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ON L_p-CONVERGENCE RATES FOR STATISTICAL FUNCTIONS WITH APPLICATION TO L-ESTIMATES

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Summary

A parameter which may be represented as a functional $T(F)$ of a distribution function F may be estimated by the "statistical function" $T(F_n)$, where F_n is the empirical distribution function. Recently, Boos and Serfling (1979, Florida State University Statistics Report No. M 499) obtained sufficient conditions for the Berry-Esseen theorem to hold for $T(F_n)$-$T(F)$ and applied the results to derive rates of convergence in L_∞ for L-estimates. The present note complements their work by obtaining the L_p-rates of convergence, $1\leqq p<\infty$ for $T(F_n)$-$T(F)$ and its application to L-estimates.

1. Introduction and a portmanteau theorem

Let a parameter θ of a distribution function (d.f.) F be represented as a functional $T(F)$. If $X_1,\cdots,X_n$ is a random sample from F, then θ may be estimated by $T(F_n)$, where $F_n(x)$ denote the empirical d.f. For many instances the random variable $n^{1/2}(T(F_n)-T(F))/\sigma(T,F)$ is asymptotically standard normal, for some positive constant $\sigma(T,F)$. Using von Mises [6] expansion of $T(F_n)-T(F)$ as a sum of two terms; a U-statistic and a remainder term, i.e. $T(F_n)-T(F)=U_n+R_n$, Boos and Serfling [2] prove that under some conditions, $\sup_x|\mathrm{P}[n^{1/2}(T(F_n)-T(F))\leqq x\sigma(T,F)]-\Phi(x)|=O(n^{-1/2})$, $n\to\infty$, where $\Phi(\cdot)$ denote the d.f. of the standard normal variate, see Theorem 2.1 (i) below. Then, they use this result to establish rates of convergence for L-estimate, i.e., $T(F)=\int_0^1 F^{-1}(u)J(u)du$.

The purpose of the present note is to show that one can obtain L_p-rates of convergence for $T(F_n)-T(F)$ for $1\leqq p<\infty$, thus when combining these rates with those of Boos and Serfling [2] we can list rates

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of convergence for L_p $1\leqq p\leqq\infty$.

For any function ϕ, define $\|\phi\|_p=\left(\int|\phi(t)|^p dt\right)^{1/p}$, $1\leqq p<\infty$ and $\|\phi\|_\infty=\sup_t|\phi(t)|$. Denote by G_n the d.f. of $n^{1/2}(T(F_n)-T(F))/\sigma(T, F)$. Then we state and prove the following portmanteau theorem.

THEOREM 1.1. *Suppose that $T(F_n)-T(F)$ can be written as V_n+R_n where*

$$V_n=n^{-2}\sum_i\sum_j h(X_i, X_j)\ , \tag{1.1}$$

with $h(\cdot, \cdot)$ a symmetric function, such that $\mathrm{E}\,h(X_1, X_2)=0$, $\mathrm{E}|h(X_1, X_2)|^3<\infty$, *and* $\mathrm{E}|h(X_1, X_1)|^2<\infty$. *Let $\{\varepsilon_n\}$ be a sequence of real numbers such that $\varepsilon_n\geqq n^{-1}$ for all $n\geqq 1$ and $\varepsilon_n\to 0$ as $n\to\infty$.*

(i) *If for some $C>0$, $\mathrm{P}[|R_n|>C\varepsilon_n]=O(\varepsilon_n^{1/2})$ and putting $\sigma(T, F)=4\,\mathrm{Var}\,g(X_1)$, with $g(x)=\mathrm{E}_F[h(X_1, X_2)\,|\,X_1=x]$, then*

$$\|G_n-\Phi\|_\infty=O(\varepsilon_n^{1/2})\ . \tag{1.2}$$

(ii) *If in addition $\mathrm{E}(R_n^2)=O(\varepsilon_n^2)$, then for any $1\leqq p<\infty$*

$$\|G_n-\Phi\|_p=O(\varepsilon_n^{1/2})\ . \tag{1.3}$$

PROOF. (i) Follows exactly that of Theorem 1.1 of Boos and Serfling [2] with obvious modifications and hence is not repeated here. (ii) Let $H_n(\cdot)$ denote the d.f. of the random variable $\sqrt{n}\,V_n/\sigma(T, F)$. We use the following inequalities:

$$\|G_n-\Phi\|_p^p\leqq\|G_n-\Phi\|_\infty^{p-1}\|G_n-\Phi\|_1 \tag{1.4}$$

and

$$\|G_n-\Phi\|_1\leqq\|H_n-\Phi\|_1+\sqrt{n}\ \mathrm{E}^{1/2}(R_n^2)/\sigma(T, F)\ . \tag{1.5}$$

A proof of (1.5) is given in the appendix. Now, from Part (i) it follows that $\|G_n-\Phi\|_\infty^{p-1}=O(\varepsilon_n^{(p-1)/2})$.

Also we have

$$\|H_n-\Phi\|_1\leqq\|K_n-\Phi\|_1+\sqrt{n}\ \mathrm{E}^{1/2}(U_n-V_n)^2/\sigma(T, F)\ , \tag{1.6}$$

where $K_n(\cdot)$ denote the d.f. of $\sqrt{n}\,U_n/\sigma(T, F)$, with $U_n=\binom{n}{2}^{-1}\sum_{1\leqq i<j\leqq n}h(X_i, X_j)$. Note that if $W_n=n^{-1}\sum_{i=1}^n h(X_i, X_i)$, then $U_n-V_n=n^{-1}(U_n-W_n)$. Hence

$$\begin{aligned}\sqrt{n}\ \mathrm{E}^{1/2}(U_n-V_n)^2&=n^{-1/2}\,\mathrm{E}^{1/2}(U_n-W_n)^2\\&\leqq 2n^{-1/2}\{[\mathrm{Var}\,(U_n)]+[\mathrm{Var}\,(W_n)]\}^{1/2}\\&=2n^{-1/2}\{O(n^{-1/2})\}=O(n^{-1})=O(\varepsilon_n)\ .\end{aligned}$$

All that is remained is to show that $\|K_n-\Phi\|_1=O(\varepsilon_n^{1/2})$. It suffices, cf. Callaert and Janssen [3], to consider $\hat{K}_n$, the d.f. of $\sqrt{n}\,U_n/2\sigma_g$. Write $U_n=\hat{U}_n+\hat{\Delta}_n$, where $\hat{U}_n=2n^{-1}\sum_{j=1}^{n} g(X_j)$ with $g(X_1)=\mathrm{E}[h(X_1, X_2)\,|\,X_1]$, and $\hat{\Delta}_n=\binom{n}{2}^{-1}\sum_{1\le i<j\le n}[h(X_i, X_j)-g(X_i)-g(X_j)]$. Let L_n denote the d.f. of $\sqrt{n}\,\hat{U}_n/2\sigma_g$. Then

$$(1.7)\qquad \|\hat{K}_n-\Phi\|_1\leqq\|L_n-\Phi\|_1+\sqrt{n}\ \mathrm{E}^{1/2}(\hat{\Delta}_p^2)/4\sigma_g^2=O(n^{-1/2})+O(n^{-1/2})\ ,$$

where the first term in the last upper bound is obtained from Theorem 4.3 of Ibragimov [5], and the second from the well-known result that $\mathrm{E}\,\hat{\Delta}_n^2=O(n^{-1})$. Thus $\|\hat{K}_n-\Phi\|_1=O(\varepsilon_n^{1/2})$ and the theorem is now proved.

Remark 1.1. Note that listing the rate in Theorem 1.1 in terms of ε_n allows the accommodation of the cases when $\mathrm{P}[|R_n|>Cn^{-1}]$ and $\mathrm{E}(R_p^2)$ do not attain the optimum rate. Note also that if we assume that $\mathrm{E}|h(X_1, X_2)|^{2+\delta}<\infty$ and $\mathrm{E}|h(X_1, X_1)|^{(2+\delta)/2}<\infty$ for some $0<\delta\leqq 1$, then $\|G_n-\Phi\|_\infty=O(\varepsilon_n^{\delta/2})$ if $\mathrm{P}[|R_n|>Cn^{-1}]=O(\varepsilon_n^{\delta/2})$ and if further $\mathrm{E}(R_n^2)=O(\varepsilon_n^{1+\delta})$, then $\|G_n-\Phi\|_p=O(\varepsilon_n^{\delta/2})$ $1\leqq p<\infty$. All the ingredients needed for the proof of this extension are in the proof of Theorem 1.1 above except that in this case $\|\hat{K}_n-\Phi\|_\infty=O(\varepsilon_n^{\delta/2})$, $0<\delta\leqq 1$. This latter results is obtained by altering the proof of Callaert and Janssen [3], for details see Ahmad [1], Theorem 2.1.

Remark 1.2. Another rate of convergence is possible to establish namely, if $\varepsilon_n=O(n^{-1})$, if $\mathrm{E}|h(X_1, X_2)|^{2+\delta}<\infty$, $\mathrm{E}|h(X_1, X_1)|^{(2+\delta)}<\infty$, $0<\delta<1$, or $\mathrm{E}\,h^2(X_1, X_2)\ln(1+|h(x_1, x_2)|)<\infty$, $\delta=0$ and if for some $C>0$, $\sum_{n=1}^{\infty}\varepsilon_n^{1-\delta/2}\,\mathrm{P}[|R_n|>C\varepsilon_n]<\infty$, then $\sum_{n=1}^{\infty}\varepsilon_n^{1-\delta/2}\|G_n-\Phi\|_\infty$. If further, $\sum_{n=1}^{\infty}\varepsilon_n^{(1-\delta)}\,\mathrm{E}^{1/2}(R_n^2)<\infty$, then $\sum_{n=1}^{\infty}\varepsilon_n^{1-\delta/2}\|G_n-\Phi\|_p<\infty$, $1\leqq p<\infty$, $0\leqq\delta<1$. Again the main points in the proof of this result is to show that $\sum_{n=1}^{\alpha} n^{-1+\delta/2}\|H_n-\Phi\|_\infty<\infty$ which is proved in Theorem 2.2 of Ahmad [1], and that $\sum_{n=1}^{\infty} n^{-1+\delta/2}\cdot\|L_n-\Phi\|_1<\infty$ which is proved by Heyde [4], Corollary 2.

2. Application to L-estimates

Consider the functional defined by $T(F)=\int_0^1 F^{-1}(u)J(u)du$ and the corresponding L-estimate $T(F_n)$. As shown by Boos and Serfling [2] it is possible to write

$$(2.1)\qquad T(F_n)-T(F)=V_n+R_n\ ,$$

where $V_n = n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n} h(X_i, X_j)$ with $h(x, y) = (1/2)[\alpha(x) + \alpha(y) + \beta(x, y)]$ where $\alpha(x) = -\int_{-\infty}^{\infty} [I(x \leqq t) - F(t)]J \circ F(t)dt$, and $\beta(x, y) = -\int_{-\infty}^{\infty} [I(x \leqq t) - F(t)][I(y \leqq t) - F(t)]J^{(1)} \circ F(t)dt$, with $J^{(1)}(\cdot)$ the first derivative of J. Note also that $R_n = T(F_n) - T(F) - V_n$ may be written as $-\int_{-\infty}^{\infty} W_{F_n, F}(x)dx$ with $W_{G,F} = K \circ G - K \circ F - J \circ F(G - F) - (1/2)(J^{(1)} \circ F)(F - G)^2$ and $K(u) = \int_{-0}^{u} J(v)dv$. Further in this case $\sigma^2(J, F) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J \circ F(x)J \circ F(u) \cdot [F(\min(x, y)) - F(x)F(y)]dxdy$.

Boos and Serfling [2] established the Berry-Esseen bound of $\sqrt{n} \cdot (T(F_n) - T(F))/\sigma(J, F)$ under two sets of conditions on J and F, viz.:

CONDITION A: Assume that J vanishes outside a closed interval $[a, b]$, $0 < a < b < 1$ and that $J^{(1)}$ exists and is such that $|J^{(1)}(x) - J^{(1)}(y)| \leqq D|x - y|^\gamma$ for some $D > 0$, $\gamma > 0$ for all x, y defined on an open interval containing $[a, b]$, i.e., $J^{(1)}$ is Lipschiz of order $\gamma > 0$.

Or

CONDITION B: Suppose that $J^{(1)}$ exists and is such that $|J^{(1)}(x) - J^{(1)}(y)| \leqq D|x - y|^\gamma$, for some $D > 0$, $\gamma > 1/3$ and all $x, y \in (0, 1)$. Assume also that $\mathrm{E}|X_1|^3 < \infty$.

Using Theorem 1.1 we shall establish L_p-rate of convergence of the order $n^{-1/2}$ for $\sqrt{n}(T(F_n) - F(F))/\sigma(J, F)$ assuming that either Condition A or B is satisfied and that $\sigma^2(J, F) > 0$.

THEOREM 2.1. *Assume that $\sigma^2(J, F) > 0$ and that either Condition* A *or* B *is satisfied. Let G_n denote the d.f. of $n^{1/2}(T(F_n) - T(F))/\sigma(J, F)$. Then*

$$\|G_n - \Phi\|_p = O(n^{-1/2}), \quad 1 \leqq p \leqq \infty. \tag{2.2}$$

PROOF. We need to establish that under either Condition A or B, $\mathrm{P}[|R_n| > Cn^{-1}] = O(n^{-1/2})$ for some $C > 0$ and $\mathrm{E}(R_n)^2 = O(n^{-2})$. But the first assertion is a consequence of Theorems 2.1 and 2.2 of Boos and Serfling [2]. Thus we shall prove the second. Since $J^{(1)}$ is Lipschiz of order γ we obtain

$$|R_n| \leqq (D/2)\int |F_n(x) - F(x)|^{2+\gamma}dx \leqq (D/2)\|F_n - F\|_\infty^\gamma \cdot \|F_n - F\|_2^2. \tag{2.3}$$

Hence

$$\mathrm{E}\, R_n^2 \leqq (D^2/4)\,\mathrm{E}\,\|F_n - F\|_\infty^\gamma \cdot \|F_n - F\|_2^4 \leqq (D^2/4)\,\mathrm{E}\,\|F_n - F\|_2^4 \leqq O(n^{-2}), \tag{2.4}$$

where $\mathrm{E}\|F_n - F\|_2^4 = O(n^{-2})$ follows from Lemma 2.1 of Boos and Serfling [2].

Remark 2.1. Since under Condition A, or Condition B, $P[|R_n|> Cn^{-1}]=O(n^{-1/2})$ and that $E|R_n^2|=O(n^{-2})$ it follows that $\sum_{n=1}^{\infty} n^{-1+\delta/2} P[|R_n|> Cn^{-1}]<\infty$, and $\sum_{n=1}^{\infty} n^{-1+\delta/2} E^{1/2}(R_n^2)<\infty$, and hence it follows that for any $0\leqq\delta<1$, $\sum_{n=1}^{\infty} n^{-1+\delta/2}\|G_n-\Phi\|_p<\infty$, $1\leqq p<\infty$, from Remark 1.2. It appears that this is the first attempt at series rate of convergence for L-estimates. Also analogous rates are obtainable for linear combinations of order statistics. $T_n=n^{-1}\sum_{i=1}^{n} J(i/(n+1))X_{(i)}$, where $X_{(1)}\leqq\cdots\leqq X_{(n)}$ denote the order statistics corresponding to $X_1,\cdots,X_n$.

Remark 2.2. Boos and Serfling indicate how the Berry-Esseen rate can be obtained for the class of M-estimates (we refer the reader to Boos and Serfling [2] for definition). Their representation $T(F_n)-T(F)=V_n+R_n$ leads to R_n satisfying $|R_n|<D_1\|F_n-F\|_\infty^3$ under certain conditions, but it can be shown that $E\|F_n-F\|_\infty^{2k}=O(n^{-k})$, $k=1, 2,\cdots$, and thus $E R_n^2=O(n^{-2})$ and thus we can obtain the rate $n^{-1/2}$ for L_p-convergence in light of Theorem 1.1.

Appendix

PROOF OF (1.5). Note that

$$\|G_n-\Phi\|_1\leqq\|G_n-H_n\|_1+\|H_n-\Phi\|_1 .$$

But

$$\begin{aligned}\|G_n-H_n\|_1&=\int_{-\infty}^{\infty}|E\, I(X_n>x)-E\, I(Y_n>x)|dx\\&\leqq E\int|I(X_n>x)-E\, I(Y_n>x)|dx\\&=E|X_n-Y_n|\leqq E^{1/2}|X_n-Y_n|^2 ,\end{aligned}$$

where $X_n=n^{1/2}\{T(F_n)-T(F)\}/\sigma(T, F)$ and $Y_n=n^{1/2}V_n/\sigma(T, F)$ with V_n defined in (1.1).

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