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JOINT MOMENTS OF THE NUMBER OF + RUNS AND THE NUMBER OF + SIGNS IN A RANDOM SEQUENCE

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Summary

In this paper we present recurrence relations for computing joint moments of the number of + runs and the number of + signs in a random sequence, and we give the results of computation of the moments of the order $\leqq 6$ using the recurrence relations and the idea of initial turning points.

1. Introduction

Let $x=(x_1,\cdots,x_N)$ be a random sequence of size N drawn from a univariate continuous distribution. Let T be the sequence of signs (+ or −) of the differences $x_{i+1}-x_i$ $(i=1,\cdots,N-1)$. In T a sequence of successive + signs not immediately preceded or followed by a + sign is called a + run or a run up. Let r be the number of + runs in T, and s be the number of + signs in T.

Moore and Wallis [4] and Mann [3] obtained the moments of s of the order $\leqq 6$. Levene and Wolfowitz [1] and Levene [2] obtained, among others, the joint moments of r and s of the order $\leqq 2$. Their method is based on the following notion of initial turning points.

Define y_i $(i=1,\cdots,N-1)$ by

$$y_i=\begin{cases} 1, & \text{if } x_{i-1}>x_i<x_{i+1}, \\ 0, & \text{otherwise}, \end{cases}$$

putting $x_0=+\infty$, and we call i an initial turning point of the sequence $x=(x_1,\cdots,x_N)$, if $y_i=1$. Similarly define z_i $(i=1,\cdots,N-1)$ by

$$z_i=\begin{cases} 1, & \text{if } x_i<x_{i+1}, \\ 0, & \text{otherwise}; \end{cases}$$

then we have

(1) $$r=\sum_{i=1}^{N-1} y_i \quad \text{and} \quad s=\sum_{i=1}^{N-1} z_i .$$

Higher moments of r and s are needed for some situations, for example, in evaluating asymptotic expansions for the distribution of r and s. But the evaluation of higher moments becomes more and more laborious as the order grows up, if we employ only the relations (1).

In Section 2 we give recurrence relations for facilitating the computation of the numerical values of the joint moments, and present, as an example of using them, another proof of the expression for the mean of r given in Levene [2]. In Section 3 we give the results of computation.

2. Recurrence relations

For a random sequence $x=(x_1,\cdots,x_N)$, denote by $\nu_{ab}^{(N)}$ the joint moment of r and s about the origin:

$$\nu_{ab}^{(N)}=\mathrm{E}(r^a s^b) ,$$

and by $\mu_{ab}^{(N)}$ the joint moment of r and s about the mean:

$$\mu_{ab}^{(N)}=\mathrm{E}[(r-\mathrm{E}(r))^a(s-\mathrm{E}(s))^b] .$$

Since x is a random sequence from a continuous distribution, both $\nu_{ab}^{(N)}$ and $\mu_{ab}^{(N)}$ have the same values, if we calculate them based on the random permutation $p=(p_1,\cdots,p_N)$ of the set of N integers $\{1,\cdots,N\}$, provided that every permutation is assigned the same probability $1/N!$.

In order to obtain the recurrence relations, we classify all the permutations $p=(p_1,\cdots,p_N)$ into subsets. First, we get N subsets A_i $(i=1,\cdots,N)$, where A_i consists of the permutations with $p_1=i$. In the second step we subdivide each A_i into $N-1$ subsets A_{ij} $(j=1,\cdots,N-1)$. A permutation $p=(p_1,\cdots,p_N)$ in A_i is subclassified into A_{ij}, if p_2 is the jth smallest integer in the set $\{1,\cdots,N\}-\{p_1\}$. In the third step we subdivide each A_{ij} in a similar way into $N-2$ subsets A_{ijk} $(k=1,\cdots,N-2)$. A permutation $p=(p_1,\cdots,p_N)$ in A_{ij} is subclassified into A_{ijk}, if p_3 is the kth smallest integer in the set $\{1,\cdots,N\}-\{p_1,p_2\}$.

Then, it is clear for $p=(p_1,\cdots,p_N)\in A_{ijk}$ that

(2) $$p_1<p_2 , \quad \text{if and only if } i\leqq j ,$$

and

(3) $$p_2<p_3 , \quad \text{if and only if } j\leqq k .$$

To obtain $\nu_{ab}^{(N)}$ we must find the sum of $r^a s^b$ over all $N!$ permutations. If the domain of summation is restricted to A_i or A_{ij} $(i=1,\cdots,$

N; $j=1,\cdots,N-1$), the resulting sum is denoted by $M_{ab}^{(N)}(i)$ or $M_{ab}^{(N)}(i, j)$ respectively. Then we have

$$\nu_{ab}^{(N)}=\frac{1}{N!}\sum_{i=1}^{N} M_{ab}^{(N)}(i)=\frac{1}{N!}\sum_{i=1}^{N}\sum_{j=1}^{N-1} M_{ab}^{(N)}(i, j) . \tag{4}$$

Example 1. In the case $N=3$ we have:

$$A_1=A_{11}\cup A_{12} , \quad A_{11}=\{(1, 2, 3)\},\ A_{12}=\{(1, 3, 2)\} ,$$
$$A_2=A_{21}\cup A_{22} , \quad A_{21}=\{(2, 1, 3)\},\ A_{22}=\{(2, 3, 1)\} ,$$
$$A_3=A_{31}\cup A_{32} , \quad A_{31}=\{(3, 1, 2)\},\ A_{32}=\{(3, 2, 1)\} ;$$

from which we have for $a=b=0$

$$M_{00}^{(3)}(i, j)=1 , \qquad \text{for all } (i, j) \text{ with } 1\leqq i\leqq 3 \text{ and } 1\leqq j\leqq 2 ,$$

while for $a+b>0$ we have

$$M_{ab}^{(3)}(1, 1)=2^b ,$$
$$M_{ab}^{(3)}(i, j)=1 , \qquad \text{if } (i, j)=(1, 2),\ (2, 1),\ (2, 2) \text{ or } (3, 1) ,$$
$$M_{ab}^{(3)}(3, 2)=0 .$$

PROPOSITION. Let $N\geqq 3$, then we have recurrence relations for M's:

$$M_{ab}^{(N)}(i, j)=\sum_{k=j}^{N-2}\sum_{d=0}^{b}\binom{b}{d}M_{ad}^{(N-1)}(j, k) + \sum_{k=1}^{j-1}\sum_{c=0}^{a}\sum_{d=0}^{b}\binom{a}{c}\binom{b}{d}M_{cd}^{(N-1)}(j, k) , \qquad \text{if } i\leqq j , \tag{5}$$

$$M_{ab}^{(N)}(i, j)=M_{ab}^{(N-1)}(j)=\sum_{k=1}^{N-2} M_{ab}^{(N-1)}(j, k) , \qquad \text{if } i>j , \tag{6}$$

where $\sum_{k=\alpha}^{\beta}$ is understood to be zero when $\alpha>\beta$.

PROOF. Let $p=(p_1,\cdots,p_N)$ be a permutation of $\{1,\cdots,N\}$ such that $p\in A_{ijk}\subset A_{ij}\subset A_i$ with $1\leqq i\leqq N$, $1\leqq j\leqq N-1$, $1\leqq k\leqq N-2$. Let r and s be the number of + runs and + signs in p, while let r' and s' denote the number of + runs and + signs in the permutation $(p_2,\cdots, p_N)$. Then we have from (2) and (3) that

$$r=\begin{cases} r'+1 , & \text{if } i\leqq j>k , \\ r' , & \text{if } i>j \text{ or } j\leqq k , \end{cases}$$

$$s=\begin{cases} s'+1 , & \text{if } i\leqq j , \\ s' , & \text{if } i>j , \end{cases}$$

from which the recurrence relations (5) and (6) are clear.

We can compute $\nu_{ab}^{(N)}$ numerically for any N, a and b, using the recurrence relations (5) and (6) and the initial conditions given in Example 1.

We can also give another proof of the formulae for $\nu_{01}^{(N)}$, $\nu_{10}^{(N)}$, $\mu_{02}^{(N)}$, $\mu_{11}^{(N)}$ and $\mu_{20}^{(N)}$ given in the literatures mentioned in Section 1. We illustrate in the following example only the proof of the formula for $\nu_{10}^{(N)}$, the proof of others being omitted to save the space. Note that the value of $\mu_{11}^{(N)}$ given in Levene [2] is in error, the correct value being 1/6 instead of 1/3.

Example 2. To find a general formula for $\nu_{10}^{(N)}$, we write down the recurrence relations (5) and (6) for $a=1$ and $b=0$:

$$\begin{aligned} M_{10}^{(N)}(i, j) &= \sum_{k=1}^{N-2} M_{10}^{(N-1)}(j, k) + \sum_{k=1}^{j-1} M_{00}^{(N-1)}(j, k) \\ &= M_{10}^{(N-1)}(j) + (j-1)\cdot(N-3)!\,, \qquad \text{if } i \leqq j\,, \\ M_{10}^{(N)}(i, j) &= M_{10}^{(N-1)}(j)\,, \qquad \text{if } i > j\,, \end{aligned}$$

from which it follows that

$$\nu_{10}^{(N)} = \frac{1}{N!} N \sum_{j=1}^{N-1} M_{10}^{(N-1)}(j) + \frac{(N-3)!}{N!} \sum_{i=1}^{N} \sum_{j=i}^{N-1} (j-1) = \nu_{10}^{(N-1)} + \frac{1}{3}$$

for $N \geqq 3$. Since $\nu_{10}^{(2)}=1/2$ and $\nu_{10}^{(3)}=5/6$, we have for $N \geqq 2$

$$\nu_{10}^{(N)} = \frac{2N-1}{6}\,.$$

3. Results of computation

We have computed $\mu_{ab}^{(N)}$ for $a+b \leqq 6$ and $N \leqq 18$, by using (4) and the recurrence relations (5) and (6). Combining these numerical values with the idea of initial turning points, we get a polynomial expression of $\mu_{ab}^{(N)}$ in N for $N \geqq N(a, b)$, where $N(a, b)$ is a positive integer dependent on a and b.

Since r and s are expressed as in (1), we have

$$\mu_{ab}^{(N)} = \sum_{i_1=1}^{N-1} \cdots \sum_{i_a=1}^{N-1} \sum_{j_1=1}^{N-1} \cdots \sum_{j_b=1}^{N-1} \mathrm{E}\,(\tilde{y}_{i_1} \cdots \tilde{y}_{i_a} \tilde{z}_{j_1} \cdots \tilde{z}_{j_b})\,, \tag{7}$$

putting $\tilde{y}_i = y_i - \mathrm{E}\,(y_i)$ $(i=1,\cdots,N-1)$ and $\tilde{z}_j = z_j - \mathrm{E}\,(z_j)$ $(j=1,\cdots,N-1)$. As (y_i, z_i) $(i=1,\cdots,N-1)$ is independent of the set of (y_j, z_j)'s with $|j-i| \geqq 3$, the expectation in the right hand side of (7) is equal to zero, if one of the indices $i_1,\cdots,i_a$, $j_1,\cdots,j_b$ differs from all others by 3 or more. Hence, (7) can be expressed as a polynomial in N of degree $\leqq[(a+b)/2]$ for N greater than or equal to some integer $N(a, b)$ dependent on a and b. The coefficients of the polynomial can be found from the $[(a+b)/2]+1$ successive values of $\mu_{ab}^{(N)}$ with $N\geqq N(a, b)$. An upper bound of $N(a, b)$ is given by $2a+b$, as illustrated in the following example.

Example 3. We shall evaluate $\mu_{22}^{(N)}$ as an example. It is expressed by (7) as

$$\mu_{22}^{(N)}=\sum_{i_1=1}^{N-1}\sum_{i_2=1}^{N-1}\sum_{j_1=1}^{N-1}\sum_{j_2=1}^{N-1}\mathrm{E}(\tilde{y}_{i_1}\tilde{y}_{i_2}\tilde{z}_{j_1}\tilde{z}_{j_2})\ . \tag{8}$$

There appear many types of expectations in (8). In the following consideration we can delete from the summation (8) the expectations for which one of $\tilde{y}_{i_1}, \tilde{y}_{i_2}, \tilde{z}_{j_1}, \tilde{z}_{j_2}$, is independent from the other three. The expectations of the remaining type appear in (8) either c_0, $c_0(N-N_0)$ or $(c_0/2)(N-N_0)(N-N_0-1)$ times for N such that $N\geqq N_0$, where c_0 and N_0 are positive integers not dependent on N but on the type of the expectation. Let N^* be the maximum among all N_0's, then $\mu_{22}^{(N)}$ can be expressed as a polynomial in N of the degree $\leqq 2$ for $N\geqq N^*$; hence we have $N(2, 2)\leqq N^*$.

Among the expectations in the right hand side of (8) the following ones give the maximum number of N_0:

$$\begin{aligned}
&\mathrm{E}(\tilde{y}_i\tilde{y}_{i+2}\tilde{z}_j\tilde{z}_{j+1}) && (2\leqq i,\ i+4\leqq j\leqq N-2)\,,\\
&\mathrm{E}(\tilde{y}_i\tilde{z}_{i+1}\tilde{y}_j\tilde{z}_{j+1}) && (2\leqq i,\ i+4\leqq j\leqq N-2)\,,\\
&\mathrm{E}(\tilde{y}_i\tilde{z}_{i+1}\tilde{z}_j\tilde{y}_{j+2}) && (2\leqq i,\ i+3\leqq j\leqq N-3)\,,\\
&\mathrm{E}(\tilde{z}_i\tilde{y}_{i+2}\tilde{y}_j\tilde{z}_{j+1}) && (1\leqq i,\ i+5\leqq j\leqq N-2)\,,\\
&\mathrm{E}(\tilde{z}_i\tilde{y}_{i+2}\tilde{z}_j\tilde{y}_{j+2}) && (1\leqq i,\ i+4\leqq j\leqq N-3)\,,\\
&\mathrm{E}(\tilde{z}_i\tilde{z}_{i+1}\tilde{y}_j\tilde{y}_{j+2}) && (1\leqq i,\ i+4\leqq j\leqq N-3)\,.
\end{aligned}$$

Note that the marginal distribution of y_1 is different from those of $y_2, \cdots, y_{N-1}$, while $z_1, \cdots, z_{N-1}$ have the same marginal distributions. Furthermore, note that (y_i, z_{i+2}) and (z_i, z_{i+2}) are both independent pairs, but (y_i, y_{i+2}) and (z_i, y_{i+2}) are both dependent pairs for $i=1, 2, \cdots, N-3$.

All the above expectations appear in (8) at least once for $N\geqq 8$, and $(1/2)(N-6)(N-7)$ times for $N\geqq N_0=6$. Other types of expectations in (8) have no greater values of N_0; thus we have $N(2, 2)\leqq N^*=6$.

From the values of $\mu_{22}^{(6)}=11/45$, $\mu_{22}^{(7)}=569/1890$ and $\mu_{22}^{(8)}=23/63$, computed by using the method described in Section 2, we have the quadratic expression $\mu_{22}^{(N)}=(7N^2+16N+114)/1890$ for $N\geqq 6$.

In the following we list polynomial expressions of $\nu_{01}^{(N)}$, $\nu_{10}^{(N)}$ and $\mu_{ab}^{(N)}$

for $a+b\leqq 6$, together with the domain of N for which these expressions are valid. As mentioned earlier, the values of $\nu_{01}^{(N)}$, $\nu_{10}^{(N)}$, $\mu_{02}^{(N)}$, $\mu_{11}^{(N)}$, $\mu_{20}^{(N)}$, $\mu_{03}^{(N)}$, $\mu_{04}^{(N)}$, $\mu_{05}^{(N)}$ and $\mu_{06}^{(N)}$ are not new.

List of Moments

$$\nu_{01}^{(N)}=\frac{N-1}{2}, \qquad N\geqq 2$$

$$\nu_{10}^{(N)}=\frac{2N-1}{6}, \qquad N\geqq 2$$

$$\mu_{02}^{(N)}=\frac{N+1}{12}, \qquad N\geqq 2$$

$$\mu_{11}^{(N)}=\frac{1}{6}, \qquad N\geqq 3$$

$$\mu_{20}^{(N)}=\frac{2N+2}{45}, \qquad N\geqq 4$$

$$\mu_{03}^{(N)}=0, \qquad N\geqq 2$$

$$\mu_{12}^{(N)}=-\frac{N+1}{45}, \qquad N\geqq 4$$

$$\mu_{21}^{(N)}=-\frac{1}{30}, \qquad N\geqq 5$$

$$\mu_{30}^{(N)}=-\frac{2N+2}{945}, \qquad N\geqq 6$$

$$\mu_{04}^{(N)}=\frac{(N+1)(5N+3)}{240}, \qquad N\geqq 4$$

$$\mu_{13}^{(N)}=\frac{5N+1}{120}, \qquad N\geqq 5$$

$$\mu_{22}^{(N)}=\frac{7N^2+16N+114}{1890}, \qquad N\geqq 6$$

$$\mu_{31}^{(N)}=\frac{14N+11}{630}, \qquad N\geqq 7$$

$$\mu_{40}^{(N)}=\frac{(N+1)(28N+6)}{4725}, \qquad N\geqq 8$$

$$\mu_{05}^{(N)}=0, \qquad N\geqq 2$$

$$\mu_{14}^{(N)}=-\frac{(N+1)(7N+1)}{630}, \qquad N\geqq 6$$

$$\mu_{23}^{(N)}=-\frac{77N+17}{2520}, \qquad N\geqq 7$$

$$\mu_{32}^{(N)}=-\frac{89N^2+112N+968}{28350}, \qquad N\geqq 8$$

$$\mu_{41}^{(N)}=-\frac{292N+157}{28350}, \qquad N\geqq 9$$

$$\mu_{50}^{(N)}=-\frac{(N+1)(88N+46)}{93555}, \qquad N\geqq 10$$

$$\mu_{06}^{(N)}=\frac{(N+1)(35N^2+28N+9)}{4032}, \qquad N\geqq 6$$

$$\mu_{15}^{(N)}=\frac{35N^2+13}{2016}, \qquad N\geqq 7$$

$$\mu_{24}^{(N)}=\frac{35N^3+223N^2+1239N-629}{37800}, \qquad N\geqq 8$$

$$\mu_{33}^{(N)}=\frac{70N^2+173N+393}{12600}, \qquad N\geqq 9$$

$$\mu_{42}^{(N)}=\frac{462N^3+1463N^2+15190N+14486}{935550}, \qquad N\geqq 10$$

$$\mu_{51}^{(N)}=\frac{308N^2+286N+113}{62370}, \qquad N\geqq 11$$

$$\mu_{60}^{(N)}=\frac{(N+1)(168168N^2-54340N+60818)}{127702575}, \qquad N\geqq 12 .$$

Appendix

The authors are suggested by the referee another method to compute the joint moments of r and s by using the following recurrence relation. Let $\mathrm{P}_N(r, s)$ denote the number of permutations of size N with r + runs and s + signs. Then the recurrence relation:

$$\mathrm{P}_N(r, s)=(r+1)\,\mathrm{P}_{N-1}(r, s)+(s-r+1)\,\mathrm{P}_{N-1}(r-1, s)+r\,\mathrm{P}_{N-1}(r, s-1)$$
$$+(N-r-s+1)\,\mathrm{P}_{N-1}(r-1, s-1)\,,$$

can be proved by an extended discussion of Mann [3].

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