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INFINITE DIVISIBILITY, COMPLETENESS AND REGRESSION PROPERTIES OF THE UNIVARIATE GENERALIZED WARING DISTRIBUTION

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Summary

This paper is concerned with properties of the univariate generalized Waring distribution such as infinite divisibility, discrete self-decomposability, completeness and regression.

1. Introduction

The univariate generalized Waring distribution (UGWD) along with many discrete distributions appearing in the statistical literature, belongs to a class of distributions whose probability generating functions (p.g.f.'s) can be expressed in terms of the Gauss hypergeometric function

$$_2F_1(\alpha, \beta; \gamma; z)=\sum_{r=0}^{\infty} \frac{\alpha_{(r)}\beta_{(r)}}{\gamma_{(r)}} \frac{z^r}{r!}, \qquad |z|<1$$

where

$$h_{(l)}=\frac{\Gamma(h+l)}{\Gamma(h)}, \qquad h>0,\ l \in R .$$

In particular, the p.g.f. of the UGWD with parameters a, k, ρ (UGWD $(a, k; \rho)$) is given by

$$(1.1) \qquad G(s)=\frac{\rho_{(k)}}{(a+\rho)_{(k)}} {}_2F_1(a, k; a+k+\rho; s)$$
$$=\frac{\rho_{(a)}}{(k+\rho)_{(a)}} {}_2F_1(k, a; a+k+\rho; s) ,$$
$$|s|\leq 1,\ a>0,\ k>0,\ \rho>0 .$$

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The UGWD has been widely used in a broad variety of scientific fields as remote and different as linguistics (e.g. Simon [14], Haight [4]), biology (e.g. Irwin [5]), bibliographic and economic studies (e.g. Kendall [8]) and accident theory (e.g. Irwin [6], [7], Xekalaki [20]). For more detailed accounts of the genesis, structural properties and applications of this distribution the reader is referred to the works of Shimizu [11], Sibuya [12], Sibuya and Shimizu [13] and Xekalaki [17], [21].

In this paper, a greater insight is given into the study of the structural properties of the UGWD and a characterization problem relating to it is considered.

In particular, it is shown that the UGWD has the important property of infinite divisibility (Section 2). More precisely, the UGWD is proved to be a discrete self decomposable distribution on the non-negative integers in the sense of Steutel and van Harn [15]. Section 3 shows that the family of univariate generalized Waring distributions is boundedly complete with respect to the parameters a, k and ρ. Finally, Section 4 is concerned with a regression property associated with the UGWD. It is shown that this property is unique and therefore characteristic for the UGWD.

2. Self-decomposability of the univariate generalized Waring distribution

Let X be a non-negative integer valued random variable (r.v.) that follows the univariate generalized Waring distribution, i.e.

$$X \sim \mathrm{UGWD}(a, k; \rho) , \qquad a, k, \rho > 0 .$$

It has been shown by Irwin [6] that the UGWD $(a, k; \rho)$ can arise from a Poisson distribution with parameter $\lambda > 0$ when λ has a gamma $(k; m)$ distribution whose scale parameter m is itself a random variable having the Beta II (Pearson Type VI) distribution with parameters a and ρ $(a, \rho > 0)$. This implies that the p.g.f. in (1.1) can be written as

$$G(s) = \mathrm{E}_Z \{e^{Z(s-1)}\} , \qquad |s| \leqq 1 \tag{2.1}$$

where $Z \stackrel{d}{=} X_1 X_2 X_3^{-1}$ with X_1, X_2, X_3 independently distributed as gamma $(k; 1)$, gamma $(a; 1)$ and gamma $(\rho; 1)$ respectively, a fact also noticed by Sibuya [12].

Bondesson [1] showed that if $Y_1, Y_2, \cdots, Y_n$ are independent gamma variables then the probability distribution of $Y = Y_1^{q_1} Y_2^{q_2} \cdots Y_n^{q_n}$, $|q_i| \geqq 1$, $i = 1, 2, \cdots, n$ is a generalized Γ-convolution—in Thorin's [16] sense—which implies that Y is self-decomposable. Letting $q_1 = q_2 = 1$ and $q_3 = -1$ it follows that the probability distribution of Z in (2.1) is self-decomposable. This means that, for every $\alpha \in (0, 1)$ there exist two independent random variables Z' and Z_α with $Z' \stackrel{d}{=} Z$ such that

$$Z \stackrel{d}{=} \alpha Z' + Z_\alpha . \tag{2.2}$$

Combining (2.1) and (2.2) we can prove the following theorem.

THEOREM 1. *The univariate generalized Waring distribution is a discrete self-decomposable distribution on* $\{0, 1, 2, \cdots\}$.

We do not provide a formal proof of this theorem as this can be thought of as a special case of a more general result concerning mixtures of the Poisson distribution with the mixing distribution as self-decomposable. This is stated and proved in the form of the following theorem.

THEOREM 2. *Let F be a self-decomposable probability distribution with $F(0)=0$ and let $H(x)$ be the probability distribution defined by*

$$H(x) = \int_0^{+\infty} e^{-\lambda} \frac{\lambda^x}{x!} dF(\lambda) , \qquad x = 0, 1, 2, \cdots . \tag{2.3}$$

Then $H(x)$ is discrete self-decomposable in the sense of Steutel and van Harn.

PROOF. Let $G(s)$ denote the p.g.f. of the probability distribution in (2.3). Since $F(\lambda)$ is self-decomposable it follows that, for every $\alpha \in (0, 1)$, $\lambda \stackrel{d}{=} \alpha\lambda' + \lambda_\alpha$ where λ', λ_α are independent random variables with $\lambda' \stackrel{d}{=} \lambda$. Then

$$\begin{aligned} G(s) &= \mathrm{E}\,\{\exp\,[(\alpha\lambda' + \lambda_\alpha)(s-1)]\} \\ &= \mathrm{E}\,\{\exp\,[\alpha\lambda'(s-1)]\}\,\mathrm{E}\,\{\exp\,[\lambda_\alpha(s-1)]\} \\ &= \mathrm{E}\,\{\exp\,[\lambda(1-\alpha+\alpha s-1)]\}\,\mathrm{E}\,\{\exp\,[\lambda_\alpha(s-1)]\} \end{aligned}$$

i.e.

$$G(s) = G(1-\alpha+\alpha s) G_\alpha(s)$$

where $G_\alpha(s)$ is the p.g.f. defined by $\mathrm{E}\,(\exp\{\lambda_\alpha(s-1)\})$. This shows that $H(x)$ satisfies Steutel and van Harn's [15] definition of a discrete self-decomposable distribution. Therefore, the theorem has been established.

Note. A result by Forst [3] using a measure theoretic approach could also be used for an alternative derivation of the above theorem.

Obviously, the result of Theorem 1 follows as an immediate consequence. To this end, the negative binomial distribution being a gamma mixture of the Poisson distribution is also discrete self-decomposable.

COROLLARY. *The* UGWD$(a, k; \rho)$ *is an infinitely divisible distribution on* $\{0, 1, 2, \cdots\}$ *with a non-increasing canonical measure* r_n, *i.e., the probabilities* p_n, $n=0, 1, 2, \cdots$ *of the* UGWD$(a, k; \rho)$ *satisfy the relationship*

$$(n+1)p_{n+1}=\sum_{k=0}^{n} p_k r_{n-k}\,, \qquad n=0, 1, 2, \cdots \tag{2.4}$$

with $r_n \geqq 0$.

Remark 1. The p.g.f. of the UGWD$(a, k; \rho)$ can be put in the form

$$G(s)=\exp\left\{-\lambda \int_s^1 \frac{1-Q(u)}{1-u}\,du\right\}$$

where $\lambda=\frac{p_1}{p_0}$ and $Q(u)=\sum_{j=1}^{+\infty} q_j u^j$ is a unique p.g.f. The probabilities q_j, $j=1, 2, \cdots$ satisfy the relationship

$$r_n=\lambda \sum_{j=n+1}^{+\infty} q_j\,, \qquad n=0, 1, 2, \cdots$$

(see Steutel and van Harn [15]) and hence they can be uniquely determined from the recurrence formula given below which is obtained from (2.4) by subtraction

$$\lambda p_0 q_n=(n+\lambda)p_n-(n+1)p_{n+1}-\lambda \sum_{j=0}^{n-1} q_j p_{n-j}\,, \qquad n=1, 2, \cdots \quad (q_0=0)$$

or, substituting for $p_n=p_0\frac{a_{(n)}k_{(n)}}{(a+k+\rho)_{(n)}n!}$

$$q_n=p_n^*\left\{n\frac{ak+\rho(a+k+\rho)}{ak(a+k+\rho+n)}-\sum_{j=0}^{n-1} q_j\binom{n}{j}\binom{a+k+\rho+n-1}{j}\Big/\binom{a+n-1}{j}\binom{k+n-1}{j}\right\}, \qquad n=1, 2, \cdots$$

where $p_n^*=\frac{p_n}{p_0}$ and $\binom{c}{j}=\frac{c(c-1)\cdots(c-j+1)}{j!}$ for any $c \in R$ and $j \in I^+ \cup \{0\}$.

Remark 2. The infinite divisibility of the UGWD$(a, k; \rho)$ can also be deduced from the fact that it is a generalized negative binomial convolution in Bondesson's [2] sense as being a mixture on q of the negative binomial distribution with parameters q and k $(0<q<1,\ k>0)$.

Remark 3. From (2.4) it follows that the difference p_n-p_{n-1} $(p_{-1}=0)$ changes sign at most once (Steutel and van Harn [15]) and therefore, the UGWD $(a, k; \rho)$ can have at most one mode (unimodal). Irwin [7] remarked that the mode, if it exists (i.e. if $a>1$, $k>1$, $\rho\leqq(a-1)\cdot(k-1)-1$), is located at the point $\left[\frac{(a-1)(k-1)}{\rho+1}\right]$. (Here $[c]$ denotes the integral part of c.)

3. Completeness of the family of univariate generalized Waring distributions

In this section the question of completeness of the family $\mathcal{W}=\{p_W(x; a, k, \rho): a>0, k>0, \rho>0\}$, of univariate generalized Waring distributions is examined. It turns out that this family enjoys the property of bounded completeness as indicated by the following theorem.

THEOREM 3. *The family $\mathcal{W}$ defined as above is boundedly complete.*

This theorem can be proved along lines similar to those followed below to establish a result concerning more general families of distributions.

THEOREM 4. *Let $\mathcal{P}=\{p(x; \theta): x=0, 1, 2, \cdots; \theta \in \Theta\}$ be a family of discrete distributions indexed by a parameter vector $\theta \in \Theta$. If there exists a θ_0 in $\bar{\Theta}$ such that*

$$\text{(u.)} \lim_{\theta\to\theta_0} \frac{p(x+1; \theta)}{p(x; \theta)}=0\ , \qquad x=0, 1, 2, \cdots \tag{3.1}$$

then $\mathcal{P}$ is boundedly complete.

PROOF. Consider an arbitrary function $g(X)$ of an r.v. X whose probability distribution belongs to the family $\mathcal{P}$. Assume that $g(x)$, $x=0, 1, 2, \cdots$ is independent of the vector θ and bounded, i.e. $|g(x)|\leqq M$, $x=0, 1, 2 \cdots$, for some $M>0$.

We will show that if

$$\mathrm{E}\,(g(X))=0 \qquad \text{for all } \theta \in \Theta \tag{3.2}$$

then, $g(x)=0$ for every $x=0, 1, 2, \cdots$.

Assume that (3.2) holds. This implies that

$$\sum_{x=0}^{+\infty} g(x)p(x; \theta)=0 \qquad \text{for all } \theta \in \Theta\ . \tag{3.3}$$

It follows then that

$$g(0)+\lim_{\theta\to\theta_0} \sum_{x=1}^{+\infty} \frac{p(x; \theta)}{p(0; \theta)} g(x)=0\ . \tag{3.4}$$

Since $g(x)$ is bounded and because of (3.1) we have that

$$|g(x)|p(x;\theta)\leqq Mp(0;\theta)\varepsilon^x, \qquad x=0,1,2,\cdots, \tag{3.5}$$

for some $M>0$, some $0<\varepsilon<1$ and all θ in $\Theta\cap\mathcal{N}(\theta_0, \delta(\varepsilon))$, $\delta(\varepsilon)>0$, where $\mathcal{N}(\theta_0, \delta(\varepsilon))$ is a deleted $\delta(\varepsilon)$-neighborhood of θ_0.

This implies that $g(x)p(x;\theta)/p(0;\theta)$ is absolutely bounded by an integrable function. Hence by the Lebesgue dominated convergence theorem, (3.3) can be written as

$$g(0)+\sum_{n=1}^{+\infty} g(x)\lim_{\theta\to\theta_0}\frac{p(x;\theta)}{p(0;\theta)}=0\ . \tag{3.6}$$

But, from (3.1) it is obvious that

$$(\text{u.})\lim_{\theta\to\theta_0}\frac{p(x+r;\theta)}{p(x;\theta)}=0 \qquad \text{for every } r\geqq 1;\ x=0,1,2,\cdots. \tag{3.7}$$

Consequently, it follows from (3.6) that

$$g(0)=0\ .$$

Assume now that

$$g(x)=0\ , \qquad x=0,1,\cdots,n, \text{ for some } n\geqq 0\ . \tag{3.8}$$

We will show that $g(n+1)=0$.

Using (3.8), equation (3.3) becomes

$$g(n+1)+\sum_{x=n+2}^{+\infty}\frac{p(x;\theta)}{p(n+1;\theta)}g(x)=0 \qquad \text{for all } \theta\in\Theta\ . \tag{3.9}$$

The general term of the series in (3.9) is an integrable function because of (3.5). Then, by the Lebesgue dominated convergence theorem it follows that

$$g(n+1)+\sum_{x=n+2}^{+\infty} g(x)\lim_{\theta\to\theta_0}\frac{p(x;\theta)}{p(n+1;\theta)}=0\ .$$

But because of (3.7) the above relationship shows that

$$g(n+1)=0$$

which completes the inductive argument.

Hence $g(x)=0$ for all $x=0,1,2,\cdots$. Therefore the theorem has been established.

The result of Theorem 3 now follows by an argument of the type used above if one replaces the requirement of uniform convergence in x of the function in (3.1) by pointwise convergence and observes that

$$0 \leq \left| \sum_{x=1}^{\infty} \frac{g(x) a_{(x)} k_{(x)}}{(a+k+\rho)_{(x)} x!} \right| \leq \sum_{x=1}^{\infty} \frac{|g(x)| a_{(x)} k_{(x)}}{(a+k+\rho)_{(x)} x!} \leq M\left(\frac{(a+\rho)_{(k)}}{\rho_{(k)}} - 1 \right) .$$

4. A characterization of the univariate generalized Waring distribution based on linear regression

Consider X, Y to be two non-negative, integer valued r.v.'s such that the conditional distribution of Y given $X=x$ is the negative hypergeometric with parameters x, m and n ($NH(x; m, n)$), i.e.,

$$\text{(4.1)} \qquad \mathrm{P}(Y=y \mid X=x) = \frac{\binom{-m}{y}\binom{-n}{x-y}}{\binom{-m-n}{x}}, \qquad m>0,\ n>0;\ y=0, 1, \cdots, x .$$

One may observe that if $X \sim \mathrm{UGWD}(a, k; \rho)$ and $Y|(X=x) \sim NH(x; m, n)$, $m>0$, $n>0$, $m+n=k$ then, the regression of X on Y is linear i.e.,

$$\text{(4.2)} \qquad \mathrm{E}(X \mid Y=y) = \frac{(\rho+m+n-1)y+an}{\rho+m-1}, \qquad y=0, 1, 2, \cdots .$$

The proof of this result is straightforward. The question now arises whether (4.2), along with (4.1), uniquely determines the distribution of the r.v. X. In other words, when $Y|(X=x) \sim NH(x; m, n)$ is the condition

$$\text{(4.3)} \qquad \mathrm{E}(X \mid Y=y) = \alpha y + \beta, \qquad y=0, 1, \cdots$$

for some constants α and β sufficient to characterize the distribution of X as UGWD?

In the sequel, it is shown that this is the case when $n=1$. (Characteristic properties of discrete distributions along these lines have been studied by, among others, Korwar [9], [10] and Xekalaki [18], [19].)

To show the above mentioned result we need to prove the following lemma.

LEMMA. *Let X be an r.v. on $\{0, 1, 2, \cdots\}$ with $\mathrm{P}(X=0)<1$. Let Y be another non-negative and integer-valued r.v. such that the conditional distribution of $Y|(X=x)$ is given by* (4.1). *Assume that* (4.3) *holds, for some constants α and β. Then* (i) $\beta>0$, (ii) $\alpha>1$.

PROOF. (i) From (4.3) and since $Y \leq X$ we have $\mathrm{E}(X \mid Y=0) \geq 0$ i.e., $\beta \geq 0$. Note however, that if $\beta=0$ then $\sum_{x=1}^{\infty} \frac{x n_{(x)}}{(m+n)_{(x)}} \mathrm{P}(X=x)=0$ which implies that $\mathrm{P}(X=x)=0$, $x=1, 2, \cdots$. But, X is non-degenerate ($\mathrm{P}(X=0)<1$). Hence $\beta>0$.

(ii) Using (4.3) and the fact that $X \geqq Y$ we obtain $\mathrm{E}(X|Y=y) \geqq y$ for every y, i.e.,

$$\beta \geqq (1-\alpha)y\ , \qquad y=0,1,2,\cdots .$$

Since $\beta>0$, this cannot hold for all the values of y unless $\alpha>1$.

THEOREM 5. *Let X, Y be two r.v.'s on $\{0,1,2,\cdots\}$ such that* $\mathrm{P}(X=0)<1$ *and*

$$\text{(4.4)} \qquad \mathrm{P}(Y=y|X=x)=\binom{m+y-1}{y}\bigg/\binom{m+x}{x}\ , \qquad y=0,1,\cdots,x;\ m>0$$

(i.e., $Y|(X=x)\sim NH(x;m,1)$). Then, the regression of X on Y is linear as in (4.3) *with $\alpha<1+m^{-1}$ if and only if* $X\sim\mathrm{UGWD}\left(\frac{\beta}{\alpha-1}, m+1; \frac{\alpha}{\alpha-1}-m\right)$.

PROOF. The "if" part follows immediately from (4.2) for $n=1$.

"Only if" Part. From (4.4) and (4.3) we have

$$\text{(4.5)} \qquad \sum_{x=y}^{\infty} x g_x=(\alpha y+\beta)\sum_{x=y}^{\infty} g_x\ , \qquad y=0,1,\cdots$$

where $g_x=x!\,\mathrm{P}(X=x)/(m+1)_{(x)}$.

Considering relation (4.5) for $y=r$ and $y=r+1$ and subtracting the resulting equations by parts we obtain

$$\text{(4.6)} \qquad [(\alpha-1)r+\beta]g_r=\alpha\sum_{x=r+1}^{\infty} g_x\ , \qquad r=0,1,2,\cdots .$$

Applying the same technique once more and since, from the lemma $\alpha>1$ (4.6) reduces to

$$g_{r+1}-\frac{(\alpha-1)r+\beta}{(\alpha-1)r+\beta+2\alpha-1}g_r=0\ , \qquad r=0,1,\cdots .$$

This is a first-order difference equation in g_r whose the solution is given by

$$g_r=g_0\frac{(\beta/(\alpha-1))_{(r)}}{((\beta+\alpha)/(\alpha-1)+1)_{(r)}}\ , \qquad r=0,1,2,\cdots .$$

Therefore,

$$\text{(4.7)} \qquad \mathrm{P}(X=r)=\mathrm{P}(X=0)\frac{(\beta/(\alpha-1))_{(r)}(m+1)_{(r)}}{((\beta+\alpha)/(\alpha-1)+1)_{(r)}}\frac{1}{r!}\ , \qquad r=0,1,\cdots$$

where, from the fact that $\frac{\alpha}{\alpha-1}-m>0$ and the condition $\sum_{r=0}^{\infty} \mathrm{P}(X=r)=1$,

$$\mathrm{P}(X=0)=\frac{(\alpha/(\alpha-1)-m)_{(m+1)}}{((\alpha+\beta)/(\alpha-1)-m)_{(m+1)}}.$$

From the lemma it is obvious that $\frac{\beta}{\alpha-1}>0$. Then, (4.7) shows that $X\sim\mathrm{UGWD}\left(\frac{\beta}{\alpha-1}, m+1; \frac{\alpha}{\alpha-1}-m\right)$. Hence, the theorem has been established.

Note that in the case $m=n=1$, (4.4) reduces to the discrete uniform distribution i.e. $\mathrm{P}(Y=y\,|\,X=x)=\frac{1}{x+1}$, $y=0, 1, \cdots, x$. Thus, Theorem 5 establishes also, indirectly, a relationship between the discrete uniform distribution and the UGWD.

At this point, it is worth noting the following interesting fact.

The UGWD $(a, k; \rho)$ is symmetrical with respect to the parameters a and k (i.e. UGWD $(a, k; \rho)\sim$UGWD $(k, a; \rho)$). Also, the negative hypergeometric distribution as given by (4.1) is invariant under the simultaneous exchange of the pairs (m, y) and $(n, x-y)$. Therefore, assuming $m=1$ in (4.1) we can characterize the distribution of X as UGWD using the linearity of the regression $\mathrm{E}(X|X-Y=z)$ as a necessary and sufficient condition.

Note also that if $Y|(X=x)\sim NH(x; m, n)$ as in (4.1) and $X\sim$UGWD $(a, m+n; \rho)$ then the random vector $(Y, X-Y)$ has a bivariate distribution with probability function

$$\text{(4.8)} \quad \mathrm{P}(Y=y, X-Y=z)=\frac{\rho_{(m+n)}}{(a+\rho)_{(m+n)}}\frac{a_{(y+z)}m_{(y)}n_{(z)}}{(a+m+n+\rho)_{(y+z)}y!z!}$$
$$y=0, 1, 2, \cdots; \ z=0, 1, 2, \cdots.$$

This is a bivariate distribution which the authoress (Xekalaki [17]) has named the bivariate generalized Waring distribution and has examined in the context of accident theory. (This distribution has also been studied from a different viewpoint by Sibuya and Shimizu [13].)

It is now clear that the result of Theorem 5 is essentially a characterization of the bivariate distribution in (4.8) when $n=1$. It follows then, because of our previous remark that assuming the linearity of either of the regressions $\mathrm{E}(X|Y=y)$ and $\mathrm{E}(X|X-Y=z)$ we can arrive at a characterization of (4.8) provided that (4.1) holds with $n=1$ or $m=1$ respectively.

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