldsymbol{\beta}_1^*)=[I(f)]^{1/2}(a,\boldsymbol{b}_1)\Gamma_{n1}',\ \boldsymbol{\beta}_2^*\boldsymbol{\beta}_2^{*\prime}=c\}\ .$$

For $\boldsymbol{\theta}_0\in\Omega$ and $\rho>0$ define

$$\Omega(\boldsymbol{\theta}_0,\rho)=\{\boldsymbol{\theta}\in\Omega:\ \boldsymbol{\theta},\boldsymbol{\theta}_0\in S(\boldsymbol{\omega},c)\ \text{for some}\ \boldsymbol{\omega}\in\Omega_0\ \text{and}\ c>0,\ \text{and}\ \|\boldsymbol{\theta}-\boldsymbol{\theta}_0\|\leqq\rho\}$$

(where $\|\ \|$ is the Euclidean norm on $\Omega$), and let $\Omega^*(\boldsymbol{\theta}_0,\rho)$ be its image under the transformation (5.1). For $\boldsymbol{\theta}\in\Omega$ let

$$\eta(\boldsymbol{\theta})=\lim_{\rho\to0}\{\mathcal{A}[\omega^*(\boldsymbol{\theta},\rho)]/\mathcal{A}[\Omega(\boldsymbol{\theta},\rho)]\}$$

where $\mathcal{A}$ denotes area. Then, by Theorem VIII of Wald ([13], p. 478), the likelihood ratio test for (1.2) is asymptotically optimal in the sense that it

(a) has asymptotically best average power with respect to the weight function $\eta(\boldsymbol{\theta})$ and the family of surfaces $\mathcal{S}=\{S(\boldsymbol{\omega},c):\ \boldsymbol{\omega}\in\Omega_0,\ c>0\}$;

(b) has asymptotically best constant power on the surface in $\mathcal{S}$;

(c) is an asymptotically most stringest test.

By Remark 4 of Section 4, it follows that with the score-generating function $\psi=\phi_f$, the $Q_n$-test is asymptotically power-equivalent to the Wald-optimal likelihood ratio test. Thus if the underlying distribution function $F$ is logistic, then the $Q_n$-test using Wilcoxon scores is asymptotically optimal; and if $F$ is normal, then the $Q_n$-test using normal scores is asymptotically optimal.

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