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DENSITY ESTIMATION FOR LINEAR PROCESSES

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Summary

Let $X_1, \cdots, X_n$ be random variables forming a realization from a linear process $X_t = \sum_{r=0}^{\infty} g_r Z_{t-r}$ where $\{Z_t\}$ is a sequence of independent and identically distributed random variables with $\mathrm{E}|Z_1|^{\varepsilon} < \infty$ for some $\varepsilon > 0$, and $g_r \to 0$ as $r \to \infty$ at some specified rate. Let X_1 have a probability density function f. It is then established that for every real x, the standard kernel type estimator $\hat{f}_n(x)$ based on X_t $(1 \leqq t \leqq n)$ is, under some general regularity conditions, asymptotically normal and converges a.s. to $f(x)$ as $n \to \infty$.

1. Introduction

Let $X_1, \cdots, X_n$ be a set of identically distributed random variables (r.v.) with a common distribution function (d.f.) F and let us assume that F admits a probability density function (p.d.f.) f at some point x. If $f(x)$ is not known it can be estimated by using kernel type density estimators $\hat{f}_n$. Several important properties of such estimators have been derived in the past (for a bibliography see Rosenblatt [4] and Wegman [6]). In most of these cases, however, the r.v.'s have been assumed to be mutually independent. Recently, attempts have been made to extend these results to other than independent r.v.'s. Rosenblatt [3] has derived some interesting results about $\hat{f}_n$ for X_t's forming a Markov sequence. Similar results exist (Ahmad [1]) for X_t's forming a ϕ-mixing process. Delecroix [2] has derived the central limit theorem for $\hat{f}_n$ when X_t's form a L_2-mixing process.

The aim of the present paper is to extend these results for X_t's when they form a linear process defined by

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(1.1) $$X_t=\sum_{r=0}^{\infty} g_r Z_{t-r}$$

where $\{Z_t\}$ is an innovation process consisting of independent and identically distributed (i.i.d.) r.v.'s, and the convergence in (1.1) is in some probability sense. Most of the important stochastic process models such as the autoregressive (AR) schemes, and the mixed autoregressive moving average (ARMA) schemes are linear processes. The primary use of density estimation is possibly its application to discriminant analysis. In fact, the density estimates may be used to derive some sample based classification rules when the observations in the sample form a linear process. In such cases it will be interesting to find out if these estimates behave in a manner similar to those based on i.i.d. observations.

In the present paper we concentrate on the one-dimensional p.d.f. f and study the properties of $\hat{f}_n$ defined in (2.1). We plan to deal with estimators of p.d.f.'s of more than one-dimension in a subsequent article.

2. Probability density estimate and its asymptotic property

We define the estimator $\hat{f}_n(x)$ of $f(x)$ by

(2.1) $$\hat{f}_n=\hat{f}_n(x)=n^{-1}\sum_{t=1}^{n}\phi(x-X_t\,;\,r_n)$$

where $\{r_n\}$ is a sequence of real numbers such that $r_n\to 0$, but $nr_n\to\infty$ as $n\to\infty$, $\phi(y\,;\,r_n)=r_n^{-1}\phi(y/r_n)$ $(-\infty<y<\infty)$ and ϕ is a nonnegative Borel function which satisfies the following condition.

A. (i) For every real y, $\phi(y)<M$ where M is used as a generic symbol which denotes a finite positive constant independent of n, (ii) $\int_{-\infty}^{\infty}\phi(y)dy<\infty$, (iii) $\lim_{y\to\pm\infty} y\phi(y)=0$, and (iv) for every real a, $\int|\phi(y+a)-\phi(y)|dy\leqq M|a|$.

B. Further assume that if φ_0 denotes the characteristic function (ch.f.) of Z_1 then

$$\int_{-\infty}^{\infty}|u\varphi_0(u)|du<\infty\ .$$

C. $\mathrm{E}(|Z_1|^{\varepsilon})<\infty$ for some $\varepsilon>0$, and if $\varepsilon\geqq 1$ then $\mathrm{E}(Z_1)=0$.

D. $\sum_{k=v}^{\infty} k|g_k|^{\beta}=O(v^{-\theta})$ for some $\theta>0$, where $\beta=\varepsilon/2$ if $\varepsilon\leqq 1$ and $\beta=1/2$ if $\varepsilon>1$.

Define

$$T_n=(nr_n)^{1/2}(\hat{f}_n-f_n) \tag{2.2}$$

where $f_n=f_n(x)=\mathrm{E}\,(\phi(x-X_1;r_n))$. We then have the following

THEOREM 2.1. *Let conditions A, B, C and D hold with $\{r_n\}$ chosen as above. Then as $n\to\infty$*

$$\mathcal{L}(T_n)\to\mathcal{N}(0,\sigma^2) \tag{2.3}$$

where $\sigma^2=f(x)\int_{-\infty}^{\infty}\phi^2(u)du$.

In order to prove this theorem we need to establish a few lemmas. We first set

$$Y_t=r_n^{1/2}(\phi(x-X_t;r_n)-f_n(x))\,. \tag{2.4}$$

Note that Y_t does, indeed, depend on n, $Y_t=Y_{tn}$.

LEMMA 2.2. *Let the conditions of Theorem 2.1 hold. Then*

$$\sum_{v=1}^{\infty}|\mathrm{E}\,Y_1Y_{1+v}|\leqq Mr_n^{\lambda} \tag{2.5}$$

for some $\lambda\in(0,1)$, where M is used here and subsequently as a generic symbol which denotes a finite positive constant, independent of n.

PROOF. Let $X_t^*=\sum_{r=0}^{t-2}g_rZ_{t-n}$ and let the d.f. of X_{1+v}^* be denoted by G_v. It is then easy to see that

$$G_v^{(s)}(y)\leqq(2\pi)^{-1}\int_{-\infty}^{\infty}|u^s\varphi_0(u)|du<M \qquad (s=0,1)\,, \tag{2.6}$$

for every real y. Let the conditional expectation of $\phi(x-X_{1+v};r_n)$ given $X_{1+v}-X_{1+v}^*=y$ be denoted by $J(y)$ and let

$$c_n=\max_{|y|\leqq\eta_n}J(y)\,,\quad d_n=\min_{|y|\leqq\eta_n}J(y)\,,\quad \eta_n=\eta_n(v)=\left(\sum_{k=v}^{\infty}|g_k|^{\delta}/r_n\right)^{1/(1+\delta)}\,,$$

where $\delta=\varepsilon$ if $0<\varepsilon\leqq2$ and $\delta=2$ if $\varepsilon\geqq2$. Then by (2.6) we have that

$$0\leqq d_n\leqq c_n\leqq M\,,\qquad 0\leqq c_n-d_n\min(M,M\eta_n)\,. \tag{2.7}$$

Now set

$$\begin{aligned} I&=\mathrm{E}\,(\phi(x-X_1;r_n)-f_n)\phi(x-X_{1+v};r_n)\,,\\ I_1&=\mathrm{E}\,(\phi(x-X_1;r_n)-f_n)\phi(x-X_{1+v};r_n)W_v \\ I_2&=I-I_1 \end{aligned} \tag{2.8}$$

where $W_v=1$ if $|X_{1+v}-X_{1+v}^*|\leqq\eta_n$ and $W_v=0$, otherwise. Note that since $\phi(x-y;r_n)\leqq Mr_n^{-1}$ for every real x, y, $f_n\geqq\mathrm{E}\,\phi(x-X_1;r_n)W_v=f_n-\mathrm{E}\,\phi(x-$

$X_1; r_n)(1-W_v)\geqq f_n - Mr_n^{-1}Q_v$ where $Q_v = P(|X_{1+v} - X_{1+v}^*| > \eta_n)$. Also $d_n(1-Q_v) \leqq \mathrm{E}\, J(X_{1+v} - X_{1+v}^*)\, W_v \leqq c_n$. Therefore

$$-(c_n-d_n)q_n - Md_n r_n^{-1}Q_v \leqq I_1 \leqq (c_n-d_n)f_n + d_n f_n Q_v\ . \tag{2.9}$$

Similarly we can show that

$$|I_2| < Mr_n^{-1}Q_v \tag{2.10}$$

and hence from (2.7), (2.9), (2.10) and the facts that $f_n \leqq M$, and $r_n \to 0$ as $n \to \infty$ we have for sufficiently large n that

$$|\mathrm{E}\, Y_1 Y_{1+v}| = r_n |I| \leqq M(r_n \eta_n + Q_v)\ . \tag{2.11}$$

Now observe that $\mathrm{E}\,|X_{1+v} - X_{1+v}^*|^\delta \leqq M \sum_{k=v}^{\infty} |g_k|^\delta$ by Theorem 2 in von Bahr and Esseen [5]. Therefore, $Q_v \leqq M \sum_{k=v}^{\infty} |g_k|^\delta \eta_n^{-\delta}$. It follows immediately that the right side of (2.11) is $\leqq Mr_n^\lambda \left(\sum_{k=v}^{\infty} |g_k|^\delta\right)^{1/(1+\delta)} \leqq Mr_n^\lambda \sum_{k=v}^{\infty} |g_k|^\lambda$ where $\lambda = \delta/(1+\delta)$. Since $\lambda \geqq \beta$, $\sum_{v=1}^{\infty} \sum_{k=v}^{\infty} |g_k|^\lambda \leqq \sum_{k=1}^{\infty} k|g_k|^\lambda < \infty$ by condition D. The result (2.5) follows easily.

LEMMA 2.3. *Let the conditions of Theorem* 2.1 *hold. Let* $\{m_n\}$, $\{t_n\}$ *and* $\{k_n\}$ *be sequences of positive integers such that* (i) $k_n = [n/(m_n + t_n)]$ *and as* $n \to \infty$, (ii) $m_n, t_n, k_n \to \infty$, $t_n/m_n \to 0$, (iii) $m_n/(nr_n)^\gamma \to 0$ *for some* $\gamma (0 < \gamma < 1/2)$ *and* (iv) $m_n^{-1} n^{1-\beta/2} r_n^{-\beta/2} t_n^{-\theta} \to 0$. *Write*

$$\begin{aligned} U_j &= n^{-1/2} \sum_{t \in A_j} Y_{t'} \\ V_j &= n^{-1/2} \sum_{t \in B_j} Y_{t'}\ , \qquad 1 \leqq j \leqq k_n \\ W &= n^{-1/2} \sum_{t \in C} Y_t \end{aligned} \tag{2.12}$$

where $A_j = \{\alpha_{j-1}+1, \cdots, \alpha_j - t_n\}$, $B_j = \{\alpha_j - t_n + 1, \cdots, \alpha_j\}$, $C = \{n - d_n + 1, \cdots, n\}$, $\alpha_j = j(m_n + t_n)$, *and* $d_n = n - k_n(m_n + t_n)$. *Then as* $n \to \infty$

$$\mathcal{L}\left(\sum_{j=1}^{k_n} U_j\right) \to \mathcal{N}(0, \sigma^2)\ , \tag{2.13}$$

and

$$\sum_{j=1}^{k_n} V_j + W \to 0 \ \textit{in probability}\ , \tag{2.14}$$

σ^2 *being defined as in* (2.3).

PROOF. Let $\varphi^{(j)}$ denote the ch.f. of $U_1, \cdots, U_j$ and let φ_j be the ch.f. of U_j. Then

(2.15) $$\left|\varphi^{(k_n)}(u,\cdots,u)-\prod_{j=1}^{k_n}\varphi_j(u)\right|\leqq\sum_{j=2}^{k_n}|\varphi^{(j)}(u,\cdots,u)-\varphi_j(u)\varphi^{(j-1)}(u,\cdots,u)|\,.$$

Now set $N_j=\exp\left(iu\sum_{r=1}^{j-1}U_r\right)-\varphi^{(j-1)}(u,\cdots,u)$, $P_j=\exp(iuU_j)$, $P_j^*=P_j-\mathrm{E}(P_j|\mathcal{N}_j)$, where $\mathcal{N}_j=\sigma\{Z_{\alpha_{j-1}-t_n+1},\cdots,Z_{\alpha_j-t_n}\}$ $(2\leqq j\leqq k_n)$. Since $\mathrm{E}(P_j|\mathcal{N}_j)$ is independent of N_j and $\mathrm{E}\,N_j=0$, the jth summand on the right hand side of (2.15) is equal to

(2.16) $$|\mathrm{E}\,N_jP_j|=|\mathrm{E}\,N_jP_j^*|\leqq|(\mathrm{E}\,|N_j|^2)^{1/2}(\mathrm{E}\,|P_j^*|^2)^{1/2}\leqq M(\mathrm{E}\,|P_1^*|^2)^{1/2}\,.$$

Again we can write $P_1=h(X_1,\cdots,X_m)$ $(m=m_n)$ where $h(y_1,\cdots,y_m)=\exp\left(iu\sum_{t=1}^{m}g(x-y_t)\right)$, $g(w)=n^{-1/2}r_n^{1/2}(\phi(w;r_n)-f_n(x))$. This implies that

(2.17) $$\mathrm{E}\,|P_1^*|^2\leqq\mathrm{E}\,|h(R_1+W_1,\cdots,R_m+W_m)-h(R_1+W_1^*,\cdots,R_m+W_m^*)|^2$$

where $R_j=\sum_{r=0}^{t_n+j-1}g_rZ_{j-r}$, $W_j=X_j-R_j$ $(1\leqq j\leqq m)$ and $(W_1^*,\cdots,W_m^*)$ is an independent copy of $(W_1,\cdots,W_m)$ and is also independent of $R_1,\cdots,R_m$. Note that $(W_1,\cdots,W_2)$ is independent $R_1,\cdots,R_m$. Again since $|\exp(ia)-1|^2\leqq M|a|^{2\beta}$ for every real a,

(2.18) $$\begin{aligned}&|h(y_1,\cdots,y_m)-h(y_1^*,\cdots,y_m^*)|^2\\&\leqq M|u|^{2\beta}n^{-\beta}r_n^{-\beta}\sum_{j=1}^{m}|\phi((x-y_j)/r_n)-\phi((x-y_j^*)/r_n)|^{2\beta}\end{aligned}$$

for any real y_t, y_t^* $(1\leqq t\leqq m)$. Therefore, by (2.17), (2.18), condition A, the fact that $\mathrm{E}\,|X|^{2\beta}\leqq\mathrm{E}^{2\beta}|X|$ for any r.v. X and that $\mathrm{E}\,|W_j|^{2\beta}\leqq M\sum_{k=t_n+j}^{\infty}|g_k|^{2\beta}$ we have the relation

(2.19) $$\begin{aligned}(\mathrm{E}\,|P_1^*|^2)^{1/2}&\leqq M|u|^{\beta}n^{-\beta/2}r_n^{-\beta/2}\sum_{j=1}^{m}\mathrm{E}^{1/2}|W_j-W_j^*|^{2\beta}\\&\leqq M|u|^{\beta}n^{-\beta/2}r_n^{-\beta/2}\sum_{j=1}^{m}\sum_{k=t_n+j}^{\infty}|g_k|^{\beta}\\&\leqq M|u|^{\beta}n^{-\beta/2}r_n^{-\beta/2}\sum_{k=t_n+1}^{\infty}k|g_k|^{\beta}\,.\end{aligned}$$

Conditions D and (i), (ii), (iv) in Lemma 2.3 will, therefore, imply that the right hand side of the inequality in (2.15)

(2.20) $$\leqq M|u|^{\beta}n^{-\beta/2}r_n^{-\beta/2}k_nt_n^{-\theta}\to 0$$

as $n\to\infty$. Therefore, in order to derive the asymptotic distribution of $\sum_{j=1}^{k_n}U_j$ we can assume that the U_j $(1\leqq j\leqq k_n)$ are i.i.d.r.v.'s. Let $\alpha=2\gamma/(1-2\gamma)$ where γ is defined as in condition (iii) of Lemma 2.3. Now note that $\mathrm{E}\,|Y_1|^{2+\alpha}\leqq Mr_n^{1+\alpha/2}\,\mathrm{E}\,((\phi(x-X_1;r_n))^{2+\alpha}+(f_n)^{2+\alpha})\leqq Mr_n^{-\alpha/2}$.

Hence

(2.21) $$\mathrm{E}\,|U_1|^{2+\alpha} \leqq Mn^{-1-\alpha/2} m_n^{2+\alpha} r_n^{-\alpha/2} .$$

Again for sufficiently large n,

(2.22) $$\mathrm{E}\,U_j^2 = n^{-1} m_n \Big(\mathrm{E}\,Y_1^2 + 2 \sum_{v=1}^{m_n} (1 - v/m_n)\, \mathrm{E}\,Y_1 Y_{1+v} \Big) \geqq \sigma^2 n^{-1} m_n/2$$

by Lemma 2.2 and the fact that

$$\mathrm{E}\,Y_1^2 = r_n\, \mathrm{E}\,\phi^2(x - X_1\,;\, r_n) - r_n f_n^2 = \sigma^2 - r_n f_n^2 \to \sigma^2 \qquad \text{as } n \to \infty .$$

(2.21) and (2.22) and condition (iii) in Lemma 2.3 will, therefore, imply that

(2.23) $$\begin{aligned} k_n^{-\alpha/2}\, \mathrm{E}\,|U_1|^{2+\alpha}/(\mathrm{E}\,U_1^2)^{1+\alpha/2} &\leqq M(nr_n)^{-\alpha/2} m_n^{1+\alpha} \\ &= M(m_n/(nr_n)^{\gamma})^{1+\alpha} \to 0 \qquad \text{as } n \to \infty . \end{aligned}$$

Hence the Liapaunov condition for the central limit theorem holds and (2.13) follows immediately. Now observe that since $k_n m_n/n \to 1$ as $n \to \infty$, $n^{-1}(k_n t_n + d_n) = 1 - k_n m_n/n \to 0$ as $n \to \infty$. Consequently

(2.24) $$\begin{aligned} \mathrm{E}\Big(\sum_{j=1}^{k_n} V_j + W \Big)^2 &\leqq n^{-1}(k_n t_n + d_n) \Big(\mathrm{E}\,Y_1^2 + 2 \sum_{v=1}^{\infty} |\mathrm{E}\,Y_1 Y_{1+v}| \Big) \\ &\leqq Mn^{-1}(k_n t_n + d_n) \to 0 , \qquad \text{as } n \to \infty \end{aligned}$$

and the result (2.14) follows.

Now since $T_n = \sum_{j=1}^{k_n} U_j + \sum_{j=1}^{k_n} V_j + W$ the result (2.3) is a direct consequence of (2.13) and (2.24). Theorem 2.1 is thus established.

Let $\psi(u) = \int \exp(iuy)\phi(y)dy$ and let φ be the ch.f. of X_1. Then assume that the following condition holds.

E. For some $q > 0$, $\lim_{u \to 0} (1 - \psi(u))/|u|^q = k_q$, $|k|_q < \infty$ and $\left| \int_{-\infty}^{\infty} \exp(-iux) \cdot |u|^q \varphi(u) du \right| < \infty$.

THEOREM 2.4. *Let the conditions of Theorem* 2.1 *and condition* E *hold. If, additonally,* $\{r_n\}$ *is such that* $nr_n^{2q+1} \to 0$ *as* $n \to \infty$, *then as* $n \to \infty$,

(2.25) $$\mathcal{L}((nr_n)^{1/2}(\hat{f}_n - f)) \to \mathcal{N}(0, \sigma^2) .$$

PROOF. Note that as $n \to \infty$,

$$\begin{aligned} (f_n - f)/r_n^q = (2\pi)^{-1} \int \exp(-iux)((\psi(r_n u) - 1)/|r_n u|^q)|u|^q \varphi(u) du \to \\ -(2\pi)^{-1} k_q \int e^{-iux} |u|^q \varphi(u) du . \end{aligned}$$

This implies that $(nr_n)^{1/2}(f_n - f) \to 0$ as $n \to \infty$, and, therefore, (2.25) will, now, follow immediately from (2.3).

It is necessary to establish that the sequences $\{m_n\}$, $\{t_n\}$ and $\{r_n\}$ can, indeed, be chosen such that the conditions on $\{r_n\}$ and (i)–(iv) in Lemma 2.3 will hold. If we take $t_n = [n^a]$, $m_n = [n^b]$ and $r_n = n^{-c}$ then for given q, ε (which determines β) and θ we have the following constraints on a, b and c. (i) $(2q+1)^{-1} < c < 1$, (ii) $0 < a < b < \gamma(1-c) < 1$ (note that $0 < \gamma < 1/2$), and $\theta a + b + \beta(1-c)/2 > 1$. It is easy to see that a sufficient condition for these constraints to hold is that $\theta > 1 + q^{-1}$.

3. Almost sure convergence

We shall now establish the following

THEOREM 3.1. *Let the conditions of Theorem* 2.1 *hold. In addition, assume that* (i) $r_n \downarrow$ *and for some* α $(0 < \alpha < 1/2)$ $n^\alpha r_n \to \infty$, (ii) *for every* $p > 1$, $k(1 - r((k+1)^p)/r(k^p)) \to$ *a finite constant as* $k \to \infty$, (iii) *for every* a, b, $0 < b < a < 1$, $\int |\phi(t) - a\phi(at)| dt \leq Mb^{-1}(1-a)$ *where* M *is independent of* a *and* b, (iv) $\int |u|^s |\varphi_0(u)| du < \infty$ $(s = 0, 1, 2)$. *Then as* $n \to \infty$

$$\hat{f}_n \to f \text{ a.s.} \tag{3.1}$$

as $n \to \infty$.

PROOF. Note that since $f_n \to f$ as $n \to \infty$, it is sufficient if we establish that $\hat{f}_n - f_n \to 0$ a.s. as $n \to \infty$. We can write $\hat{f}_n - f_n = S_{nn}/n$ where $S_{nm} = \sum_{t=1}^{n} (\phi(x - X_t; r_m) - f_m)$. Let $n_k = [k^p]$ where p is any number $\in ((1-\alpha)^{-1}, \alpha^{-1})$, α being as given in (ii) above and $k = 1, 2, \cdots$. Then since $\mathrm{E}\, Y_{1n}^2 \leq M$ and (2.5) holds we have that $\mathrm{V}(S_{n_k n_k}/n_k) = \mathrm{E}(\sum Y_{tn_k})^2 / n_k^2 r_{n_k}$ $\leq M(n_k r_{n_k})^{-1} \leq Mk^{-p(1-\alpha)}$ by condition (i) above. Hence

$$S_{n_k n_k}/n_k \to 0 \text{ a.s.} \tag{3.2}$$

as $k \to \infty$. Let n be any integer. Then $n_k \leq n < n_{k+1}$ for some k and if we set $C_k = \max_{n_k \leq n < n_{k+1}} |S_{nn} - S_{n_k n}|$, $D_k = \max_{n_k \leq n < n_{k+1}} |S_{n_k n} - S_{n_k n}|$ then

$$|S_{nn}/n| \leq |S_{n_k n_k}/n_k| + C_k/n_k + D_k/n_k\,. \tag{3.3}$$

It is easy to show that

$$\mathrm{E}\, C_k^2/n_k^2 \leq \sum_{n=n_k}^{n_{k+1}} \mathrm{E}\Big(\sum_{t=n_k+1}^{n} Y_{tn}\Big)^2 \Big/ n_k^2 r_n \leq \sum_{n=n_k}^{n_{k+1}} (n - n_k)/n_k^2 r_n\,.$$

Again we can conclude from condition (ii) above that $r_n/r_{n_k} \to 1$ as k

$\rightarrow \infty$. Therefore $\mathrm{E}\, C_k^2/n_k^2 \leqq M(n_{k+1}-n_k)^2/n_k^2 r_{n_k} \leqq M k^{p\alpha-2}$. Since $p<\alpha^{-1}$ we have that

$$C_k/n_k \rightarrow 0 \text{ a.s.} \tag{3.4}$$

as $n \rightarrow \infty$. Similarly from (3.6) below we conclude that $\mathrm{E}\, D_k^2/n_k^2 \leqq \sum_{n=n_k}^{n_{k+1}} \mathrm{E}\,(S_{n_k n}-S_{n_k n})^2 \leqq M(n_{k+1}-n_k)(r_{n_k}-r_{n_{k+1}})/n_k r_{n_k}^2 \leqq M k^{p\alpha-2}$. Therefore

$$D_k/n_k \rightarrow 0 \text{ a.s.} \tag{3.5}$$

as $k \rightarrow \infty$. (3.1) now follows easily from the results (3.2)-(3.5).

LEMMA 3.2. *Let the conditions of Theorem* 3.1 *hold. Then*

$$\mathrm{E}\,(S_{n_k n}-S_{n_k n_k})^2 \leqq M(r_{n_k}-r_{n_{k+1}})n_k/r_{n_k}^2\,. \tag{3.6}$$

PROOF. The proof follows details similar to those in Lemma 2.2. First note that $S_{n_k n}-S_{n_k n_k}$ is the sum of n_k terms. The expectation of the sum of squares term in $\mathrm{E}\,(S_{n_k n}-S_{n_k n_k})^2$ can easily be shown to be less than or equal to $n_k \mathrm{E}\,(\phi(x-X_1;\, r_n)-\phi(x-X_1;\, r_{n_k}))^2 \leqq r_n^{-1}\int(\phi(t)-\gamma_k\phi(\gamma_k t))^2\cdot f(x-r_n t)dt \leqq M r_n^{-1}(1-\gamma_k)n_k \leqq M(r_{n_k}-r_{n_{k+1}})n_k/r_{n_k}^2$ where $1 \geqq \gamma_k = r_n/r_{n_k} \geqq r_{n_{k+1}}/r_{n_k} \rightarrow 1$ as $k \rightarrow \infty$ by virtue of conditions (i)-(iv) above. If we now replace $\phi(x-X_1;\, r_n)$, $\phi(x-X_{1+v};\, r_n)$ and f_n in the expression for I in (2.8) by $\phi(x-X_1;\, r_n)-\phi(x-X_1;\, r_{n_k})$, $\phi(x-X_{1+v};\, r_n)-\phi(x-X_{1+v};\, r_{n_k})$ and $f_n-f_{n_k}$ respectively then by routine analysis and following the same sequence of arguments as led to (2.11) and eventually to (2.5) we can establish that the expectation of the sum of the cross products in $\mathrm{E}\,(S_{n_k n}-S_{n_k n_k})^2$ is $\leqq M(r_{n_k}-r_{n_{k+1}})n_k r_n^{\lambda}/r_{n_k}^2$. The result (3.6) will then follow immediately.

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