Ann. Inst. Statist. Math.
35 (1983), Part A, 329-341

UNIFORM ASYMPTOTIC JOINT NORMALITY OF A SET OF INCREASING NUMBER OF SAMPLE QUANTILES

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(Received May 18, 1982)

Summary

Type $(B)_d$ asymptotic normality of the joint distribution of sample quantiles is investigated when the number of sample quantiles increases as the sample size increases. This paper aims at a refinement of the original results by Ikeda and Matsunawa [3].

1. Introduction

Let $X_{n1}<X_{n2}<\cdots<X_{nn}$ be order statistics from a continuous distribution with pdf $f(x)$ and cdf $F(x)$. Let $0<\lambda_1<\lambda_2<\cdots<\lambda_k<1$ be any given spacing of k probability levels, and take a corresponding set of k $(<n)$ sample quantiles $X_{n(k)}=(X_{nn_1}, X_{nn_2}, \cdots, X_{nn_k})'$, with $n_i=[(n+1)\lambda_i]$, $i=1, 2, \cdots, k$. Let the corresponding population quantiles be ξ_i $(=F^{-1}(\lambda_i))$, $i=1, 2, \cdots, k$. Mosteller [4] then showed that for fixed λ_i's and k the asymptotic joint distribution of $\sqrt{n}(X_{nn_i}-\xi_i)$, $i=1, 2, \cdots, k$, is k-dimensional normal, in a sense of type $(M)_d$ in our terminology, with zero mean vector and covariance matrix $\Sigma_{(k)}=[\lambda_i(1-\lambda_i)/f(\xi_i)f(\xi_j)]$, $i\leqq j$, provided that $f(x)$ is differentiable in the neighborhoods of ξ_i and that $f(\xi_i)\neq 0$.

Later on, Weiss [5] got a result on asymptotic distribution in a strong sense, of a set of increasing number of sample quantiles, which was extended to a more general situation.

Ikeda and Matsunawa [3] have proved the type $(B)_d$ asymptotic joint normality of increasing number of sample quantiles. In the first step, they proved the result in case of uniform distribution over (0, 1): Let $U_{n1}<U_{n2}<\cdots<U_{nn}$ be order statistics based on a random sample of size n drawn from a uniform distribution over (0, 1). Select k order statistics, $U_{nn_1}<U_{nn_2}<\cdots<U_{nn_k}$, and put $U_{n(k)}=(U_{nn_1}, U_{nn_2}, \cdots, U_{nn_k})'$, where k and $(n_1, n_2, \cdots, n_k)$ may depend on n as $n\to\infty$. By evaluating the K-L information, they proved that under the condition, $k/\min_{1\leqq i\leqq k+1}(n_i-n_{i-1})\to 0$, $(n\to\infty)$, $U_{n(k)}$ and $Z_{n(k)}$ are asymptotically equivalent

in the sense of type $(B)_d$ as $n \to \infty$, where $Z_{n(k)}=(Z_{n1}, Z_{n2}, \cdots, Z_{nk})'$ stands for a normal random variable with mean vector $l_{n(k)}=(l_{n1}, l_{n2}, \cdots, l_{nk})'$, $l_{ni}=n_i/(n+1)$, $i=1, 2, \cdots, k$, and covariance matrix $L_{n(k)}=[l_{ni}(1-l_{nj})/(n+2)]$, $i \leqq j$. Here we have taken a convention $n_0=0$, $n_{k+1}=n+1$.

In the second step, they utilized the above result to get a result in more general situation. Let $X_{n(k)}$ be as before, but in this case k and $(n_1, n_2, \cdots, n_k)$ may depend on n as $n \to \infty$. Also, let $Y_{n(k)}=(Y_{n1}, Y_{n2}, \cdots, Y_{nk})'$ be a normal random variable with mean vector $s_{n(k)}=(s_{n1}, s_{n2}, \cdots, s_{nk})'$, with $s_{ni}=F^{-1}(l_{ni})$, $i=1, 2, \cdots, k$ and covariance matrix $S_{n(k)}=[l_{ni}(1-l_{nj})/(n+2)f_{ni}f_{nj}]$, $i \leqq j$, with $f_{ni}=f(s_{ni})$. Then, they have proved that, under certain conditions, $X_{n(k)}$ and $Y_{n(k)}$ are asymptotically equivalent in the sense of type $(B)_d$ as $n \to \infty$.

However, the conditions still remain unsatisfactory and are not convenient to practical use. Moreover, in contrast with the case of uniform distribution, there exists a stronger condition for the spacing of $(n_1, n_2, \cdots, n_k)$, i.e., $k^2/\min_{1 \leqq i \leqq k+1} (n_i - n_{i-1}) \to 0$, $(n \to \infty)$.

In the present paper, we improve these points and present the refined conditions under which $X_{n(k)} \sim Y_{n(k)}$ $(B)_d$ holds as $n \to \infty$. In the next section, the outline of our proof is stated. In Section 3, we give a new result. A generalization of this result is considered in Section 4.

2. Statement of the outline of the proof

Let $X_{n1}<X_{n2}<\cdots<X_{nn}$ be order statistics based on a random sample of size n drawn from a continuous distribution over the real line, whose pdf and cdf being given by $f(x)$ and $F(x)$, respectively. Choose k, $X_{nn_1}<X_{nn_2}<\cdots<X_{nn_k}$, and put

$$X_{n(k)}=(X_{nn_1}, X_{nn_2}, \cdots, X_{nn_k})' . \tag{2.1}$$

First, we will make the following assumption.

Assumption 3.1. The support of $f(x)$ is identical to the entire real line: $D_f=(-\infty, \infty)$.

Let $U_{n(k)}$ and $Z_{n(k)}$ be the same as in Section 1. Then, the transformed variable

$$F(X_{n(k)}) \equiv (F(X_{nn_1}), F(X_{nn_2}), \cdots, F(X_{nn_k}))' \tag{2.2}$$

is identically distributed with $U_{n(k)}$, or

$$F^{-1}(U_{n(k)}) \equiv (F^{-1}(U_{nn_1}), F^{-1}(U_{nn_2}), \cdots, F^{-1}(U_{nn_k}))' \tag{2.3}$$

is identically distributed with $X_{n(k)}$.

Let us consider a trancation of $Z_{n(k)}$ over the domain $A_{(k)}=\{z_{(k)}|0<z_i<1;\ i=1,2,\cdots,k\}$, and denote it by $Z^*_{n(k)}$. The pdf of $Z^*_{n(k)}$, $p^*_n(z_{(k)})$, say, is given by

$$(2.4)\qquad p^*_n(z_{(k)})=\begin{cases} p_n(z_{(k)})/r_n\,, & \text{if } z_{(k)}\in A_{(k)} \\ 0\,, & \text{otherwise}\end{cases}$$

where we have used the symbol $p_n(z_{(k)})$ as the pdf of $Z_{n(k)}$ and we have put

$$(2.5)\qquad r_n=P^{Z_{n(k)}}(A_{(k)})\,.$$

Then, it is easy to see that $Z_{n(k)}\sim Z^*_{n(k)}$ $(B)_d$ holds as $n\to\infty$. Indeed it is evident that $r_n\to 1$, $(n\to\infty)$, and

$$\begin{aligned} I(Z^*_{n(k)}:\ Z_{n(k)}) &= \int_{A_{(k)}} p^*_n \log(p^*_n/p_n)d\mu_{(k)} \\ &= \int_{A_{(k)}} p^*_n(-\log r_n)d\mu_{(k)} = -\log r_n\to 0\,, \qquad (n\to\infty) \end{aligned}$$

which implies the required result. On the other hand, as we have already mentioned in Section 1, Ikeda and Matsunawa [3] have proved that, under the condition

$$(2.6)\qquad k/\min_{1\le i\le k+1}(n_i-n_{i-1})\to 0\,,\qquad (n\to\infty)\,,$$

it holds that $U_{n(k)}\sim Z_{n(k)}$ $(B)_d$, $(n\to\infty)$. Since the notion of asymptotic equivalence is reflexive and transitive in the sense of any given type, under the condition (2.6) it holds also that $U_{n(k)}\sim Z^*_{n(k)}$ $(B)_d$, $(n\to\infty)$. Further, let us put

$$(2.7)\qquad Y^*_{n(k)}\equiv F^{-1}(Z^*_{n(k)})\equiv(F^{-1}(Z^*_{n1}),\ F^{-1}(Z^*_{n2}),\ \cdots,\ F^{-1}(Z^*_{nk}))'\,.$$

Then, $X_{n(k)}\sim Y^*_{n(k)}$ $(B)_d$ holds as $n\to\infty$, under the same condition.

Finally, let $Y_{n(k)}=(Y_{n1},Y_{n2},\cdots,Y_{nk})'$ be a normal random variable with mean vector

$$(2.8)\qquad s_{n(k)}=(s_{n1},s_{n2},\cdots,s_{nk})'\,,\qquad \text{with } s_{ni}=F^{-1}(l_{ni})\,,\ i=1,2,\cdots,k$$

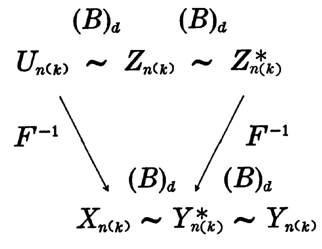
and covariance matrix

$$(2.9)\qquad S_{n(k)}=\frac{1}{n+2}\begin{bmatrix} \dfrac{l_{n1}(1-l_{n1})}{f^2_{n1}} & \dfrac{l_{n1}(1-l_{n2})}{f_{n1}f_{n2}} & \cdots & \dfrac{l_{n1}(1-l_{nk})}{f_{n1}f_{nk}} \\ & \dfrac{l_{n2}(1-l_{n2})}{f^2_{n2}} & \cdots & \dfrac{l_{n2}(1-l_{nk})}{f_{n2}f_{nk}} \\ * & & \ddots & \vdots \end{bmatrix},$$

$$\left[\qquad\qquad\qquad\qquad \frac{l_{nk}(1-l_{nk})}{f_{nk}^2} \right]$$

where we have put $f_{ni}=f(s_{ni})=f(F^{-1}(l_{ni}))$, $i=1, 2, \cdots, k$. Then, if one can show that $Y_{n(k)} \sim Y^*_{n(k)}$ $(B)_d$, $(n \to \infty)$, with possibly some additional conditions, it holds that $X_{n(k)} \sim Y_{n(k)}$ $(B)_d$ as $n \to \infty$. We will investigate this problem in the next section.

The following diagram indicates the relations among the variables introduced so far.



3. Type $(B)_d$ asymptotic equivalence of $Y_{n(k)}$ and $Y^*_{n(k)}$

The pdf's of $Y_{n(k)}$ and $Y^*_{n(k)}$ are given respectively by

$$q_n(y_{(k)})=\left(\frac{1}{\sqrt{2\pi}}\right)^k \frac{1}{|S_{n(k)}|^{1/2}} \exp\left[-\frac{1}{2}(y_{(k)}-s_{n(k)})' S^{-1}_{n(k)}(y_{(k)}-s_{n(k)})\right] \quad (-\infty<y_i<\infty,\ i=1, 2, \cdots, k), \tag{3.1}$$

and

$$\begin{aligned} q^*_n(y_{(k)})=&\frac{1}{r_n}\left(\frac{1}{\sqrt{2\pi}}\right)^k \frac{1}{|L_{n(k)}|^{1/2}} \\ &\times \exp\left[-\frac{1}{2}(F(y_{(k)})-F(s_{n(k)}))' L^{-1}_{n(k)}(F(y_{(k)})-F(s_{n(k)}))\right] \\ &\times \prod_{i=1}^{k} f(y_i), \qquad (-\infty<y_i<\infty,\ i=1, 2, \cdots, k). \end{aligned} \tag{3.2}$$

We will evaluate the K-L information

$$\begin{aligned} I(Y_{n(k)}:\ Y^*_{n(k)}) &\equiv \mathrm{E}\,[\log\,[q_n(Y_{n(k)})/q^*_n(Y_{n(k)})]] \\ &\equiv \int_{R_{(k)}} q_n \log\,(q_n/q^*_n) d\mu_{(k)}\,. \end{aligned} \tag{3.3}$$

From the relation that $|L_{n(k)}|=|S_{n(k)}|\cdot\left\{\prod_{i=1}^{k} f_{ni}\right\}^2$, it is seen that

$$\begin{aligned} \log\,[q_n(y_{(k)})/q^*_n(y_{(k)})] = &\log r_n - \sum_{i=1}^{k} \{\log f(y_i) - \log f_{ni}\} \\ &+\frac{1}{2}\{(F(y_{(k)})-F(s_{n(k)}))' L^{-1}_{n(k)}(F(y_{(k)})-F(s_{n(k)})) \end{aligned} \tag{3.4}$$

$$-(y_{(k)}-s_{n(k)})'S_{n(k)}^{-1}(y_{(k)}-s_{n(k)})\} .$$

Before proceeding to the calculation, we will make the following assumption.

ASSUMPTION 4.1. $f(x)$ is twice differentiable and $f''(x)$ is bounded and continuous over the entire real line.

Then, we get

$$\log f(y_i)-\log f_{ni}=\frac{f'_{ni}}{f_{ni}}(y_i-s_{ni})+\frac{1}{2}R_{ni}^* \tag{3.5}$$

and

$$\begin{aligned}(F(y_{(k)})-F(s_{n(k)}))'L_{n(k)}^{-1}(F(y_{(k)})-F(s_{n(k)}))\\ =(y_{(k)}-s_{n(k)})'S_{n(k)}^{-1}(y_{(k)}-s_{n(k)})+\alpha'_{n(k)}L_{n(k)}^{-1}\beta_{n(k)}\\ +\alpha'_{n(k)}L_{n(k)}^{-1}\gamma_{n(k)}+\frac{1}{4}\beta'_{n(k)}L_{n(k)}^{-1}\beta_{n(k)}+\beta'_{n(k)}L_{n(k)}^{-1}\gamma_{n(k)}+\gamma'_{n(k)}L_{n(k)}^{-1}\gamma_{n(k)}\end{aligned} \tag{3.6}$$

where we have put

$$\begin{aligned}&\alpha_{n(k)}=(\alpha_{n1}, \cdots, \alpha_{nk})' ,\\ &\beta_{n(k)}=(\beta_{n1}, \cdots, \beta_{nk})' ,\\ &\gamma_{n(k)}=(\gamma_{n1}, \cdots, \gamma_{nk})' ,\\ &\alpha_{ni}=f_{ni}(y_i-s_{ni}) , && i=1, 2, \cdots, k ,\\ &\beta_{ni}=f'_{ni}(y_i-s_{ni})^2 , && i=1, 2, \cdots, k ,\\ &\gamma_{ni}=R_{ni} , && i=1, 2, \cdots, k ,\\ &f'_{ni}=f'(s_{ni}) , && i=1, 2, \cdots, k ,\\ &R_{ni}^*=R_{ni}^*(y_i)=\psi(y_i^*)(y_i-s_{ni})^2 , && i=1, 2, \cdots, k ,\\ &R_{ni}=R_{ni}(y_i)=\frac{1}{6}f''(y_i^{**})(y_i-s_{ni})^3 , && i=1, 2, \cdots, k ,\\ &\psi(x)=\{f(x)f''(x)-(f'(x))^2\}/f^2(x) ,\end{aligned} \tag{3.7}$$

and y_i^* and y_i^{**} are some values between y_i and s_{ni}.

Now, we will make the following

ASSUMPTION 4.2. The function, $\psi(x)\equiv\{f(x)f''(x)-(f'(x))^2\}/f^2(x)$, is bounded uniformly for all x in $(-\infty, \infty)$.

Then, from (3.5) it is seen that

(3.8) $$\left|\sum_{i=1}^{k} \mathrm{E}\,[\log f(Y_{ni})-\log f_{ni}]\right| = \frac{1}{2}\left|\sum_{i=1}^{k} \mathrm{E}\,[R_{ni}^{*}]\right| \leqq \frac{M_1}{2(n+2)} \sum_{i=1}^{k} \frac{l_{ni}(1-l_{ni})}{f_{ni}^2},$$

where $M_1 = \sup_{-\infty<x<\infty} |\psi(x)|$.

For notational simplicity, let us put

(3.9) $$\eta_i = 1/(l_{ni+1}-l_{ni}), \qquad i=1, 2, \cdots, k,$$

and hence

(3.10) $$L_{n(k)}^{-1} = (n+2)\begin{bmatrix} \eta_1+\eta_0 & -\eta_1 & & & 0 \\ -\eta_1 & \eta_2+\eta_1 & -\eta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\eta_{k-1} \\ 0 & & & -\eta_{k-1} & \eta_k+\eta_{k-1} \end{bmatrix}.$$

The moments of the RHS of (3.6) will be evaluated as follows. First, since

(3.11) $$\begin{aligned} \alpha_{n(k)}' L_{n(k)}^{-1} \beta_{n(k)} = (n+2)\Big\{ & \sum_{i=1}^{k} (\eta_i+\eta_{i-1}) f_{ni} f_{ni}'(y_i-s_{ni})^3 \\ & -\sum_{i=1}^{k-1} \eta_i f_{ni} f_{ni}'(y_i-s_{ni})(y_{i+1}-s_{ni+1})^2 \\ & -\sum_{i=1}^{k-1} \eta_i f_{ni+1} f_{ni}'(y_{i+1}-s_{ni+1})(y_i-s_{ni})^2 \Big\}, \end{aligned}$$

we have

(3.12) $$\mathrm{E}\,[\alpha_{n(k)}' L_{n(k)}^{-1} \beta_{n(k)}] = 0\,.$$

Second, since

(3.13) $$\begin{aligned} \alpha_{n(k)}' L_{n(k)}^{-1} \gamma_{n(k)} = (n+2)\Big\{ & \sum_{i=1}^{k} (\eta_i+\eta_{i-1}) f_{ni}(y_i-s_{ni}) R_{ni} \\ & -\sum_{i=1}^{k-1} \eta_i f_{ni}(y_i-s_{ni}) R_{ni+1} \\ & -\sum_{i=1}^{k-1} \eta_i f_{ni+1}(y_{i+1}-s_{ni+1}) R_{ni} \Big\}, \end{aligned}$$

and by the Schwarz inequality

(3.14) $$\begin{aligned} |\mathrm{E}\,[(Y_{ni}-s_{ni})R_{ni}]| &\leqq \frac{3M_2}{(n+2)^2}\cdot\sigma_{ni}^4, \\ |\mathrm{E}\,[(Y_{ni}-s_{ni})R_{ni+1}]| &\leqq \frac{\sqrt{15}\,M_2}{6(n+2)^2}\cdot\sigma_{ni}\sigma_{ni+1}^3, \\ |\mathrm{E}\,[(Y_{ni+1}-s_{ni+1})R_{ni}]| &\leqq \frac{\sqrt{15}\,M_2}{6(n+2)^2}\cdot\sigma_{ni+1}\sigma_{ni}^3, \end{aligned}$$

where we have put $M_2=\sup_{-\infty<x<\infty}|f''(x)|$, $\sigma_{ni}^2=l_{ni}(1-l_{ni})/f_{ni}^2$, $i=1, 2, \cdots, k$, it follows from (3.13) that

$$|\mathrm{E}\,[\alpha'_{n(k)}L_{n(k)}^{-1}\gamma_{n(k)}]|\leqq\frac{3M_2}{n+2}\sum_{i=1}^{k}(\eta_i+\eta_{i-1})f_{ni}\sigma_{ni}^2 +\frac{\sqrt{15}\,M_2}{n+2}\sum_{i=1}^{k-1}\eta_i f_{ni}\sigma_{ni}\sigma_{ni+1}^3 +\frac{\sqrt{15}\,M_2}{n+2}\sum_{i=1}^{k-1}\eta_i f_{ni+1}\sigma_{ni}^3\sigma_{ni+1}\,. \tag{3.15}$$

Third, since

$$\frac{1}{4}\beta'_{n(k)}L_{n(k)}^{-1}\beta_{n(k)}=\frac{n+2}{4}\left\{\sum_{i=1}^{k}(\eta_i+\eta_{i-1})(f'_{ni})^2(y_i-s_{ni})^4 -2\sum_{i=1}^{k-1}\eta_i f'_{ni}f'_{ni+1}(y_i-s_{ni})^2(y_{i+1}-s_{ni+1})^2\right\}, \tag{3.16}$$

we have

$$\left|\mathrm{E}\left[\frac{1}{4}\beta'_{n(k)}L_{n(k)}^{-1}\beta_{n(k)}\right]\right|\leqq\left|\frac{n+2}{4}\sum_{i=1}^{k}(\eta_i+\eta_{i-1})(f'_{ni})^2\,\mathrm{E}\,(Y_{ni}-s_{ni})^4\right| \leqq\frac{3}{4(n+2)}\sum_{i=1}^{k}(\eta_i+\eta_{i-1})(f'_{ni})^2\sigma_{ni}^4\,. \tag{3.17}$$

Fourth,

$$\beta'_{n(k)}L_{n(k)}^{-1}\beta_{n(k)}=(n+2)\left\{\sum_{i=1}^{k}(\eta_i+\eta_{i-1})f'_{ni}(y_i-s_{ni})^2R_{ni} -\sum_{i=1}^{k-1}\eta_i f'_{ni}(y_i-s_{ni})^2R_{ni+1} -\sum_{i=1}^{k-1}\eta_i f'_{ni+1}(y_{i+1}-s_{ni+1})^2R_{ni}\right\}. \tag{3.18}$$

Here, we see that

$$|\mathrm{E}\,[(Y_{ni}-s_{ni})^2R_{ni}]|\leqq\frac{\sqrt{105}\,M_2}{2(n+2)^{5/2}}\cdot\sigma_{ni}^5\,, \tag{3.19}$$

$$|\mathrm{E}\,[(Y_{ni}-s_{ni})^2R_{ni+1}]|\leqq\frac{\sqrt{5}\,M_2}{2(n+2)^{5/2}}\cdot\sigma_{ni}^2\sigma_{ni+1}^3\,, \tag{3.20}$$

and

$$|\mathrm{E}\,[(Y_{ni+1}-s_{ni+1})^2R_{ni}]|\leqq\frac{\sqrt{5}\,M_2}{2(n+2)^{5/2}}\cdot\sigma_{ni+1}^2\sigma_{ni}^3\,. \tag{3.21}$$

Hence we get

$$(3.22)\qquad |\mathrm{E}\,[\beta'_{n(k)}L_{n(k)}\gamma_{n(k)}]|\leqq\frac{\sqrt{105}\,M_2}{2(n+2)^{3/2}}\sum_{i=1}^{k}(\eta_i+\eta_{i-1})|f'_{ni}|\sigma_{ni}^5 +\frac{\sqrt{5}\,M_2}{2(n+2)^{3/2}}\sum_{i=1}^{k-1}\eta_i|f'_{ni}|\sigma_{ni}^2\sigma_{ni+1}^3 +\frac{\sqrt{5}\,M_2}{2(n+2)^{3/2}}\sum_{i=1}^{k-1}\eta_i|f'_{ni+1}|\sigma_{ni+1}^2\sigma_{ni}^3 .$$

Finally,

$$(3.23)\qquad \gamma'_{n(k)}L^{-1}_{n(k)}\gamma_{n(k)}=(n+2)\left\{\sum_{i=1}^{k}(\eta_i+\eta_{i-1})R_{ni}^2-2\sum_{i=1}^{k-1}\eta_iR_{ni}R_{ni+1}\right\}.$$

Here

$$(3.24)\qquad \mathrm{E}\,[R_{ni}^{*}]\leqq\frac{5M_2^2}{12(n+2)^3}\cdot\sigma_{ni}^3 ,$$

$$(3.25)\qquad \mathrm{E}\,[R_{ni}R_{ni+1}]\leqq\frac{5M_2^2}{12(n+2)^3}\cdot\sigma_{ni}^3\sigma_{ni+1}^3 .$$

Hence

$$(3.26)\qquad |\mathrm{E}\,[\gamma'_{n(k)}L^{-1}_{n(k)}\gamma_{n(k)}]|\leqq\frac{5M_2^2}{12(n+2)^2}\left\{\sum_{i=1}^{k}(\eta_i+\eta_{i-1})\sigma_{ni}^3+2\sum_{i=1}^{k-1}\eta_i\sigma_{ni}^3\sigma_{ni+1}^3\right\}.$$

Thus, summarizing the results so far obtained, it is seen that

$$(3.27)\qquad I(Y_{n(k)}:\,Y^{*}_{n(k)})$$
$$\leqq\log r_n+\frac{M_1}{2(n+2)}\sum_{i=1}^{k}\sigma_{ni}^2+\frac{3M_2}{n+2}\sum_{i=1}^{k}(\eta_i+\eta_{i-1})f_{ni}\sigma_{ni}^4$$
$$+\frac{\sqrt{15}\,M_2}{n+2}\sum_{i=1}^{k-1}\eta_if_{ni}\sigma_{ni}\sigma_{ni+1}^3+\frac{\sqrt{15}\,M_2}{n+2}\sum_{i=1}^{k-1}\eta_if_{ni+1}\sigma_{ni}^3\sigma_{ni+1}$$
$$+\frac{3}{4(n+2)}\sum_{i=1}^{k}(\eta_i+\eta_{i-1})|f'_{ni}|^2\sigma_{ni}^4+\frac{\sqrt{105}\,M_2}{2(n+2)^{3/2}}\sum_{i=1}^{k}(\eta_i+\eta_{i-1})|f'_{ni}|\sigma_{ni}^5$$
$$+\frac{\sqrt{5}\,M_2}{2(n+2)^{3/2}}\sum_{i=1}^{k-1}\eta_i|f'_{ni}|\sigma_{ni}^2\sigma_{ni+1}^3+\frac{\sqrt{5}\,M_2}{2(n+2)^{3/2}}\sum_{i=1}^{k-1}\eta_i|f_{ni+1}|\sigma_{ni}^3\sigma_{ni+1}^2$$
$$+\frac{5M_2^2}{12(n+2)^2}\left\{\sum_{i=1}^{k}(\eta_i+\eta_{i-1})\sigma_{ni}^6+2\sum_{i=1}^{k-1}\eta_i\sigma_{ni}^3\sigma_{ni+1}^3\right\}.$$

Since $\eta_i=1/(l_{ni+1}-l_{ni})$ and $l_{ni}=n_i/(n+1)$ for each i, it holds that

$$(3.28)\qquad \frac{1}{n+2}\sum_{i=0}^{k}\eta_i=\frac{n+1}{n+2}\sum_{i=0}^{k}\frac{1}{n_{i+1}-n_i}\leqq\left(1-\frac{1}{n+2}\right)\sum_{i=0}^{k}\frac{1}{n_{i+1}-n_i}\leqq\sum_{i=0}^{k}\frac{1}{n_{i+1}-n_i}.$$

Let us put $w_n=\sum_{i=0}^{k}1/(n_{i+1}-n_i)$, $\sigma_n^2=\max_{1\leqq i\leqq k}\sigma_{ni}^2$, $M_3=\sup_{-\infty<x<\infty}f(x)$ and $M_4=$

$\sup_{-\infty<x<\infty}|f'(x)|$, it then follows that

$$(3.29)\qquad I(Y_{n(k)}:Y^*_{n(k)})\leqq \log r_n+\frac{k\sigma_n^2M_1}{2(n+2)}+w_n\sigma_n^4\left\{2(3+\sqrt{15})M_2M_3+\frac{3}{2}M_4^2\right\}$$
$$+w_n\frac{\sigma_n^5}{\sqrt{n+2}}(\sqrt{105}+\sqrt{5})M_2M_4+\frac{5w_n\sigma_n^6M_2^2}{3(n+2)}\,.$$

Thus we get the following theorem.

THEOREM 3.1. *Suppose that the assumptions* 2.1, 3.1 *and* 3.2 *are fulfilled. Then, in order that* $X_{n(k)}\sim Y_{n(k)}\ (B)_d$, $(n\to\infty)$, *it is sufficient that the following conditions are satisfied simultaneously:*

$$(3.30)\qquad w_n\equiv\sum_{i=0}^{k}\frac{1}{n_{i+1}-n_i}\to 0\,,\qquad (n\to\infty)\,,$$

and

$$(3.31)\qquad w_n\cdot\sigma_n^4\to 0\,,\qquad (n\to\infty)\,,$$

where we have put

$$\sigma_n^2=\max_{1\leqq i\leqq k}\sigma_{ni}^2\,,\quad \sigma_{ni}^2=l_{ni}(1-l_{ni})/f_{ni}^2\,,\quad i=1,2,\cdots,k\,.$$

It should be noted that the assumptions 2.1 and 3.1 in the theorem are satisfied for a wider class of probability distributions including normal, Cauchy and Laplace: $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, $\pi^{-1}/(1+x^2)$, $\frac{1}{2}e^{-|x|}$.

4. Extension of Theorem 3.1

Now we generalize the result of Theorem 3.1 to the case where D_f, the support of $f(x)$, is not necessarily the entire real line.

ASSUMPTION 4.1. The support of $f(x)$ is identical to an open interval: $D_f=(a,b)$, where a and b are extended real.

Let $U_{n(k)}$, $Z_{n(k)}$, $X_{n(k)}$, $Z^*_{n(k)}$, $Y^*_{n(k)}$ and $Y_{n(k)}$ be the same as before. Note that the range of $Y^*_{n(k)}$ is now restricted to the k-dimensional open cube: $C_{(k)}=(a,b)^k$.

Let $\bar{Y}_{n(k)}$ be the truncation of $Y_{n(k)}$ over the set $C_{(k)}$, and define

$$(4.1)\qquad \rho_n\equiv\mathrm{P}\{Y_{n(k)}\in C_{(k)}\}\,.$$

Then, the pdf of $\bar{Y}_{n(k)}$ is given by

$$(4.2)\qquad \bar{q}_n=\begin{cases} q_n/\rho_n\,, & \text{on } C_{(k)}\\ 0\,, & \text{elsewhere}\end{cases}$$

if $\rho_n \to 1$, $(n \to \infty)$, then it holds that $Y_{n(k)} \sim \bar{Y}_{n(k)}$ $(B)_d$, $(n \to \infty)$. Hence, we will first derive a condition which satisfies this requirement.

By using the Chebychev inequality, we get

$$(4.3) \qquad \begin{aligned} 1-\rho_n &= \mathrm{P}\{Y_{n(k)} \notin C_{(k)}\} \\ &\leqq k \cdot \max_{1<i<k} \mathrm{P}\{|Y_{ni}-s_{ni}| \geqq \min(|a-s_{ni}|, |b-s_{ni}|)\} \\ &\leqq \frac{k}{n+2} \cdot \max_{1 \leqq i \leqq k} \frac{\sigma_{ni}^2}{\min(|a-s_{ni}|^2, |b-s_{ni}|^2)}, \end{aligned}$$

the vanishing of which imples our requirement.

We will next find the conditions under which $\bar{Y}_{n(k)} \sim Y^*_{n(k)}$ $(B)_d$, $(n \to \infty)$. For this, we evaluate the K-L information

$$(4.4) \qquad \begin{aligned} I(\bar{Y}_{n(k)}: Y^*_{n(k)}) &= \int_{R_{(k)}} \bar{q}_n \log(\bar{q}_n/q_n^*) d\mu_{(k)} \\ &= \frac{1}{\rho_n} \int_{C_{(k)}} q_n \log(q_n/q_n^*) d\mu_{(k)} - \log \rho_n, \end{aligned}$$

the vanishing of which implies the required result. But, since $\rho_n \to 1$, $(n \to \infty)$, under the condition that the last member of (5.3) tends to zero as $n \to \infty$, this is equivalent to the condition

$$(4.5) \qquad I^*(\bar{Y}_{n(k)}: Y^*_{n(k)}) = \int_{C_{(k)}} q_n \log(q_n/q_n^*) d\mu_{(k)} \to 0, \qquad (n \to \infty).$$

Parallel with the previous section, we will make the following assumptions.

ASSUMPTION 4.2. $f(x)$ is twice differentiable and $f''(x)$ is bounded and continuous over (a, b).

ASSUMPTION 4.3. The function, $\varphi(x) \equiv \{f(x)f''(x)-(f'(x))^2\}/f^2(x)$, is bounded uniformly for all x in (a, b).

Let us designate the integral operator $\int_{C_{(k)}} \cdot q_n d\mu_{(k)}$ by $\mathrm{E}^*[\cdot]$. Then, one can see that the calculation goes similarly to that of deriving the preceding theorem. In fact, since $\mathrm{E}^*[\phi(y_{(k)})] \leqq \mathrm{E}[\phi(y_{(k)})]$ provided $\phi(y_{(k)}) \geqq 0$, this is much the same as in the previous section, except for the case $\mathrm{E}^*[\alpha'_{n(k)} L^{-1}_{n(k)} \beta_{n(k)}]$.

In that case, from (3.11) we have

$$(4.6) \qquad \begin{aligned} &|\mathrm{E}[\alpha'_{n(k)} L^{-1}_{n(k)} \beta_{n(k)}] - \mathrm{E}^*[\alpha'_{n(k)} L^{-1}_{n(k)} \beta_{n(k)}]| \\ &\leqq (n+2) M_2 M_3 \Big\{ \sum_{i=1}^{k} (\eta_i + \eta_{i-1}) \int_{y_i \notin (a,b)} |y_i - s_{ni}|^3 dP^{Y_{ni}} \\ &\quad + \sum_{i=1}^{k-1} \eta_i \int_{(y_i, y_{i+1}) \notin (a,b)^2} |y_i - s_{ni}| |y_{i+1} - s_{ni+1}|^2 dP^{(Y_{ni}, Y_{in+1})} \end{aligned}$$

$$+\sum_{i=1}^{k-1}\eta_i\int_{(y_i,y_{i+1})\notin(a,b)^2}|y_i-s_{ni}|^2|y_{i+1}-s_{ni+1}|dP^{(Y_{ni},Y_{ni+1})}\Big\}.$$

Here, by Schwarz's inequality

$$\int_{y_i\notin(a,b)}|y_i-s_{ni}|^3dP^{Y_{ni}}\leqq\left(\int_{|y_i-s_{ni}|\geqq\zeta_{ni}}dP^{Y_{ni}}\right)^{1/2}\left(\int_{|y_i-s_{ni}|\geqq\zeta_{ni}}(y_i-s_{ni})^6dP^{Y_{ni}}\right)^{1/2},$$

and by Chebychev's inequality

$$\int_{|y_i-s_{ni}|\geqq\zeta_{ni}}dP^{Y_{ni}}\leqq\frac{\sigma_{ni}^2}{(n+2)\zeta_{ni}^2},$$

where $\zeta_{ni}=\min(|a-s_{ni}|,\ |b-s_{ni}|)$. Also

$$\int_{|y_i-s_{ni}|\geqq\zeta_{ni}}(y_i-s_{ni})^6dP^{Y_{ni}}\leqq\int_{-\infty<y_i<\infty}(y_i-s_{ni})^6dP^{Y_{ni}}=\frac{15\sigma_{ni}^6}{(n+2)^3},$$

hence

$$\int_{y_i\notin(a,b)}|y_i-s_{ni}|^3dP^{Y_{ni}}\leqq\frac{\sqrt{15}\,\sigma_{ni}^4}{(n+2)^2\zeta_{ni}}. \tag{4.7}$$

Also, we have

$$\begin{aligned}(4.8)\quad &\int_{(y_i,y_{i+1})\notin(a,b)^2}|y_i-s_{ni}||y_{i+1}-s_{ni+1}|^2dP^{(Y_{ni},Y_{ni+1})}\\ &\leqq\left(\int_{(y_i,y_{i+1})\notin(a,b)^2}dP^{(Y_{ni},Y_{ni+1})}\right)^{1/2}\\ &\quad\times\left(\int_{(y_i,y_{i+1})\notin(a,b)^2}|y_i-s_{ni}|^2|y_{i+1}-s_{ni+1}|^4dP^{(Y_{ni},Y_{ni+1})}\right)^{1|2}\\ &\leqq\left(\frac{\sigma_{ni}^2}{(n+2)\zeta_{ni}^2}\right)^{1/2}c_i\left(\frac{\sigma_{ni}^2\sigma_{ni+1}^4}{(n+2)^3}\right)^{1/2}=c_i\frac{\sigma_{ni}^2\sigma_{ni+1}^2}{(n+2)^2\zeta_{ni}},\end{aligned}$$

and similarly

$$\int_{(y_i,y_{i+1})\notin(a,b)^2}|y_i-s_{ni}|^2|y_{i+1}-s_{ni+1}|dP^{(Y_{ni},Y_{ni+1})}\leqq c_i'\frac{\sigma_{ni}^2\sigma_{ni+1}^2}{(n+2)^2\zeta_{ni}}. \tag{4.9}$$

From (3.12), (4.6), (4.7), (4.8) and (4.9) it follows that

$$|\mathrm{E}^*[\alpha'_{n(k)}L^{-1}_{n(k)}\beta_{n(k)}]|\leqq c\cdot w_n\cdot\max_{1<i<k}\left\{\frac{\sigma_{ni}^4}{\min(|a-s_{ni}|,\ |b-s_{ni}|)}\right\}, \tag{4.10}$$

where c is a constant independent of n.

Summarizing the results thus obtained, we can state the following

THEOREM 4.1. *Under the assumptions* 4.1, 4.2 *and* 4.3, *in order that* $X_{n(k)}\sim Y_{n(k)}\ (B)_d$, $(n\to\infty)$, *it is sufficient that the following conditions are satisfied simultaneously:*

(4.11) $$w_n \equiv \sum_{i=0}^{k} \frac{1}{n_{i+1}-n_i} \to 0\,, \qquad (n \to \infty)\,,$$

(4.12) $$w_n \cdot \sigma_n^4 \to 0\,, \qquad (n \to \infty)\,,$$

and

(4.13) $$w_n \cdot \max_{1 \leq i \leq k} \frac{\sigma_{ni}^4}{\zeta_{ni}^2} \to 0\,, \qquad (n \to \infty)\,.$$

where $\zeta_{ni}=\min(|a-s_{ni}|,\ |b-s_{ni}|)$.

In the case of uniform distribution, this theorem gives an equivalent to Theorem 3.1 as will be seen below.

Suppose the basic distribution be uniform, $U(a, b)$. Without loss of generality, one can assume that $a=0$ and $b=1$. Then, since $\sigma_{ni}^2= l_{ni}(1-l_{ni})$ is less than or equal to unity, the condition (4.12) is a consequence of the condition (3.11). Also, since $s_{ni}=l_{ni}$, we have

$$\frac{\sigma_{ni}^4}{\zeta_{ni}^2}=\frac{l_{ni}^2(1-l_{ni}^2)}{\min(l_{ni}^2,\ (1-l_{ni})^2)}=\max(l_{ni}^2,\ (1-l_{ni})^2) \leq 1\,.$$

Therefore, (4.11) implies (4.13). $f(x)=1$ $(0 \leq x \leq 1)$, 0 (otherwise), satisfies the assumptions of the theorem. Thus, the sole condition (4.11) implies the asymptotic $(B)_d$ normality of $X_{n(k)}$ as $n \to \infty$, which is nothing but the result of Ikeda and Matsunawa [3].

If $a \to -\infty$ and $b \to +\infty$, then $\zeta_{ni}^2 \to \infty$, whatsoever the spacing s_{ni}'s should be. Hence the condition (4.13) of the above theorem becomes trivial, which shows that Theorem 4.1 is a generalization of Theorem 3.1.

The followings are the immediate consequences from Theorem 4.1.

COROLLARY 4.1. *Suppose that* $0<M \leq f(x)$ *for some* M, *and* $f'(x)$, $f''(x)$ *and* $\phi(x)$ *are uniformly bounded over a finite interval* $D_f=(a, b)$. *Then the condition* (4.11) *implies the asymptotic* $(B)_d$ *normality of* $X_{n(k)}$ *as* $n \to \infty$.

COROLLARY 4.2. *Suppose that* $0<M \leq f(x) \leq M'$ *for some* M *and* M', *and* $f'(x)$ *and* $f''(x)$ *are uniformly bounded over a finite* $D_f=(a, b)$. *Then the sole condition* (4.11) *implies the asymptotic* $(B)_d$ *normality of* $X_{n(k)}$ *as* $n \to \infty$.

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