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ON THE MOST POWERFUL QUANTILE TEST OF THE SCALE PARAMETER

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Summary

The most powerful test of the null hypothesis H_0: $\sigma=\sigma_0$ versus the alternative hypothesis H_1: $\sigma=\sigma_1$ based on a few selected sample quantiles is proposed here where σ is the scale parameter of the distribution and the location parameter μ is known. The quantiles are chosen from a large sample that is either complete or censored (singly-censored or doubly-censored). The relationship between the proposed test and the asymptotically best linear unbiased estimate (ABLUE) of the scale parameter is discussed.

1. Introduction

Given a large ordered sample of size n

$$X_{(1)}<X_{(2)}<\cdots<X_{(n)}$$

from a distribution with the probability density function $(1/\sigma)f[(X-\mu)/\sigma]$ where σ is the scale parameter and μ is the known location parameter, we want to find the most powerful test in testing the null hypothesis

$$H_0: \quad \sigma=\sigma_0$$

against the alternative hypothesis

$$H_1: \quad \sigma=\sigma_1 \ (>\sigma_0)$$

or

$$H_2: \quad \sigma=\sigma_1 \ (<\sigma_0)$$

based on a few sample quantiles selected from the given sample above.

Ogawa [13], [14] proposed a t-test based on a few selected order

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statistics to test the scale parameter of exponential distribution. He used the least squares procedure. Eisenberger [3], [4], [5], [6] and Eisenberger and Posner [7] found an approximate normal test to test the location parameter and the scale parameter separately and jointly based on one, two, four, six and eight sample quantiles. When they use two or more quantiles, the test statistic is a linear combination of symmetric quantiles. Of all cases considered, the samples are taken from a normal distribution and quantiles are selected from a complete sample. Cheng [2] discovered an asymptotically uniformly most powerful test in testing the location parameter using a few selected sample quantiles.

In this article, we will deal with the testing of the scale parameter while the location parameter is known. The sample is taken from any continuous distribution whose p.d.f. is of the form, $(1/\sigma)f\cdot[(X-\mu)/\sigma]$, the selection of the quantiles to be used will also be discussed. The quantiles could be chosen from a complete or a censored sample.

The relationship between the most powerful test proposed here and the asymptotically best linear unbiased estimate (ABLUE) of the scale parameter will be discussed in Section 5.

2. The notations

Let

$$X_{(1)}<X_{(2)}<\cdots<X_{(n)} \tag{1}$$

be a set of n order statistics taken from a distribution with the p.d.f. and the c.d.f. $(1/\sigma)f[(X-\mu)/\sigma]$ and $F[(X-\mu)/\sigma]$ respectively. The value y_λ which satisfies

$$\int_{-\infty}^{y_\lambda} f(y)dy=\lambda\,, \qquad 0<\lambda<1$$

is called the population quantile of order λ.

A set of real numbers $\{\lambda_i\}$ satisfying $0\equiv\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_k<\lambda_{k+1}\equiv1$ is called a spacing. Define $n_j=[n\lambda_j+1]$, where $[\gamma]$ denotes the integer part of the number γ. Then we call

$$X_{(n_1)}<X_{(n_2)}<\cdots<X_{(n_k)}\,, \qquad k\leqq n\,, \tag{2}$$

the sample quantiles chosen from the sample (1).

Let $\boldsymbol{X}$ be the vector of k sample quantiles (2), $\boldsymbol{u}$ the vector of the corresponding k population quantiles and $\mathbf{1}$ the unit vector of k elements, i.e.

$$\boldsymbol{X}=\begin{bmatrix} X_{(n_1)} \\ X_{(n_2)} \\ \vdots \\ X_{(n_k)} \end{bmatrix} \qquad \boldsymbol{u}=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix} \qquad \mathbf{1}=\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

where u_j is such that

$$\int_{-\infty}^{u_j} f(y)dy=\lambda_j .$$

We also denote $\frac{\sigma^2}{n}\boldsymbol{\Gamma}$ the covariance matrix of the sample quantiles (2), i.e. $\boldsymbol{\Gamma}=[\gamma_{ij}]$, and $\gamma_{ij}=\frac{\lambda_i(1-\lambda_j)}{f_i f_j}$ for $j\geqq i$, and $\gamma_{ij}=\gamma_{ji}$ for $j<i$, $i, j =1, 2, \cdots, k$, and $f_j=f(u_j)$, $j=1, 2, \cdots, k$.

Mosteller [11] showed that the limiting distribution of the k sample quantiles in (2) is asymptotically normal with p.d.f.

$$L(\boldsymbol{X})=(n/2\Pi\sigma^2)^{k/2}\Gamma^{-1/2} \exp \{(-n/2\sigma^2)(\boldsymbol{X}-\mu\mathbf{1}-\sigma\boldsymbol{u})'\boldsymbol{\Gamma}^{-1}(\boldsymbol{X}-\mu\mathbf{1}-\sigma\boldsymbol{u})\} ,$$

where $\Gamma=\det \boldsymbol{\Gamma}$.

Now that

$$\mathbf{1}'\boldsymbol{\Gamma}^{-1}\mathbf{1}=\sum_{i=1}^{k+1} \frac{(f_i-f_{i-1})^2}{\lambda_i-\lambda_{i-1}}\equiv K_1 ,$$

$$\boldsymbol{u}'\boldsymbol{\Gamma}^{-1}\boldsymbol{u}=\sum_{i=1}^{k+1} \frac{(f_i u_i-f_{i-1}u_{i-1})^2}{\lambda_i-\lambda_{i-1}}\equiv K_2 ,$$

$$\mathbf{1}'\boldsymbol{\Gamma}^{-1}\boldsymbol{u}=\sum_{i=1}^{k+1} \frac{(f_i-f_{i-1})(f_i u_i-f_{i-1}u_{i-1})}{\lambda_i-\lambda_{i-1}}\equiv K_3$$

where $f_0\equiv f_{k+1}\equiv 0$, $f_0 u_0\equiv f_{k+1}u_{k+1}\equiv 0$.

3. The ABLUE σ^*

Ogawa [12], [14] first obtained the asymptotically best linear unbiased estimate (ABLUE) σ^* of the scale parameter σ when the location parameter μ is known:

$$\sigma^*=\sum_{i=1}^{k} b_i X_{(n_i)}-\mu\frac{K_3}{K_2}$$

where

$$b_i=\frac{f_i}{K_2}\left[\frac{f_i u_i-f_{i-1}u_{i-1}}{\lambda_i-\lambda_{i-1}}-\frac{f_{i+1}u_{i+1}-f_i u_i}{\lambda_{i+1}-\lambda_i}\right], \qquad i=1, 2, \cdots, k ,$$

and the asymptotic variance of σ^* is

$$\mathrm{AV}(\sigma^*)=\frac{\sigma^2}{n}\cdot\frac{1}{K_2}\,.$$

The spacing $\{\lambda_i\}$ which minimizes $\mathrm{AV}(\sigma^*)$, i.e. equivalently maximizing K_2 with respect to $\lambda_1, \lambda_2, \cdots, \lambda_k$ is called the optimum spacing for the ABLUE σ^* which will determine the ranks of the quantiles to be selected.

4. The most powerful test

We want to test the null hypothesis

$$H_0:\quad \sigma=\sigma_0$$

against the alternative hypothesis

$$H_1:\quad \sigma=\sigma_1\ (>\sigma_0)$$

based on k sample quantiles (2). Let the known value of μ be μ_0.

Applying the famous Neyman-Pearson Lemma, the best critical region (BCR) is

$$(3)\quad \frac{L(\boldsymbol{X}|H_1)}{L(\boldsymbol{X}|H_0)}=\left(\frac{\sigma_0}{\sigma_1}\right)^k \exp\left\{-\frac{n}{2}\left[\frac{1}{\sigma_1^2}(\boldsymbol{X}-\mu_0\mathbf{1}-\sigma_1\boldsymbol{u})'\boldsymbol{\Gamma}^{-1}(\boldsymbol{X}-\mu_0\mathbf{1}-\sigma_1\boldsymbol{u})\right.\right.$$
$$\left.\left.-\frac{1}{\sigma_0^2}(\boldsymbol{X}-\mu_0\mathbf{1}-\sigma_0\boldsymbol{u})'\boldsymbol{\Gamma}^{-1}(\boldsymbol{X}-\mu_0\mathbf{1}-\sigma_0\boldsymbol{u})\right]\right\}$$
$$=\left(\frac{\sigma_0}{\sigma_1}\right)^k \exp\left\{-\frac{n}{2}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\left[\left(\boldsymbol{X}-\mu_0\mathbf{1}-\frac{\sigma_0\sigma_1}{\sigma_0+\sigma_1}\boldsymbol{u}\right)'\right.\right.$$
$$\left.\left.\times\boldsymbol{\Gamma}^{-1}\left(\boldsymbol{X}-\mu_0\mathbf{1}-\frac{\sigma_0\sigma_1}{\sigma_0+\sigma_1}\boldsymbol{u}\right)-\left(\frac{\sigma_0\sigma_1}{\sigma_0+\sigma_1}\right)^2\boldsymbol{u}'\boldsymbol{\Gamma}^{-1}\boldsymbol{u}\right]\right\}\geqq c$$

where c is a constant.

Let $\boldsymbol{Y}=\boldsymbol{X}-\mu_0\mathbf{1}-\frac{\sigma_0\sigma_1}{\sigma_0+\sigma_1}\boldsymbol{u}$, then (3) can be simplified to

$$\boldsymbol{Y}'\boldsymbol{\Gamma}^{-1}\boldsymbol{Y}\geqq c^*$$

where

$$c^*=\frac{(\sigma_0\sigma_1)^2}{\sigma_1^2-\sigma_0^2}\cdot\frac{2}{n}\ln\left[\left(\frac{\sigma_1}{\sigma_0}\right)^k c\right]+\left(\frac{\sigma_0\sigma_1}{\sigma_0+\sigma_1}\right)^2\boldsymbol{u}'\boldsymbol{\Gamma}^{-1}\boldsymbol{u}\,.$$

Since

$$\boldsymbol{X}\sim N\left(\mu\mathbf{1}+\sigma\boldsymbol{u},\ \frac{\sigma^2}{n}\boldsymbol{\Gamma}\right)$$

hence

$$\boldsymbol{Y} \sim N\left(\left[\sigma - \frac{\sigma_0\sigma_1}{\sigma_0+\sigma_1}\right]\boldsymbol{u},\ \frac{\sigma^2}{n}\boldsymbol{\Gamma}\right).$$

Johnson and Kotz ([9], pp. 178–179) has stated a theorem which was proved by Mäkeläinen [10].

Let $\boldsymbol{X}'=(X_1, \cdots, X_n)$ be a random normal vector with mean vector $\boldsymbol{\xi}$ and dispersion matrix $\boldsymbol{V}$. Let $\boldsymbol{A}$ be an $n\times n$ real symmetric matrix and assume that $\operatorname{tr}(\boldsymbol{AV})=\gamma\neq 0$. Then $Q(X)=\boldsymbol{X}'\boldsymbol{AX}$ has a noncentral χ^2 distribution if and only if $\boldsymbol{VAVAV}=\boldsymbol{VAV}$,

$$\boldsymbol{\xi}'\boldsymbol{AVA\xi}=\boldsymbol{\xi}'\boldsymbol{A\xi} \quad \text{and} \quad \boldsymbol{VAVA\xi}=\boldsymbol{VA\xi}\,.$$

The degree of freedom and the noncentrality parameter are γ and $\boldsymbol{\xi}'\boldsymbol{A\xi}/2$ respectively.

It can be easily checked that all the conditions mentioned in the theorem above are satisfied by $\boldsymbol{Y}$ hence $\frac{n}{\sigma^2}Q=\frac{n}{\sigma^2}(\boldsymbol{Y}'\boldsymbol{\Gamma}^{-1}\boldsymbol{Y})$ will have noncentral χ^2 distribution with k degrees of freedom and the noncentrality parameter

$$\begin{aligned}\nu&=\frac{1}{2}\frac{n}{\sigma^2}\left(\sigma-\frac{\sigma_0\sigma_1}{\sigma_0+\sigma_1}\right)^2\boldsymbol{u}'\boldsymbol{\Gamma}^{-1}\boldsymbol{u}\\&=\frac{n}{2\sigma^2}\left(\sigma-\frac{\sigma_0\sigma_1}{\sigma_0+\sigma_1}\right)^2K_2\\&=\frac{nK_2}{2}\left(1-\frac{\sigma_0\sigma_1}{\sigma[\sigma_0+\sigma_1]}\right)^2.\end{aligned}$$

Define q_α=the 100αth population quantile of a noncentral χ^2 distribution with k degrees of freedom and the noncentrality parameter $\nu_0=\frac{nK_2}{2}\left(1-\frac{\sigma_1}{\sigma_0+\sigma_1}\right)^2=\frac{nK_2}{2}\left(\frac{\sigma_0}{\sigma_0+\sigma_1}\right)^2$. Hence the BCR of the level α, $0<\alpha<1$, is

$$Q\geqq\frac{\sigma_0^2}{n}q_{1-\alpha}$$

or

$$\left(\boldsymbol{X}-\mu_0\boldsymbol{1}-\frac{\sigma_0\sigma_1}{\sigma_0+\sigma_1}\boldsymbol{u}\right)'\boldsymbol{\Gamma}^{-1}\left(\boldsymbol{X}-\mu_0\boldsymbol{1}-\frac{\sigma_0\sigma_1}{\sigma_0+\sigma_1}\boldsymbol{u}\right)\geqq\frac{\sigma_0^2}{n}q_{1-\alpha}$$

this proves the following theorem:

THEOREM 1. *The most powerful test T_1 in testing H_0: $\sigma=\sigma_0$ against H_1: $\sigma=\sigma_1$ $(>\sigma_0)$ at the level of significance α is*

$$\left(\boldsymbol{X}-\mu_0\boldsymbol{1}-\frac{\sigma_0\sigma_1}{\sigma_0+\sigma_1}\boldsymbol{u}\right)'\boldsymbol{\Gamma}^{-1}\left(\boldsymbol{X}-\mu_0\boldsymbol{1}-\frac{\sigma_0\sigma_1}{\sigma_0+\sigma_1}\boldsymbol{u}\right)\geqq\frac{\sigma_0^2}{n}q_{1-\alpha}$$

where q_α *is the* 100α*th quantile of a noncentral* χ^2 *distribution with* k *degrees of freedom and the noncentrality parameter* $\nu_0=\frac{nK_2}{2}\left(\frac{\sigma_0}{\sigma_0+\sigma_1}\right)^2$.

Similarly, we have

THEOREM 2. *The most powerful test* T_2 *in testing* H_0: $\sigma=\sigma_0$ *against* H_2: $\sigma=\sigma_1$ $(<\sigma_0)$ *at the level of significance* α *is*

$$\left(\boldsymbol{X}-\mu_0\boldsymbol{1}-\frac{\sigma_0\sigma_1}{\sigma_0+\sigma_1}\boldsymbol{u}\right)'\boldsymbol{\Gamma}^{-1}\left(\boldsymbol{X}-\mu_0\boldsymbol{1}-\frac{\sigma_0\sigma_1}{\sigma_0+\sigma_1}\boldsymbol{u}\right)\leqq\frac{\sigma_0^2}{n}q_\alpha$$

where q_α *is the* 100α*th quantile of a noncentral* χ^2 *distribution with* k *degrees of freedom and the noncentrality parameter* $\nu_0=\frac{nK_2}{2}\left(\frac{\sigma_0}{\sigma_0+\sigma_1}\right)^2$.

5. The power of the test

For the test T_1, the power of the test is

$$P_1=\Pr\left(\frac{n}{\sigma_1^2}Q\geqq\frac{1}{d^2}q_{1-\alpha}\right)$$

where $\frac{n}{\sigma_1^2}Q$ has a noncentral χ^2 distribution with k d.f.'s and the noncentrality parameter

$$\nu_1=\frac{nK_2}{2}\left(\frac{\sigma_0}{\sigma_0+\sigma_1}\right)^2\cdot\left(\frac{\sigma_1}{\sigma_0}\right)^2=d^2\nu_0 \quad\text{and}\quad d=\frac{\sigma_1}{\sigma_0}.$$

Similarly, the power of the test T_2 is

$$P_2=\Pr\left(\frac{n}{\sigma_1^2}Q\leqq\frac{1}{d^2}q_\alpha\right)$$

where $\frac{n}{\sigma_1^2}Q$ has a noncentral χ^2 distribution with k d.f.'s and the noncentrality parameter

$$\nu_2=\frac{nK_2}{2}\left(\frac{\sigma_0}{\sigma_0+\sigma_1}\right)^2\cdot\left(\frac{\sigma_1}{\sigma_0}\right)^2=d^2\nu_0.$$

For the test T_1, $d>1$ and for the test T_2, $d<1$.

It can be shown that, both P_1 and P_2 are increasing functions of ν_0 for fixed α, n and k. And

$$\nu_0=\frac{nK_2}{2}\cdot\left(\frac{\sigma_0}{\sigma_0+\sigma_1}\right)^2=\frac{n}{2}\cdot\frac{1}{(1+d)^2}\cdot K_2.$$

In order to obtain the maximum power of the test, we have to maximize ν_0 and in turn, to maximize K_2. Therefore, the optimum spacing

$\{\lambda_i\}$ in obtaining the ABLUE σ^* will provide the maximum power and hence the most powerful test.

6. Remarks

(1) As it can be seen, the test statistic depends on the parameter value under the alternative hypothesis, hence there is no uniformly most powerful test in testing a simple hypothesis against a composite hypothesis.

(2) From the results we have obtained above, we can see that, as long as there is an optimum spacing for the ABLUE σ^* of a given distribution, there will be a most powerful test of the scale parameter (simple versus simple) when the other parameters are known. As we know, there exists the ABLUE σ^* of the scale parameter of the following distributions—normal, Cauchy, logistic, exponential, Weibull, double exponential, gamma, Rayleigh, and Pareto.

(3) Furthermore, the test can be applied not only to the complete sample but also to the censored samples where some observations are missing so long as the ABLUE σ^* exists.

(4) In evaluating the value of the power, one can use Johnson's [8] table for quantiles with upper tail probability=.999, .9975, .995, .99, .975, .95, .9, .75, .5, .25, .1, .05, .025, .01, .005, .0025, .001 with the degree of freedom=1(1)12, 15, 20 and the square root of the noncentrality parameter=.2(.2)6.0. For any other cases, one may use the computer program written by Bargmann and Ghosh [1] to do the calculation.

(5) It can easily be shown that both tests T_1 and T_2 are unbiased.

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