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SIMULTANEOUS ESTIMATION OF PARAMETERS IN EXPONENTIAL FAMILIES

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Summary

The paper considers estimation of the natural parameter vector or the mean vector from independent distributions each belonging to the one-parameter discrete or absolutely continuous exponential family. The usual estimators (maximum likelihood, minimum variance unbiased or best invariant) are improved simultaneously under various weighted squared error losses.

1. Introduction

Let $X_1, \cdots, X_p$ be $p\,(\geqq 3)$ independent normal variables with means $\theta_1, \cdots, \theta_p$ and unit variances. Stein [12] proved the inadmissibility of usual estimator $\boldsymbol{X}=(X_1, \cdots, X_p)'$ of $\boldsymbol{\theta}=(\theta_1, \cdots, \theta_p)'$ under squared error loss. An explicit estimator dominating $\boldsymbol{X}$ was produced by James and Stein [10].

In this paper, we consider losses of the form

$$L_{\boldsymbol{d}}(\boldsymbol{\theta}, \boldsymbol{a})=\sum_{i=1}^{p} d_i(\theta_i-a_i)^2 , \qquad d_i>0 \ (i=1, \cdots, p) , \tag{1.1}$$

where

$$\boldsymbol{d}=(d_1, \cdots, d_p)' .$$

Several authors considered improvement on $\boldsymbol{X}$ under losses of the form (1.1) for a single $\boldsymbol{d}$ (see, for example, Berger [3], where other references are cited). Brown [4] and Shinozaki [11] considered improvement on $\boldsymbol{X}$ simultaneously under losses of the form $L_{\boldsymbol{d}}$ for various $\boldsymbol{d}$.

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The present paper extends the results in Section 5 of Brown [4] and Theorem 1 of Shinozaki [11] in several directions. (Note that the other main result of Shinozaki [11] is his Theorem 2.)

First, in Section 2, we consider estimation of the natural parameter vector from p independent distributions each belonging to the one-parameter exponential family absolutely continuous with respect to Lebesgue measure. Assuming losses of the form (1.1), a class of estimators is produced dominating the minimum variance unbiased estimator or some constant multiple of it simultaneously for various $\boldsymbol{d} \in \mathscr{D}$ (where $\mathscr{D}$ will be specified later). The general results are illustrated with the estimation of the natural parameter vector for the normal and the gamma distributions. Next, in this section, a subfamility of the general exponential family of distributions is considered, and a class of estimators dominating the minimum variance unbiased estimator of the mean vector is produced, once again under losses of the form (1.1).

Next, in Section 3, we consider the discrete exponential family of distributions, and address a similar problem. Once again, a general class of estimators dominating the minimum variance unbiased estimator is produced.

Our method of proof uses the integration by parts technique (or its discrete analogue) of Stein (see Stein [13], Stein [14] or Hwang [9]) in contrast to the techniques used by Brown [4] or Shinozaki [11] in proving the corresponding results in the normal case.

2. Estimators in the absolutely continuous case

Consider p independent random variables X_i $(1\leqq i\leqq p)$, X_i having pdf (with respect to Lebesgue measure on (a_i, b_i), a_i and b_i being possibly infinite)

$$f_{\theta_i}(x_i)=\pi_i(\theta_i)\rho_i(x_i)\exp(-\theta_i r_i(x_i))\,, \qquad 1\leqq i\leqq p\,, \tag{2.1}$$

where $\rho_i(x_i)>0$ and $r_i(x_i)$ is monotone in x_i, and is absolutely continuous in (a_i, b_i). It is desired to estimate $\boldsymbol{\theta}$. This problem is addressed in Hudson [8], Berger [2] and Ghosh and Parsian [6]. For any estimator $\boldsymbol{\delta}^0(\boldsymbol{X})=(\delta_1^0(\boldsymbol{X}),\cdots,\delta_p^0(\boldsymbol{X}))'$ of $\boldsymbol{\theta}$, consider the competitor $\boldsymbol{\delta}(\boldsymbol{X})=(\delta_1(\boldsymbol{X}),\cdots,\delta_p(\boldsymbol{X}))'$ with $\delta_i(\boldsymbol{X})=\delta_i^0(\boldsymbol{X})-q_i(\boldsymbol{X})\phi_i(\boldsymbol{X})$, where $q_i(\boldsymbol{X})=r_i'(X_i)\cdot\exp(s_i(\boldsymbol{X}))/\rho_i(X_i)$, and $s_i(\boldsymbol{X})$ is defined as $s_i^{i(1)}(\boldsymbol{X})=\partial s_i(\boldsymbol{X})/\partial X_i=\delta_i^0(\boldsymbol{X})r_i'(X_i)$. The following regularity assumptions are made.

(CI) $\quad \mathrm{E}_{\boldsymbol{\theta}}\,[q_i^2(\boldsymbol{X})\phi_i^2(\boldsymbol{X})]<\infty$.

(CII) $\quad \delta_i(\boldsymbol{x})$ is absolutely continuous as a function of x_i in any compact subset of (a_i, b_i) for almost all $x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_p$;

(CIII) $\lim_{x_i \to a_i} \phi_i(\boldsymbol{x}) \exp(s_i(x) - \theta_i r_i(x_i)) = \lim_{x_i \to b_i} \phi_i(\boldsymbol{x}) \exp(s_i(\boldsymbol{x}) - \theta_i r_i(x_i)) = 0$ for all θ_i;

(CIV) $\mathrm{E}_{\boldsymbol{\theta}} |\phi_i^{i(1)}(\boldsymbol{X}) \exp(s_i(X))/\rho_i(X_i)| < \infty$ for all $\boldsymbol{\theta}$.

These conditions are usually satisfied and are easy to verify (see for example Ghosh and Parsian [6]). Under the loss (1.1), the above conditions together with

(CV) $R(\boldsymbol{\theta}, \boldsymbol{\delta}^0) < \infty$ for all $\boldsymbol{\theta}$

imply that (cf. Berger]2])

$$R(\boldsymbol{\theta}, \boldsymbol{\delta}) - R(\boldsymbol{\theta}, \boldsymbol{\delta}^0) = \mathrm{E}_{\boldsymbol{\theta}}\, \Delta(\boldsymbol{X}) , \tag{2.2}$$

where $\Delta(\boldsymbol{x}) = 2 \sum_{i=1}^{p} d_i (q_i(\boldsymbol{x})/r_i'(x_i)) \phi_i^{i(1)}(\boldsymbol{x}) + \sum_{i=1}^{p} d_i q_i^2(\boldsymbol{x}) \phi_i^2(\boldsymbol{x})$, $\phi_i^{i(1)}(\boldsymbol{x}) = \partial \phi_i(\boldsymbol{x})/\partial x_i$.

Note that the UMVUE of θ_i is $k_i(X_i) = (r_i'(X_i))^{-1}(\rho_i'(X_i)/\rho_i(X_i)) + d/dX_i \cdot (r_i'(X_i))^{-1}$. Note that for any specified $c_i > 1$, $c_i k_i(X_i)$ is inadmissible, it being dominated by $k_i(X_i)$. Consider $\delta_i^0(\boldsymbol{X}) = c_i k_i(X_i)$ for some specified $c_i \in (0, 1]$, $1 \leqq i \leqq p$. The calculations of Ghosh and Parsian [6] show that when $r_i(x_i) \uparrow$ in x_i,

$$q_i(\boldsymbol{x}) = (r_i'(x_i)/\rho_i(x_i))^{1-c_i} , \qquad 1 \leqq i \leqq p , \tag{2.3}$$

while if $r_i(x_i) \downarrow$ in x_i,

$$q_i(\boldsymbol{x}) = -(-r_i'(x_i)/\rho_i(x_i))^{1-c_i} , \qquad 1 \leqq i \leqq p . \tag{2.4}$$

In either case, one can write

$$\Delta(\boldsymbol{x}) = 2 \sum_{i=1}^{p} d_i \nu_i(x_i) \phi_i^{i(1)}(\boldsymbol{x}) + \sum_{i=1}^{p} d_i w_i(x_i) \phi_i^2(\boldsymbol{x}) , \tag{2.5}$$

for some appropriately defined $\nu_i(x_i)$ and $w_i(x_i)$. We are now in a position to prove the main result of this section.

THEOREM 1. *Assume $\nu_i^{-1}(x_i)$ is integrable with $g_i'(x_i) = \nu_i^{-1}(x_i)$. Let $S = \sum_{i=1}^{p} |g_i(x_i)|^{\beta}$ for some $\beta > 0$. It is assumed that*

(i) *there exists a constant $K > 0$ such that*

$$\sum_{i=1}^{p} w_i(x_i) g_i^2(x_i) \leqq KS ; \tag{2.6}$$

(ii) *there exist constants a_i (>0) such that*

$$\inf_{\boldsymbol{d} \in \mathcal{D}} \left[\sum_{i=1}^{p} d_i a_i - \beta \max_{1 \leqq i \leqq p} (d_i a_i) \Big/ \max_{1 \leqq i \leqq p} (d_i a_i^2) \right] = a_0 > 0 , \tag{2.7}$$

where $\mathcal{D}$ is some subset of $(R^+)^p$ for which (2.7) *holds. Then for any $\tau(S)$ satisfying*

(iii) $\tau(S) \uparrow$ *in* S;

(iv) $0<\tau(S)<2K^{-1}a_0$, *defining*

$$\phi_i(\boldsymbol{x})=-\frac{a_i\tau(S)}{S}g_i(x_i)\ , \qquad 1\leqq i\leqq p,\ \boldsymbol{\phi}(\boldsymbol{x})=(\phi_1(\boldsymbol{x}),\cdots,\phi_p(\boldsymbol{x}))'$$

provides a solution to the differential inequality $\Delta(\boldsymbol{x})<0$. Consequently, under losses of the form (1.1) *with $\boldsymbol{d}\in\mathcal{D}$, $\delta^0(\boldsymbol{X})$ is improved by $\boldsymbol{\delta}(\boldsymbol{X})=(\delta_1^0(\boldsymbol{X})-q_1(\boldsymbol{X})\phi_1(\boldsymbol{X}),\cdots,\delta_p^0(\boldsymbol{X})-q_p(\boldsymbol{X})\phi_p(\boldsymbol{X}))'$.*

Remark 1. Assumptions (iii) and (iv) can always be made. For example, take $\tau(S)$ to be any constant in the interval $(0, 2K^{-1}a_0)$.

Remark 2. For any given set of d_i's ($d_i>0$, $1\leqq i\leqq p$), it is not necessarily possible to find a_i's ($a_i>0$, $1\leqq i\leqq p$) such that (2.7) holds (see for example Remark 2 of Shinozaki [11]). In fact, Shinozaki [11] points out that his Theorem 1 is useful when the d_i's are not extremely different among the loss functions.

PROOF OF THEOREM 1. First note that

$$\phi_i^{i(1)}(\boldsymbol{x})=-\frac{a_i\tau(S)}{S}g_i'(x_i)+a_i\left(-\frac{\tau'(S)}{S}+\frac{\tau(S)}{S^2}\right)\beta|g_i(x_i)|^\beta g_i'(x_i)\ . \tag{2.8}$$

Hence, combining (2.5) and (2.8) one gets,

$$\begin{aligned}\Delta(\boldsymbol{x})=&\,2\sum_{i=1}^p d_ia_i\left[-\frac{\tau(S)}{S}+\left(-\frac{\tau'(S)}{S}+\frac{\tau(S)}{S^2}\right)\beta|g_i(x_i)|^\beta\right]\\&+\sum_{i=1}^p d_ia_i^2w_i(x_i)g_i^2(x_i)\frac{\tau^2(S)}{S^2}\ .\end{aligned} \tag{2.9}$$

In view of the assumptions (2.6), (2.7), (iii) and (iv), it follows from (2.9) that for $\boldsymbol{d}\in\mathcal{D}$,

$$\begin{aligned}\Delta(\boldsymbol{x})\leqq&\,2\left(\sum_{i=1}^p d_ia_i\right)\left(-\frac{\tau(S)}{S}\right)+2\left(\max_{1\leqq i\leqq p} d_ia_i\right)\beta\frac{\tau(S)}{S}\\&+K\left(\max_{1\leqq i\leqq p} d_ia_i^2\right)\frac{\tau^2(S)}{S}\\=&-\frac{\tau(S)}{S}\left[2\left(\sum_{i=1}^p d_ia_i-\beta\max_{1\leqq i\leqq p} d_ia_i\right)-K\left(\max_{1\leqq i\leqq p} d_ia_i^2\right)\tau(S)\right]\\\leqq&-\frac{\tau(S)}{S}\left(\max_{1\leqq i\leqq p} d_ia_i^2\right)(2a_0-K\tau(S))<0\ .\end{aligned} \tag{2.10}$$

The proof of Theorem 1 is complete.

Some important applications of Theorem 1 are given below.

Example 1. Suppose $X_1, \cdots, X_p$ $(p \geqq 3)$ are independent normal variables with means $\theta_1, \cdots, \theta_p$ and unit variances. Then $r_i(x_i) = -x_i$, $\delta_i^0(x_i) = x_i$ so that $s_i^{i(1)}(x) = -x_i$. Hence, $s_i(x_i) = -1/2 \cdot x_i^2$. Also, since $\rho_i(x_i) = \exp(-1/2x_i^2)$, $q_i(x) = -1$. Hence, $\nu_i(x_i) = w_i(x_i) = 1$. Thus, $g_i(x_i) = x_i$. Consequently, (2.6) hold with $S = \sum_{i=1}^{p} x_i^2$ and $K = 1$. Now, take $\phi_i(\boldsymbol{x}) = -(a_i \tau(S)/S) x_i$ where a_i's satisfy (2.7). In order that (CI)–(CV) hold, it is assumed that $\tau(S)$ is differentiable, and $\mathrm{E}_{\boldsymbol{\theta}} |\tau'(S)| < \infty$ for all $\boldsymbol{\theta}$. If, in addition, $\tau(S) \uparrow$ in S and $0 < \tau(S) < 2a_0$, $(X_1 + \phi_1(\boldsymbol{X}), \cdots, X_p + \phi_p(\boldsymbol{X}))$ dominates $\boldsymbol{X}$ under losses of the form (1.1) simultaneously for $\boldsymbol{d} \in \mathscr{D}$. This example includes as a special case Theorem 1 of Shinozaki [11]. Also, if $\mathscr{D} = \{\boldsymbol{d}\colon d_1 = \cdots = d_p = d\ (>0)\}$ taking $a_1 = \cdots = a_p = 1$ so that $a_0 = p - 2$, one gets a class of estimators dominating $\boldsymbol{X}$ (for $p \geqq 3$) as proposed by Baranchik [1] and Strawderman [15].

Example 2. Let $X_1, \cdots, X_p$ be $p\ (\geqq 2)$ independent gamma variables, X_i having pdf

$$f_{\theta_i}(x_i) = \exp(-\theta_i x_i) \theta_i^{\alpha_i} x_i^{\alpha_i - 1} / \Gamma(\alpha_i)\ , \tag{2.11}$$

$x_i > 0$, $\theta_i > 0$, $\alpha_i > 2$, $1 \leqq i \leqq p$. Then the minimum mean squared error estimator of θ_i in the class of all estimators of the form k_i/X_i is $(\alpha_i - 2)/X_i$. Such estimators are admissible in one dimension under quadratic loss. Now, in this case $r_i(x_i) = x_i$, $\rho_i(x_i) = x_i^{\alpha_i - 1}$ so that $q_i(x_i) = x_i^{-1}$, $w_i(x_i) = x_i^{-2}$. Now, $g_i(x_i) = 1/2 x_i^2$ so that (2.6) holds with $S = 1/2 \sum_{i=1}^{p} x_i^2$ and $K = 1/2$. Hence, for a_i's satisfying (2.7), define $\phi_i(\boldsymbol{x}) = -a_i \tau(S)/S\ (1/2 x_i^2) = -\left(a_i \tau(S) \middle/ \sum_1^p x_i^2\right) x_i^2$, where (i) $\tau(S) \uparrow$ in S, (ii) $\mathrm{E}(\tau'(S)) < \infty$ and (iii) $0 < \tau(S) < 4a_0$, a_0 being defined in (2.7). Now for $p \geqq 2$, $((\alpha_1 - 2)/X_1, \cdots, (\alpha_p - 2)/X_p)$ is dominated by $\left((\alpha_1 - 2)/X_1 + \left(a_1 \tau(S) \middle/ \sum_1^p X_i^2\right) X_1, \cdots, ((\alpha_p - 2)/X_p + \left(a_p \tau(S) \middle/ \sum_1^p X_i^2\right) X_p\right)$ under losses of the form (1.1) simultaneously for $\boldsymbol{d} \in \mathscr{D}$.

Finally, in this section, we consider the following subfamily of the general exponential family of distributions with pdf's given by

$$f_\theta(x) = \exp[\mu(\theta) b(x) - \chi(\theta)] m^{-1}(x) \exp\left[-\int x m^{-1}(x) dx\right]\ , \tag{2.12}$$

with $\mu(\theta) = \mathrm{E}_\theta(X)$ and $b'(x) = m^{-1}(x)$. This particular subfamily was considered by Hudson [8]. Suppose now $X_1, \cdots, X_p$ $(p \geqq 3)$ are independent, X_i having pdf given by $f_{\theta_i}(x)$ with support (w_1, w_2) (w_1 and w_2 being possibly infinite). The problem is to improve on the UMVUE $\boldsymbol{X}$ of $\boldsymbol{\theta}$.

Hudson [8] has shown that under the conditions

$$\text{(CVI)}\quad \lim_{x_i\to w_1} \phi_i(\boldsymbol{x}) \exp\left[\mu(\theta_i)b(x_i)-\int x_i m^{-1}(x_i)dx_i\right]$$
$$=\lim_{x_i\to w_2} \phi_i(\boldsymbol{x}) \exp\left[\mu(\theta_i)b(x_i)-\int x_i m^{-1}(x_i)dx_i\right]=0\ ,$$
$$1\leqq i\leqq p,\ \text{for almost all } x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_p\ ,$$

and

$$\text{(CVII)}\quad \mathrm{E}_{\boldsymbol{\theta}}|m(X_i)\phi_i^{i(1)}(\boldsymbol{X})|<\infty\ ,\qquad 1\leqq i\leqq p,\ \text{for all } \boldsymbol{\theta}\ ,$$

one gets the identity

$$\mathrm{E}_{\boldsymbol{\theta}}\left[(X_i-\mu(\theta_i))\phi_i(\boldsymbol{X})\right]=\mathrm{E}_{\boldsymbol{\theta}}\left[m(X_i)\phi_i^{i(1)}(\boldsymbol{X})\right]\ ,\qquad 1\leqq i\leqq p\ . \tag{2.13}$$

Hence, if $\boldsymbol{\delta}^*(\boldsymbol{x})=(\delta_1^*(\boldsymbol{x}),\cdots,\delta_p^*(\boldsymbol{x}))$ with $\delta_i^*(\boldsymbol{x})=x_i+\phi_i(\boldsymbol{x})$, then,

$$\sum_{i=1}^p d_i\,\mathrm{E}_{\boldsymbol{\theta}}\left[\delta_i^*(\boldsymbol{X})-\mu(\theta_i)\right]^2-\sum_{i=1}^p d_i\,\mathrm{E}_{\boldsymbol{\theta}}\left[X_i-\mu(\theta_i)\right]^2=\mathrm{E}_{\boldsymbol{\theta}}\,\Delta(\boldsymbol{X})\ , \tag{2.14}$$

where

$$\Delta(\boldsymbol{x})=2\sum_{i=1}^p d_i m(x_i)\phi_i^{i(1)}(\boldsymbol{x})+\sum_{i=1}^p d_i\phi_i^2(\boldsymbol{x})\ . \tag{2.15}$$

It follows that (2.15) is also of the form (2.5) with $\nu_i(x_i)=m(x_i)$ and $w_i(x_i)=1$. Let $g_i(x_i)=\int \nu_i^{-1}(x_i)dx_i=b(x_i)$. Now, taking $S=\sum_{i=1}^p d^2(x_i)$, it follows that (2.6) holds with $K=1$. If (2.7) holds now with $\beta=2$, define $\phi_i(\boldsymbol{X})=-(a_i\tau(S)/S)b(X_i)$, $1\leqq i\leqq p$, where $\tau(S)\uparrow$ in S, $\mathrm{E}[\tau'(S)]<\infty$ and $0<\tau(S)<2a_0$. Then, $(X_1+\phi_1(\boldsymbol{X}),\cdots,X_p+\phi_p(\boldsymbol{X}))$ dominates $\boldsymbol{X}$ under losses os the form (1.1) when $\boldsymbol{d}\in\mathcal{D}$.

To see an application of the above result, consider the following example which appears in Hudson [8], and in Ghosh and Parsian [6].

Example 3. Let $X_1,\cdots,X_p$ be independent, X_i having pdf

$$f_{\theta_i}(x_i)=\exp(-x_i)x_i^{\theta_i-1}/\Gamma(\theta_i)\ ;\qquad x_i>0,\ \theta_i>0,\ 1\leqq i\leqq p\ . \tag{2.16}$$

In this case, $\mathrm{E}_{\theta_i}(X_i)=\theta_i$. The above class of pdf's is of the form (2.12) with $b(x_i)=\log x_i$, and $m^{-1}(x_i)=x_i^{-1}$. Thus, (2.6) holds with $K=1$ and $S=\sum_{i=1}^p(\log x_i)^2$. Hence, if (2.7) holds, for any $\tau(S)\uparrow$ in S with $\mathrm{E}[\tau'(S)]<\infty$ and $0<\tau(S)<2a_0$, $(X_1+\phi_1(\boldsymbol{X}),\cdots,X_p+\phi_p(\boldsymbol{X}))$ dominates $\boldsymbol{X}$ with $\phi_i(\boldsymbol{X})=-(a_i\tau(S)/S)\log X_i$, $1\leqq i\leqq p$, $p\geqq 3$.

3. Discrete exponential family

Let $X_1, \cdots, X_p$ be p independent random variables, X_i having probability function (pf)

$$f_{\theta_i}(x_i)=\pi_i(\theta_i)t_i(x_i)\theta_i^{x_i}\,, \qquad x_i=0, 1, \cdots;\ 1\leqq i\leqq p\,. \tag{3.1}$$

The problem is to estimate $\boldsymbol{\theta}$ under losses of the form

$$L_{\boldsymbol{d}}(\boldsymbol{\theta}, \boldsymbol{u})=\sum_{i=1}^{p} d_i(\theta_i-u_i)^2/\theta_i^{m_i}\,, \tag{3.2}$$

where m_i's are *known positive* integers.

The UMVUE of θ_i is $\delta_i^0(X_i)=t_i(X_i-1)/t_i(X_i)$, where $t_i(-1)$ is defined as zero. Our goal in this section is to improve on the estimator $\boldsymbol{\delta}^0(\boldsymbol{X})=(\delta_1^0(X_1), \cdots, \delta_p^0(X_p))'$ of $\boldsymbol{\theta}$ under losses of the form (3.2) simultaneously for various $\boldsymbol{d}\in\mathcal{D}$, where $\mathcal{D}$ is some suitable subset of $(R^+)^p$. Consider the competing estimators of the form $\boldsymbol{\delta}(\boldsymbol{X})=(\delta_1(\boldsymbol{X}), \cdots, \delta_p(\boldsymbol{X}))'$ where $\delta_i(\boldsymbol{X})=\delta_i^0(X_i)+\phi_i(\boldsymbol{X})$, $i=1, \cdots, p$. If $\mathrm{E}\,\phi_i^2(\boldsymbol{X})<\infty$ for all $i=1, \cdots, p$, under (3.2) it follows that (cf. Hwang [9]).

$$R(\boldsymbol{\theta}, \boldsymbol{\delta})-R(\boldsymbol{\theta}, \boldsymbol{\delta}^0)=2\,\mathrm{E}_{\boldsymbol{\theta}}\, u(\boldsymbol{X})\,, \tag{3.3}$$

where $u(\boldsymbol{x})=\sum_{i=1}^{p} d_iv_i(x_i)\Delta_i\psi_i(\boldsymbol{x})+\sum_{i=1}^{p} d_iw_i(x_i)\psi_i^2(\boldsymbol{x})$; $v_i(x_i)=t_i(x_i+m_i-1)/t_i(x_i)$, $w_i(x_i)=1/2\cdot t_i(x_i+m_i)/t_i(x_i)$, $\psi_i(\boldsymbol{x})=\phi_i(\boldsymbol{x}+m_i\boldsymbol{e}_i)$, where $\boldsymbol{e}_i$ is a vector with ith element 1 and the rest zeroes, and $\Delta_i\psi_i(\boldsymbol{x})=\psi_i(\boldsymbol{x})-\psi_i(\boldsymbol{x}-\boldsymbol{e}_i)$. We want to obtain solutions to the difference inequality $u(\boldsymbol{x})<0$. Assume that $v_i(x_i)>0$ for all $x_i=0, 1, \cdots$. One can see that this condition holds in important special cases of Poisson and negative binomials. Define

$$h_i(x_i)=\sum_{k=0}^{x_i} v_i^{-1}(k)\,, \qquad x_i=0, 1, \cdots\,. \tag{3.4}$$

Let $m_i(x_i)=b_ih_i(x_i)$ for some $b_i>0$; $M=b_0+\sum_{i=1}^{p} m_i(x_i)$ *for some* $b_0\geqq 0$ and $M_i=b_0+\sum_{\substack{j=1\\ \neq i}}^{p} m_j(x_j)+m_i(x_i-1)=M-\Delta_i m_i(x_i)$. Suppose now there exist $a_i>0$ such that

$$\inf_{\boldsymbol{d}\in\mathcal{D}} \frac{\sum_{i=1}^{p} d_ia_i-\max_{1\leq i\leq p} d_ia_i}{\max_{1\leq i\leq p} d_ia_i^2}=a_0>0\,. \tag{3.5}$$

If the elements of $\mathcal{D}$ are bounded way from zero, the existence of such a_i's is automatically guaranteed. In this case the class $\mathcal{D}$ is much wider than in the previous section. Define now

$$\psi_i(x) = -ca_i h_i(x_i)/M \quad \text{where } 0<c<2a_0,\ 1\leq i\leq p\,. \tag{3.6}$$

The main result of this section is now as follows.

THEOREM 2. *Assume that* (3.5) *and the following condition*

$$\sum_{i=1}^{p} w_i(x_i)h_i^2(x_i) \leq 1/2\cdot M \tag{3.7}$$

hold. Then $\boldsymbol{\psi}(\boldsymbol{x})=(\psi_1(\boldsymbol{x}),\cdots,\psi_p(\boldsymbol{x}))'$ *defined in* (3.6) *provides a solution to* $u(\boldsymbol{x})<0$, *where* $u(\boldsymbol{x})$ *is defined in* (3.3). *Consequently* $\boldsymbol{\delta}(\boldsymbol{x})=(\delta_1^0(X_1)+\phi_1(\boldsymbol{X}),\cdots,\delta_p^0(X_p)+\phi_p(\boldsymbol{X}))'$ *dominates* $\boldsymbol{\delta}^0(\boldsymbol{X})$, *where* $\phi_i(\boldsymbol{X})=\psi_i(\boldsymbol{X}-m_i\boldsymbol{e}_i)$. $1\leq i\leq p$.

PROOF. In view of (3.7),

$$\sum_{i=1}^{p} d_i w_i(x_i)\psi_i^2(x) = c^2 \sum_{i=1}^{p} d_i a_i^2 w_i(x_i)h_i^2(x_i)/M^2 \leq \frac{1}{2}c^2(\max_{1\leq i\leq p} d_i a_i^2)/M\,. \tag{3.8}$$

Also

$$\begin{aligned} \Delta_i\psi_i(\boldsymbol{x}) &= ca_i\left[\frac{h_i(x_i-1)}{M_i}-\frac{h_i(x_i)}{M}\right] = ca_i\frac{h_i(x_i-1)M-h_i(x_i)M_i}{MM_i} \\ &= ca_i\left[\frac{h_i(x_i-1)\Delta_i m_i(x_i)-\Delta_i h_i(x_i)M_i}{MM_i}\right]. \end{aligned} \tag{3.9}$$

Hence

$$\begin{aligned} \sum_{i=1}^{p} d_i v_i(x_i)\Delta_i\psi_i(\boldsymbol{x}) &= cM^{-1}\sum_{i=1}^{p} d_i a_i + cM^{-1}\sum_{i=1}^{p} b_i h_i(x_i-1)d_i a_i M_i^{-1} \\ &\leq -cM^{-1}\left(\sum_{i=1}^{p} d_i a_i - \max_{1\leq i\leq p} d_i a_i\right), \end{aligned} \tag{3.10}$$

where one uses the monotonicity of $m_i(x_i)$ (i.e. of $h_i(x_i)$) in order to obtain the last inequality. Combining (3.8) and (3.10), and using (3.5), one gets,

$$\begin{aligned} u(\boldsymbol{x}) &\leq -cM^{-1}\left(\sum_{i=1}^{p} d_i a_i - \max_{1\leq i\leq p} d_i a_i\right) + \frac{1}{2}c^2M^{-1}\max_{1\leq i\leq p} d_i a_i^2 \\ &\leq -cM^{-1}\max_{1\leq i\leq p} d_i a_i^2\left(a_0-\frac{1}{2}c\right)<0\,. \end{aligned} \tag{3.11}$$

The proof of the theorem is complete.

Example 4. Let $X_1,\cdots,X_p$ $(p\geq 2)$ be independent, $X_i\sim$ Poisson (θ_i), $i=1,\cdots,p$. Consider the loss (3.2) with $m_1=\cdots=m_p=1$. This is a generalized version of the loss considered by Clevenson and Zidek [5], and Ghosh and Parsian [7]. Note that in this case, $v_i(x_i)=1$ so that $h_i(x_i)=x_i+1$; also, $w_i(x_i)=1/2\cdot(x_i+1)^{-1}$. Taking $b_i=1$ so that $m_i(x_i)=h_i(x_i)=x_i+1$, it follows that (3.7) holds. Now if (3.5) holds,

$$\phi_i(\boldsymbol{x}) = -\frac{ca_i}{b_0+\sum_{i=1}^{p}(x_i+1)}(x_i+1) = -\frac{ca_i}{b_0+p+\sum_{1}^{p}x_i}(x_i+1)\ , \qquad (1\leqq i\leqq p)$$

provide a solution to $u(\boldsymbol{x})<0$. Consequently,

$$(X_1+\phi_1(\boldsymbol{X}),\cdots, X_p+\phi_p(\boldsymbol{X})) \quad \text{with} \quad \phi_i(\boldsymbol{X}) = -\frac{ca_i}{b_0+p-1+\sum_{1}^{p}X_i}X_i$$

dominates $\boldsymbol{X}$ under losses of the form (3.2) simultaneously for $\boldsymbol{d}\in\mathcal{D}$. This includes in particular the corresponding results of Clevenson and Zidek [5] and of Ghosh and Parsian [7].

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