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THE POLYTOPAL ASSOCIATION SCHEME

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Summary

The polytopal association scheme for PBIB designs is introduced and studied utilizing the concept of clustering of treatments.

1. Introduction

In [1], a family of m-associate class partially balanced association schemes (PBAS(m)) was defined by considering n treatments as being clustered at each of the s vertices of a regular s-sided polygon. This *polygonal association scheme* and the resulting partially balanced incomplete block (PBIB) designs were studied in detail for $s=3$, 4, 5 and 6.

In the present paper, the ideas begun in [1] are extended to general s-sided polygons (Section 4), to p-dimensional polytopes (Section 3), and to the five regular polyhedra in three dimensional space (Section 5).

2. Clustering and the polytopal association scheme

In [3], it was shown how a PBAS having $m+1$ associate classes could be constructed from a PBAS having m associate classes. The procedure involves replacing each of the s treatments of the original PBAS by n treatments. The resulting *clustered* PBAS in $v=ns$ treatments will be referred to as a CPBAS$(m+1)$. The relation between the parameters of the PBAS(m) and the CPBAS$(m+1)$ are given in [3]. The characteristic roots of the resulting PBIB$(m+1)$ design in terms of those of the PBIB(m) design are given by the following theorem.

THEOREM 2.1. *Let N denote the incidence matrix of a* PBIB(m) *design and let NN' have characteristic roots $\theta_0, \theta_1, \theta_2, \cdots, \theta_m$ with multi-*

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plicities $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_m$. If N^* denotes the incidence matrix of the corresponding clustered PBIB$(m+1)$ *design, then the characteristics roots of $N^*N^{*\prime}$ are $\theta_i^*=n\theta_i$ with multiplicities $\alpha_i^*=\alpha_i$ for $i=0, 1, 2, \cdots, m$, and $\theta_{m+1}^*=0$ with multiplicity $\alpha_{m+1}^*=v(n-1)$.*

This theorem is easily established by noting that $N^*=N\otimes J_n$, the Kronecker product of N with a column vector of n 1's. Also of note is that $\theta_{m+1}^*=r-\lambda_{m+1}=0$ so that $\lambda_{m+1}=r$, where λ_i denotes the number of blocks in the design in which two treatments that are ith associates jointly appear.

The *regular polytopal association scheme* provides an example of the usefulness of the clustering principle. It can be defined as follows. Consider an s-vertexed regular polytope in p-dimensional space, and suppose n treatments are clustered at each of the s vertices, so that $v=ns$. Two treatments belonging to adjacent (i.e. one-edge-away) vertices will be called first associates; two treatments "two-edges-away" will be second associates; . . . ; two treatments "$(m-1)$-edges-away" will be $(m-1)$th associates; and two treatments in the same cluster will be mth associates.

The important properties of a PBAS are contained in its parameters, namely the number of treatments involved (v), the number of associate classes (m), the number of ith associates of each treatment (n_i, $i=1, 2, \cdots, m$), and the matrices $P_i=(p_{jk}^i)$ in which p_{jk}^i denotes the number of treatments that are simultaneously jth associates of one treatment (say t_1) and kth associates of another treatment (say t_2), where t_1 and t_2 are themselves ith associates (for $i, j, k=1, 2, \cdots, m$). The important properties of the corresponding PBIB design are the characteristic roots $\theta_0, \theta_1, \cdots, \theta_m$ of NN', where N again denotes the incidence matrix of the design, together with their multiplicities $\alpha_0, \alpha_1, \cdots, \alpha_m$. In the next three sections, these properties are given for regular polytopal designs.

3. The regular polytopal association scheme for general p

For $p>4$ there are only three regular polytopes each of which has an analogue in 2, 3 and 4-dimensional space. These three are the *regular simplex*, the *cross polytope*, and the *measure polytope*. The designs corresponding to these polytopes will be examined in this section.

Regular simplex. The regular simplex in p-dimensions has $s=p+1$ vertices, each vertex being joined to every other by an edge (see Fig. 1). For $p=2$ the regular simplex is an equilateral triangle and for $p=3$ it is a regular tetrahedron. The parameters of the regular simplex design are given in Table 1.

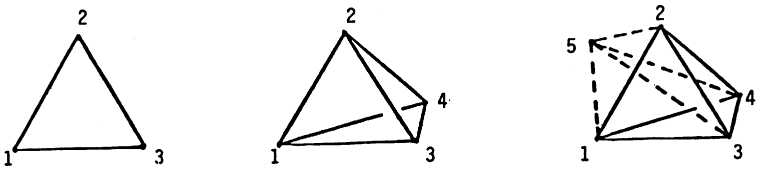

Fig. 1. The regular simplex in p-space for $p=2, 3, 4$

Table 1. Parameters of the regular simplex design

$v=n(p+1)$	$m=2$
$n_1=np$	$n_2=n-1$
$P_1=\begin{bmatrix} n(p-1) & n-1 \\ n-1 & 0 \end{bmatrix}$	$P_2=\begin{bmatrix} np & 0 \\ 0 & n-2 \end{bmatrix}$
$\theta_0=rk$	$\alpha_0=1$
$\theta_1=r-n\lambda_1+(n-1)\lambda_2$	$\alpha_1=p$
$\theta_2=r-\lambda_2$	$\alpha_2=(n-1)(p+1)$

Cross polytope. The cross polytope in p-dimensions has $s=2p$ vertices, each vertex being joined to $2(p-1)$ other vertices by an edge. The remaining vertices are two-edges-away (see Fig. 2). For $p=2$ the cross polytope is a square and for $p=3$ it is a regular octahedron. The parameters of the cross polytope design are given in Table 2.

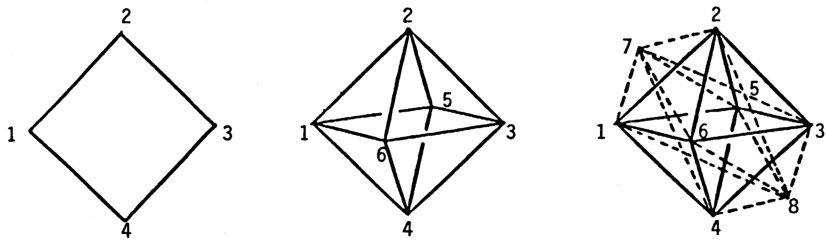

Fig. 2. The cross polytope in p-space for $p=2, 3, 4$

Table 2. Parameters of the cross polytope design

$v=2np$	$m=3$	
$n_1=2n(p-1)$	$n_2=n$	$n_3=n-1$
$P_1=\begin{bmatrix} 2n(p-2) & n & n-1 \\ n & 0 & 0 \\ n-1 & 0 & 0 \end{bmatrix}$		
$P_2=\begin{bmatrix} 2n(p-1) & 0 & 0 \\ 0 & 0 & n-1 \\ 0 & n-1 & 0 \end{bmatrix}$		
$P_3=\begin{bmatrix} 2n(p-1) & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & n-2 \end{bmatrix}$		
$\theta_0=rk$		$\alpha_0=1$
$\theta_1=r-2n\lambda_1+n\lambda_2+(n-1)\lambda_3$		$\alpha_1=p-1$
$\theta_2=r-n\lambda_2+(n-1)\lambda_3$		$\alpha_2=p$
$\theta_3=r-\lambda_3$		$\alpha_3=2p(n-1)$

Measure polytope. The p-dimensional analogue of the cube is the measure polytope. It has $s=2^p$ vertices and $p2^{p-1}$ edges with each vertex being joined to p others. Other vertices are at most p-edges-away (see Fig. 3). For $p=2$ the measure polytope is a square and for $p=3$ it is a cube. The parameters of the measure polytope design are given in Table 3.

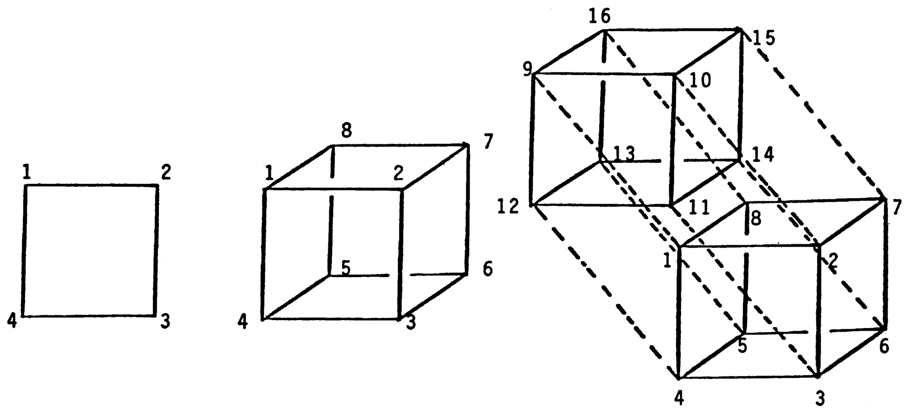

Fig. 3. The measure polytope in p-space for $p=2, 3, 4$

Table 3. Parameters of the measure polytope design

$$v=n2^p \qquad m=p+1$$

$$n_i=n\binom{p}{i}, \qquad i=1, 2, \cdots, p \quad n_{p+1}=n-1$$

$P_{p+1}=\text{diag}(n_1, n_2, \cdots, n_p, n-2)$
and, for $i=1, 2, \cdots, p$,
$P_i=(p^i_{jk})$ is a symmetric matrix with

$$p^i_{jk}=\begin{cases} n\binom{i}{i/2}\binom{p-i}{k-i/2} & \text{for } j=k=1, 2, \cdots, p \\ 0 & \text{for } j=k=p+1 \\ n\binom{i}{\frac{k-j+i}{2}}\binom{p-i}{\frac{k+j-i}{2}} & \text{for } j<k=1, 2, \cdots, p \\ (n-1)\delta_{ij} & \text{for } j=1, 2, \cdots, p, \ k=p+1 \end{cases}$$

in which δ_{ij} is the Kronecker delta and $\binom{a}{b}$ is assumed to be 0 if b is not an integer or if $b>a$ or if $b<0$.

$$\theta_0=rk \qquad \alpha_0=1$$

$$\theta_i=r+\sum_{j=1}^{p+1}\lambda_j z_{ij} \qquad \alpha_i=\binom{p}{i} \qquad \text{for } i=1, 2, \cdots, p$$

$$\theta_{p+1}=r-\lambda_{p+1} \qquad \alpha_{p+1}=2^p(n-1)$$

in which $z_{ij}=\sum_{k=\max(0,\, i+j-p)}^{\min(i,j)}\binom{i}{k}\binom{p-i}{j-k}(-1)^k$
for $i, j=1, 2, \cdots, p$
and $z_{i,p+1}=n-1$ for $i=1, 2, \cdots, p$.

4. The polygonal association scheme

For $p=2$, polytopes become polygons and give rise to the polygonal association scheme introduced in [1]. Let n treatments be clustered at each of the s vertices of a regular s-sided polygon, for $s=3, 4, 5, \cdots$. The resulting polygonal association scheme has parameters as given in Table 4.

Table 4. Parameters of polygonal Designs

$v=ns \qquad m=(s+1)/2 \quad$ for s odd

$\qquad\qquad m=(s+2)/2 \quad$ for s even

For $i=1, 2, \cdots, m-1$,

$$P_i=\left[\begin{array}{c|c} Q_i & R_i \\ \hline R_i & 0 \end{array}\right]$$

in which $Q_i=(q^i_{jk})$ is an $(m-1)\times(m-1)$ symmetic matrix with $q^i_{jk}=n(\delta_{|j-k|,i}+\delta_{j+k,i}+\delta_{j+k,s-i})$ and $R_i=(r_{j,1})$ is an $(m-1)\times 1$ column vector with $r_{j,1}=(n-1)\delta_{i,j}$,

and $P_m=\text{diag}\,(2n, 2n, \cdots, 2n, n-2) \quad$ for s odd

$\qquad =\text{diag}\,(2n, 2n, \cdots, 2n, n, n-2) \quad$ for s even.

$\theta_0=rk$

$\theta_i=r+\sum_{j=1}^{m} n_j\lambda_j \cos(2\pi ij/s) \quad$ for $i=1, 2, \cdots, m-1$

$\theta_m=r-\lambda_m$

with $\alpha_0=1$, $\alpha_m=(n-1)s$, and

for s odd, $\alpha_i=2$ for $i=1, 2, \cdots, m-1$ and

for s even, $\alpha_i=2$ for $i=1, 2, \cdots, m-2$

$\alpha_{m-1}=1.$

One interesting feature of these designs is that, with proper numbering of the ns treatments, the matrix NN' becomes a *circulant* matrix. Suppose treatment number t $(t=1, 2, \cdots, ns)$ is assigned coordinates (α, β) $(\alpha=1, 2, \cdots, s;\ \beta=1, 2, \cdots, n)$ such that $t=(\beta-1)s+\alpha$ so that the treatment numbered t is the βth treatment in the cluster of n treatments at the αth vertex. Two treatments (α, β) and (α', β') will then be kth associates $(k=1, 2, \cdots, m-1)$ if $|\alpha-\alpha'|=k$ and will be mth associates if $\alpha=\alpha'$.

With this numbering, the matrix NN' for a polygonal design is a real symmetric circulant matrix. Its characteristic roots have the form

$$\theta_i=\sum_{j=1}^{ns} a_j \cos(2\pi ij/ns)\ , \qquad i=1, 2, \cdots, ns$$

where $a_1, a_2, \cdots, a_{ns}$ are the entries in the first row of NN' (see [2]). Observing that the a_j values are each r, λ_1, $\lambda_2, \cdots$, or λ_m, and that certain characteristic roots are repeated, the characteristic roots of NN' and their multiplicities can be written as in Table 4.

5. The polyhedral association scheme

There are five regular polyhedra in 3-space, each giving rise to a PBAS in a similar way. The tetrahedron, octahedron and hexahedron are special cases of the regular simplex, cross polytope and measure polytope, respectively, with $p=3$. The icosahedron and dodecahedron designs have parameters given in Tables 5 and 6 respectively.

Table 5. Parameters of the icosahedron design

$v=12n \qquad m=4$

$n_1=5n \qquad n_2=5n \qquad n_3=n \qquad n_4=n-1$

$$P_1=\begin{bmatrix} 2n & 2n & 0 & n-1 \\ 2n & 2n & n & 0 \\ 0 & n & 0 & 0 \\ n-1 & 0 & 0 & 0 \end{bmatrix} \qquad P_2=\begin{bmatrix} 2n & 2n & n & 0 \\ 2n & 2n & 0 & n-1 \\ n & 0 & 0 & 0 \\ 0 & n-1 & 0 & 0 \end{bmatrix}$$

$$P_3=\begin{bmatrix} 0 & 5n & 0 & 0 \\ 5n & 0 & 0 & 0 \\ 0 & 0 & 0 & n-1 \\ 0 & 0 & n-1 & 0 \end{bmatrix} \qquad P_4=\begin{bmatrix} 5n & 0 & 0 & 0 \\ 0 & 5n & 0 & 0 \\ 0 & 0 & n & 0 \\ 0 & 0 & 0 & n-2 \end{bmatrix}$$

$\theta_0=rk \qquad \alpha_0=1$

$\theta_1=r-n\lambda_1-n\lambda_2+n\lambda_3+(n-1)\lambda_4 \qquad \alpha_1=5$

$\theta_2=r+\sqrt{5}\,n\lambda_1-\sqrt{5}\,n\lambda_2-n\lambda_3+(n-1)\lambda_4 \qquad \alpha_2=3$

$\theta_3=r-\sqrt{5}\,n\lambda_1+\sqrt{5}\,n\lambda_2-n\lambda_3+(n-1)\lambda_4 \qquad \alpha_3=3$

$\theta_4=r-\lambda_4 \qquad \alpha_4=12(n-1)$

Table 6. Parameters of the dodecahedron design

$v=20n \qquad m=6$

$n_1=3n \qquad n_2=6n \qquad n_3=6n \qquad n_4=3n \qquad n_5=n \qquad n_6=n-1$

$$P_1=\begin{bmatrix} 0 & 2n & 0 & 0 & 0 & n-1 \\ 2n & 2n & 2n & 0 & 0 & 0 \\ 0 & 2n & 2n & 2n & 0 & 0 \\ 0 & 0 & 2n & 0 & n & 0 \\ 0 & 0 & 0 & n & 0 & 0 \\ n-1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad P_2=\begin{bmatrix} n & n & n & 0 & 0 & 0 \\ n & n & 2n & n & 0 & n-1 \\ n & 2n & n & n & n & 0 \\ 0 & n & n & n & 0 & 0 \\ 0 & 0 & n & 0 & 0 & 0 \\ 0 & n-1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_3=\begin{bmatrix} 0 & n & n & n & 0 & 0 \\ n & 2n & n & n & n & 0 \\ n & n & 2n & n & 0 & n-1 \\ n & n & n & 0 & 0 & 0 \\ 0 & n & 0 & 0 & 0 & 0 \\ 0 & 0 & n-1 & 0 & 0 & 0 \end{bmatrix} \qquad P_4=\begin{bmatrix} 0 & 0 & 2n & 0 & n & 0 \\ 0 & 2n & 2n & 2n & 0 & 0 \\ 2n & 2n & 2n & 0 & 0 & 0 \\ 0 & 2n & 0 & 0 & 0 & n-1 \\ n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n-1 & 0 & 0 \end{bmatrix}$$

$$P_5=\begin{bmatrix} 0 & 0 & 0 & 3n & 0 & 0 \\ 0 & 0 & 6n & 0 & 0 & 0 \\ 0 & 6n & 0 & 0 & 0 & 0 \\ 3n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & n-1 \\ 0 & 0 & 0 & 0 & n-1 & 0 \end{bmatrix} \qquad P_6=\begin{bmatrix} 3n & 0 & 0 & 0 & 0 & 0 \\ 0 & 6n & 0 & 0 & 0 & 0 \\ 0 & 0 & 6n & 0 & 0 & 0 \\ 0 & 0 & 0 & 3n & 0 & 0 \\ 0 & 0 & 0 & 0 & n & 0 \\ 0 & 0 & 0 & 0 & 0 & n-2 \end{bmatrix}$$

$\theta_0=rk$	$\alpha_0=1$
$\theta_1=r-3n\lambda_2+3n\lambda_3-n\lambda_5+(n-1)\lambda_6$	$\alpha_1=4$
$\theta_2=r+n\lambda_1-2n\lambda_2-2n\lambda_3+n\lambda_4+n\lambda_5+(n-1)\lambda_6$	$\alpha_2=5$
$\theta_3=r-2n\lambda_1+n\lambda_2+n\lambda_3-2n\lambda_4+n\lambda_5+(n-1)\lambda_6$	$\alpha_3=4$
$\theta_4=r-\sqrt{5}\,n\lambda_1+2n\lambda_2-2n\lambda_3+\sqrt{5}\,n\lambda_4-n\lambda_5+(n-1)\lambda_6$	$\alpha_4=3$
$\theta_5=r+\sqrt{5}\,n\lambda_1+2n\lambda_2-2n\lambda_3-\sqrt{5}\,n\lambda_4-n\lambda_5+(n-1)\lambda_6$	$\alpha_5=3$
$\theta_6=r-\lambda_6$	$\alpha_6=20(n-1)$

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