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A NOTE ON UNIFORM APPROXIMATION TO DISTRIBUTIONS OF EXTREME ORDER STATISTICS

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Summary

The error bound of an approximation to the distribution of the kth largest order statistic as established by Reiss (1981, *Adv. Appl. Prob.*, **13**, 533–547) is improved by making use of an asymptotic expansion of length two in Reiss (the same as above).

1. Introduction

It is well known (see e.g. [1]) that the limit distribution of the kth largest order statistic $Z_{n-k+1:n}$ of a sample of size n is a gamma distribution P_k if the uniform distribution, say, Q on $[0,1]$ is the underlying probability measure. The classical standardization for this limit law is given by $n(Z_{n-k+1:n}-1)$. A result of Ikeda and Matsunawa [2] suggests that this result holds true uniformly over all Borel sets if $k\equiv k(n)$ fulfills the condition $k(n)/n\to 0$ as $n\to\infty$. Reiss [3] also proves asymptotic expansions arranged in powers of k/n. The asymptotic expansions of length 2 is given by

$$(1.1)\qquad \left|Q^n\{n(Z_{n-k+1:n}-1)\in B\}-\int_B [1-((x+k)^2-k)/(2n-3k)]dP_k(x)\right| \leqq C_2\left(\frac{k}{n}\right)^2$$

uniformly over all Borel sets B where C_2 is a universal constant and P_k has the Lebesgue-density p_k given by $p_k(x)=(|x|^{k-1}/(k-1)!)\, e^x$ for $x<0$. It is immediate from (1.1) that for some universal constant $C_1>0$:

$$(1.2)\qquad |Q^n\{n(Z_{n-k+1:n}-1)\in B\}-P_k(B)|\leqq C_1\frac{k}{n}$$

uniformly over all Borel sets B. Moreover, it was proved in [3] that a different standardization leads to a more accurate approximation if

k is large. Combining both results we shall be able to improve the error bound of the second approximation by omitting a term $\exp(-k^{1/2}/4)$. This possibility was already suggested by the numerical computations in [3], p. 542.

2. The result

The main idea of the following approach is to choose constants $a_{n,k}$ and $b_{n,k}$ such that the first two moments of the distribution of $a_{n,k}\cdot(Z_{n-k+1:n}-b_{n,k})$ are accurate approximations to those of the approximating gamma distribution P_k. If $k\geqq(4\log n)^2$ then the following result is an immediate consequence of Theorem 2.9 in [3].

THEOREM 2.1. *There exists a constant $C>0$ such that for every positive integer n and $k\in\{1,\cdots,[n/2]\}$ the following inequality holds true:*

$$\left|Q^n\left\{\left[\frac{n^{3/2}}{(n-k)^{1/2}}\left(Z_{n-k+1:n}-\frac{n-k}{n}\right)-k\right]\in B\right\}-P_k(B)\right|\leqq C\frac{k^{1/2}}{n}$$

uniformly over all Borel sets B.

PROOF. It remains to prove the assertion for $k<(4\log n)^2$. Set $p_{k,2}(x)=1-((x+k)^2-k)/(2n-3k)$. Define $h(x):=(n/(n-k))^{1/2}x+k((n/(n-k))^{1/2}-1)$ for every real number x. The inverse of h is $g(x)=((n-k)/n)^{1/2}x-k(1-((n-k)/n)^{1/2})$.

Since $n^{3/2}(n-k)^{-1/2}(Z_{n-k+1:n}-(n-k)/n)-k=h(n(Z_{n-k+1:n}-1))$ we obtain from (1.1) that

$$\left|Q^n\left\{\left[\frac{n^{3/2}}{(n-k)^{1/2}}\left(Z_{n-k+1:n}-\frac{n-k}{n}\right)-k\right]\in B\right\}-P_k(B)\right|$$
$$\leqq|P_{k,2}\{h\in B\}-P_k(B)|+O((k/n)^2)$$

where $P_{k,2}$ is the approximating measure in (1.1) (with P_k-density $p_{k,2}$). By assumption we have $(k/n)^2\leqq Ck^{1/2}/n$. Moreover $|P_{k,2}\{h\in B\}-P_k(B)|\leqq 1/2\int|((n-k)/n)^{1/2}p_k(g(x))p_{k,2}(g(x))-p_k(x)|dx$.

Notice that $g(x)>0$ if $x\geqq k((n/n-k))^{1/2}-1)$. We split the above integral into two parts, say I_1 and I_2, namely from $-\infty$ to 0 and from 0 to $k((n/(n-k))^{1/2}-1)$. Estimates for I_1 and I_2 will establish the assertion.

To show that $I_2=O(k^{1/2}/n)$ is trivial and can be left to the reader. Ad I_1: We have $((n-k)/n)^{1/2}=1-k/(2n)+O((k/n)^2)$. Hence we can replace in I_1 the expression $((n-k)/n)^{1/2}$ by $1-k/(2n)$, $g(x)$ by $(1-k/(2n))x-k^2/(2n)$ which is equal to $[x-k(x+k)/(2n)]$, and $e^{g(x)}$ and $|g(x)|^{k-1}$ by the two leading terms in their series representations. The omitted terms are of order $O((k/n)^2)$. Thus we obtain for $k \geqq 2$ (the case of $k=1$ is treated in an analogous way):

$$I_1=\int_{-\infty}^{0}\left|\left(1-\frac{k}{2n}\right)\left(|x|^{k-1}+\frac{(k-1)k(x+k)|x|^{k-2}}{2n}\right)\left(1-\frac{k(x+k)}{2n}\right)\right.$$

$$\left.\cdot\left(1-\frac{(x+k)^2-k}{2n-3k}+\frac{k(x+k)^2}{n(2n-3k)}\right)-|x|^{k-1}\right|\frac{e^x}{(k-1)!}dx+O\left(\left(\frac{k}{n}\right)^2\right)$$

$$=\frac{1}{2n(k-1)}\int_{-\infty}^{0}|x(x+k)^2+kx(x+k)+(k-1)k(x+k)|$$

$$\cdot\frac{|x|^{k-2}e^x}{(k-2)!}dx+O\left(\left(\frac{k}{n}\right)^2\right).$$

Thus an application of the Cauchy-Schwarz inequality yields (notice that the terms involving k^4, k^5 and k^6 vanish):

$$I_1 \leqq \frac{(10k^3+14k^2-24k)^{1/2}}{2n(k-1)}+O\left(\left(\frac{k}{n}\right)^2\right)=O\left(\frac{k^{1/2}}{n}\right).$$

The proof is complete.

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