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LOWER BOUND OF RISK IN LINEAR UNBIASED ESTIMATION AND ITS APPLICATION

CZESŁAW STĘPNIAK

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Summary

Lower bound of risk in linear unbiased estimation and its connection with the existence of a uniformly minimum variance linear unbiased estimator is considered.

1. Introduction

In a linear model with a known covariance the lower bound of squared risk for unbiased estimation of a parametric function coincides with the variance of a best linear unbiased estimator. For further results see Rao [5].

In general linear model the lower bound is a function of unknown variance components. Some properties of the function will be established and its connection with the existence of a uniformly minimum variance linear unbiased estimator will be given.

2. Definitions and notations

Let X be a random vector taking values in a finite-dimensional inner product space $\{H, (\cdot, \cdot)\}$, and having a finite second order moment $\mathrm{E}\,\|X\|^2$. Then there exists a vector E_x and a self-adjoint non-negative definite (n.n.d.) operator Σ_x such that $\mathrm{E}\,(h, X)=(h, E_x)$ and $\mathrm{Cov}\,\{(h, X), (h', X)\}=(h, \Sigma_x h')$ for all $h, h' \in H$ (cf. Kruskal [1]).

Our assumptions are:

$$E_x = T\beta \tag{1}$$

and

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$$\Sigma_x = V\gamma \tag{2}$$

where T and V are given linear operators: T from an inner product space $\{K, \langle\cdot, \cdot\rangle\}$ in H and V from an inner product space $\{L, [\cdot, \cdot]\}$ in the space of self-adjoint operators on H, while β and γ are unknown parameters.

It is not assumed that β and γ vary independently of each other and that $V\gamma$ is non-singular. The only assumption is that the span of possible values of β fills the whole space K, while γ is an element of a given set Γ in L. The reasons of the assumptions are similar as in Olsen, Seely and Birkes [4].

Let us consider the sets:

$$R_+(V) = \{V\gamma:\ V\gamma \text{ is n.n.d.},\ \gamma \in L\}$$

and

$$\Gamma(V) = \{\gamma \in L:\ V\gamma \in R_+(V)\}\ .$$

We note that $R_+(V)$ is a closed convex cone in the space of self-adjoint operators on H, provided $R_+(V)$ is non-empty. Thus $\Gamma(V)$ is a closed convex cone in L as an inverse image of $R_+(V)$ by a linear transformation.

It is assumed that the set Γ, i.e. the set of possible values of γ, is a subset of $\Gamma(V)$.

We shall say that a random vector X *is subject to a linear model* $L(T\beta, V\gamma;\ \gamma \in \Gamma)$ if the conditions (1) and (2) are satisfied. Note that any variance components model may be written in such a form.

Throughout this paper, the usual operator notation will be used. Among others, the symbols T^*, $R(T)$ and T^+ will denote, respectively, the adjoint, the range and the Moore-Penrose generalized inverse of the operator T. It will be convenient to denote by V_γ the value of the operator V at the point γ.

3. Lower bound of risk in linear unbiased estimation

Consider an experiment where the observed vector X is subject to a linear model $L(T\beta, V\gamma;\ \gamma \in \Gamma)$. We are interested in estimation of a parameter function $\langle k, \beta\rangle$ by estimators of the form (h, X). Problem of quadratic estimation for variance components may be reduced also to the form by a respective specification (cf. Seely [7]).

It is well known that (h, X) is an unbiased estimator of $\langle k, \beta\rangle$, if and only if, $T^*h = k$. Thus $\langle k, \beta\rangle$ possesses a linear unbiased estimator, if and only if, $k \in R(T^*)$.

A linear unbiased estimator of $\langle k, \beta\rangle$ is said to be *locally best at* γ

if it minimizes $\mathrm{Var}_\gamma(h, X)=(h, V_\gamma h)$ with respect to $h \in H$, under the condition $T^*h=k$. A linear unbiased estimator with minimal variance for all $\gamma \in \Gamma$, providing it exists, is called a *uniformly minimum variance linear unbiased estimator* (UMVLUE).

It is known that a locally best estimator at γ always exists and is of the form (h_0, X), where

$$h_0=W_\gamma^+T(T^*W_\gamma^+T)^+k \tag{3}$$

for $W_\gamma=V_\gamma+TT^*$.

Moreover, the variance of the estimator at the considered point γ is equal to

$$\mathrm{Var}_\gamma(h_0, X)=\langle k, [(T^*W_\gamma^+T)^+-I]k\rangle \tag{4}$$

(cf. Rao [5], p. 300). If $R(T)\subset R(V_\gamma)$ then the operator W_γ in the formula (3) may be replaced by V_γ and (4) reduces to

$$\mathrm{Var}_\gamma(h_0, X)=\langle k, (T^*V_\gamma^+T)^+k\rangle . \tag{4'}$$

The lower bound of risk in linear unbiased estimation of $\langle k, \beta\rangle$ is defined by

$$\pi_k(\gamma)=\inf_{T^*h=k}(h, V_\gamma h)$$

provided that $k \in R(T^*)$. By (4)

$$\pi_k(\gamma)=\langle k, [(T^*W_\gamma^+T)^+-I]k\rangle .$$

Now we present some properties of the lower bound.

THEOREM 1. (a) *$\pi_k(\gamma)$ is a concave functional on $\Gamma(V)$ and homogeneous, in the sense that $\pi_k(c\gamma)=c\pi_k(\gamma)$ for every $c>0$.*
(b) *$\pi_k(\gamma)$ is differentiable (in the sense of Fréchet) in the relative interior of $\Gamma(V)$.*

Remark. For all γ belonging to the relative interior of $\Gamma(V)$ the operators V_γ have the same range (see LaMotte [3], Lemma 2).

PROOF OF THE THEOREM. Concavity follows from the relations:

$$\begin{aligned}\pi_k(c\gamma+[1-c]\gamma')&=\inf_{T^*h=k}(h, V_{c\gamma+(1-c)\gamma'}h)\\ &\geqq c\inf_{T^*h=k}(h, V_\gamma h)+(1-c)\inf_{T^*h=k}(h, V_{\gamma'}h)\\ &=c\pi_k(\gamma)+(1-c)\pi_k(\gamma')\end{aligned}$$

for every $c \in [0, 1]$ and $\gamma, \gamma' \in \Gamma(V)$. The property $\pi_k(c\gamma)=c\pi_k(\gamma)$ is evident.

Differentiability follows from the fact that π_k is a superposition of differentiable operators (linear, and Moore-Penrose generalized inverse on a set of operators having the same range).

Denote by $\nabla\pi_k(\gamma_0)$ the gradient (Fréchet derivative) of π_k at a point γ_0 belonging to the relative interior of $\Gamma(V)$. By Theorem 1(b) and Theorem 25.1 in Rockafellar [6] we obtain

COROLLARY. *An unbiased estimator* (h, X) *of* $\langle k, \beta\rangle$ *is locally best at* γ_0, *if and only if,* $\mathrm{Var}_\gamma(h, X)=\pi_k(\gamma_0)+[\nabla\pi_k(\gamma_0), \gamma-\gamma_0]$.

THEOREM 2. *A UMVLUE of* $\langle k, \beta\rangle$ *exists, if and only if, the functional* π_k *is additive on the minimal convex cone* C *spanned on* Γ.

PROOF. Let (h, X) be a UMVLUE of $\langle k, \beta\rangle$. Then $\mathrm{Var}_\gamma(h, X)=\pi_k(\gamma)$ for every $\gamma\in C$. Thus π_k is additive by additivity of $\mathrm{Var}_\gamma(h, X)$.

On the other hand, if π_k is additive, then, by homogeneity of π_k, there exists a linear functional on L such that π_k is its restriction to C. Thus π_k is differentiable and its derivative does not depend on γ. Therefore any locally best estimator is UMVLUE.

The Theorem is a complement of a well known result of Kruskal [2] concerning the existence of UMVLUE's for all estimable functions in a linear model.

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AGRICULTURAL UNIVERSITY, LUBLIN

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