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INEQUALITIES FOR FUNCTIONS OF ORDER STATISTICS UNDER AN ADDITIVE MODEL

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Summary

This work contains inequalities concerning random variables of the form $\sum_{i=1}^{n} c_i\phi_i(X_{(i)})$ where: (a) $X_{(1)}\leqq\cdots\leqq X_{(n)}$ are the order statistics of a random vector $X=(X_1,\cdots,X_n)$ under an additive model, (b) ϕ_i, $i=1,\cdots,n$ are real-valued functions satisfying certain monotonicity, and convexity or concavity conditions, and (c) c_i, $i=1,\cdots,n$ is a nondecreasing or nonincreasing sequence of constants. A special case is that $\phi=\phi_1=\cdots=\phi_n$ is increasing, convex (or linear) and $c_1\leqq\cdots\leqq c_n$ (concave (or linear) and $c_1\geqq\cdots\geqq c_n$). Inequalities are first obtained under the additive (or location) model $\boldsymbol{X}=\boldsymbol{Y}+\boldsymbol{\delta}$ and then extended to the model $\boldsymbol{X}=\boldsymbol{Y}+\boldsymbol{Z}$, where $\boldsymbol{\delta}$ is a real vector and $\boldsymbol{Z}$ is a random vector. The inequalities obtained are in terms of the ordered components of $\boldsymbol{Y}$, $\boldsymbol{\delta}$, and $\boldsymbol{Z}$. Majorization is an important tool in the derivation of these inequalities. One use of these inequalities is to extend the applicability of the large number of known results for random vectors with i.i.d. components to random vectors with dependent and/or heterogeneously distributed components. Several applications are included by way of illustration.

1. Introduction

The study of order statistics seems to have been fully developed in the statistical literature. However, only a few results are available for ordered random variables from dependent or heterogeneous distributions. For these distributions the problem can often be very complicated due to dependence or the lack of symmetry. Thus inequalities which yield bounds become useful. With the aid of inequalities, existing results and statistical tables for the i.i.d. (independent and iden-

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tically-distributed) case can be applied to give bounds for the dependent or heterogeneous case. It is for this reason inequalities play an important role in the area of order statistics. For a survey of existing results in this area, the reader is referred to David [2] and Marshall and Olkin ([5], pp. 348–355).

This paper concerns some new probability and moment inequalities for linear combinations of functions of order random variables with dependent or (and) heterogeneous components under an additive model. The model we consider is $\boldsymbol{X}=\boldsymbol{Y}+\boldsymbol{\delta}$, where $\boldsymbol{Y}=(Y_1,\cdots,Y_n)$ is an n-dimensional random vector whose density function satisfies certain conditions, and $\boldsymbol{\delta}=(\delta_1,\cdots,\delta_n)$ is a real vector. Let $X_{(1)}\leqq\cdots\leqq X_{(n)}$ denote the order statistics of the components of $\boldsymbol{X}=(X_1,\cdots,X_n)$. Let $\phi_i: R^1\rightarrow R^1$ $(i=1,\cdots,n)$ be real-valued monotone functions, and let $c_1,\cdots,c_n$ denote given real numbers. Our results give inequalities for the distribution function of and the expectation of the random variable $\sum_{i=1}^{n} c_i\phi_i(X_{(i)})$ when the ϕ_i's are convex and $0\leqq c_1\leqq\cdots\leqq c_n$, or when the ϕ_i's are concave and $c_1\geqq\cdots\geqq c_n\geqq 0$. The inequalities depend on the order statistics of the elements of $\boldsymbol{Y}$ and on a partial ordering of the real vector $\boldsymbol{\delta}$, and they show how the parameters are related. If $Y_1,\cdots,Y_n$ are i.i.d. random variables, then under reasonable conditions the bound yielded by the inequalities is attainable when $\delta_1=\cdots=\delta_n$ holds given their sum. Our result also yields inequalities for linear combinations of order statistics under this additive model when the coefficients $c_1,\cdots,c_n$ are in ascending (or descending) order but not necessarily nonnegative. This includes the following statistics: the maximum (max), the minimum (min), the range, max-$\bar{X}$ ($\bar{X}$ is the sample mean), and min-$\bar{X}$. The reason we pay special attention to such linear combinations is partially because they are frequently encountered in statistical applications, and partially due to the fact that the mathematical tool we adopt is especially useful for this case. Our tool includes certain inequalities via majorization theory and rearrangements. For references, see Hardy, Littlewood and Pólya [3], Marshall and Olkin [5].

2. The main results

Before proceeding, we recall the following concepts for the partial ordering of random variables (U_1 and U_2 denote two univariate random variables).

B_1 (Pointwise inequality). $U_1\geqq U_2$ a.s.

B_2 (Stochastic inequality). $U_1\overset{\text{st}}{\geqq}U_2$, that is,

$$\mathrm{P}\,[U_1\leqq t]\leqq \mathrm{P}\,[U_2\leqq t] \qquad \text{for all } t\,.$$

B_3 (Moment inequality). $\mathrm{E}\,h(U_1)\geqq \mathrm{E}\,h(U_2)$ for all h which are monotonically nondecreasing such that the expectations exist.

B_4 (Mean inequality). $\mathrm{E}\,U_1\geqq \mathrm{E}\,U_2$ (if the means exist).

It is well known that $B_1\Longrightarrow B_2\Longleftrightarrow B_3\Longrightarrow B_4$. Since the inequalities given in this section are either pointwise or stochastic, it is therefore understood that the moment inequality (and mean inequality) will follow as special consequences.

Now let us consider the following additive model

$$\mathbf{X}=\mathbf{Y}+\boldsymbol{\delta}\ , \tag{2.1}$$

where $\mathbf{Y}=(Y_1,\cdots,Y_n)$ is a random vector with density $g(\mathbf{y})$, and $\boldsymbol{\delta}=(\delta_1,\cdots,\delta_n)$ a real vector. Let

$$X_{(1)}\leqq\cdots\leqq X_{(n)},\ Y_{(1)}\leqq\cdots\leqq Y_{(n)},\ \delta_{(1)}\leqq\cdots\leqq\delta_{(n)}\ , \tag{2.2}$$

$$X_{[1]}\geqq\cdots\geqq X_{[n]},\ Y_{[1]}\geqq\cdots\geqq Y_{[n]},\ \delta_{[1]}\geqq\cdots\geqq\delta_{[n]} \tag{2.3}$$

denote the ordered components of $\mathbf{X}$, $\mathbf{Y}$ and $\boldsymbol{\delta}$. For $i=1,\cdots,n$ let ϕ_i: $R^1\to R^1$ be real-valued functions, and consider the statistic $\sum_{i=1}^n c_i\phi_i(X_{(i)})$. We first note a simple fact when a constant is added to each of the components of $\mathbf{Y}$.

Fact 2.1. Under the additive model given in (2.1) if $\delta_1=\cdots=\delta_n=\bar{\delta}$ (say), then for every $\phi_1,\cdots,\phi_n$ and every given set of real numbers $c_1,\cdots,c_n$ we have

$$\sum_{i=1}^n c_i\phi_i(X_{(i)})=\sum_{i=1}^n c_i\phi_i(Y_{(i)}+\bar{\delta})\ \text{a.s.} \tag{2.4}$$

In the following we consider the nontrivial case in which the δ_i values are not necessarily equal. The first theorem (Theorem 2.1) concerns a rearrangement inequality (the reader is referred to Chapter 10 of Hardy, Littlewood and Pólya [3] for a discussion of rearrangement theory); it will be stated under the following conditions:

Condition 2.1. (a) For $i=1,\cdots,n$ and $x\in R^1$, $\phi_i(x): R^1\to R^1$ is a function of x for each i and is monotonically nondecreasing both in x and in i. (b) For each i, $(d/dx)\phi_i(x)\equiv\phi_i'(x)$ exists for $x\in R^1$, and is monotonically nondecreasing in i for each x.

Condition 2.2. Condition 2.1 holds with "convex" in (a) replaced by "concave" and with "nondecreasing in i" in (b) replaced by "nonincreasing in i".

Note that in many applications the special case $\phi_1=\cdots=\phi_n=\phi$ is of great interest. In this case Condition 2.1 (Condition 2.2) reduces to: the function $\phi(x): R^1 \to R^1$ is nondecreasing and is a convex (concave) function of x. This of course, includes linear functions.

THEOREM 2.1. *Let* $\boldsymbol{X}$, $\boldsymbol{Y}$ *and* $\boldsymbol{\delta}$ *be as defined in* (2.1).
(a) *If Condition* 2.1 *is satisfied and if* $0 \leqq c_1 \leqq \cdots \leqq c_n$ (*Condition* 2.2 *is satisfied and if* $c_1 \geqq \cdots \geqq c_n \geqq 0$), *then*

$$\sum_{i=1}^{n} c_i\phi_i(Y_{(i)}+\delta_{(i)}) \geqq (\leqq) \sum_{i=1}^{n} c_i\phi_i(X_{(i)}) \geqq (\leqq) \sum_{i=1}^{n} c_i\phi_i(Y_{(i)}+\delta_{[i]}) \ a.s. \tag{2.5}$$

(*Note that in* (2.5) *the* Y'*s and the* δ'*s are similarly ordered on the left-hand side, and are reversely ordered on the right-hand side.*)
(b) *If* $\phi_i(x)=ax+b$, $a \geqq 0$ *for* $i=1,\cdots,n$, *and if* $c_1 \leqq \cdots \leqq c_n$ ($c_1 \geqq \cdots \geqq c_n$), *then* (2.5) *also holds.*

PROOF. The proof follows immediately from two lemmas (Lemmas 2.1 and 2.2) which are stated and proved below.

LEMMA 2.1. (a) *If Condition* 2.1 *is satisfied and if* $0 \leqq c_1 \leqq \cdots \leqq c_n$ (*Condition* 2.2 *is satisfied and if* $c_1 \geqq \cdots \geqq c_n \geqq 0$), *then* $\sum_{i=1}^{n} c_i\phi_i(x_{(i)})$ *is a nondecreasing and Schur-convex* (*Schur-concave*) *function of* $\boldsymbol{x}=(x_1,\cdots,x_n)$. (b) *If* $\phi_i(x)=ax+b$, $a \geqq 0$ *for* $i=1,\cdots,n$, *and if* $c_1 \leqq \cdots \leqq c_n$ ($c_1 \geqq \cdots \geqq c_n$), *then* $\sum_{i=1}^{n} c_i\phi_i(x_{(i)})$ *is a nondecreasing and Schur-convex* (*Schur-concave*) *function of* $\boldsymbol{x}$.

PROOF. (a) Assume that Condition 2.1 is satisfied and that $0 \leqq c_1 \leqq \cdots \leqq c_n$. Then from the facts that $\phi_i'(x) \geqq 0$, $\phi_i'(x) \uparrow i$, the inequalities

$$c_i\phi_i'(x_1) \geqq c_i\phi_{i-1}'(x_1) \geqq c_{i-1}\phi_{i-1}'(x_1) \geqq c_{i-1}\phi_{i-1}'(x_2)$$

hold for all i and all $x_1 \geqq x_2$. The assertion now follows from Proposition 3.H.2 in Marshall and Olkin ([5], p. 92). The proof for the other case is similar. (b) Assume that $\phi_i(x)=ax+b$, $a \geqq 0$ for $i=1,\cdots,n$. By definition of majorization $\boldsymbol{x} \succ \boldsymbol{y}$ implies that

$$\Delta_r \equiv \sum_{i=r}^{n} \phi_i(x_{(i)}) - \sum_{i=r}^{n} \phi_i(y_{(i)}) \geqq 0$$

holds for $r \geqq 2$ and $\Delta_1=0$. Therefore if $\boldsymbol{x} \succ \boldsymbol{y}$, then from the identity (with $c_0 \equiv 0$)

$$\sum_{i=1}^{n} c_i\phi_i(x_{(i)}) - \sum_{i=1}^{n} c_i\phi_i(y_{(i)}) = \sum_{r=1}^{n} (c_r - c_{r-1})\Delta_r$$

and the monotonicity of c_i $(i=1,\cdots,n)$ this difference is $\geqq 0$.

Note that in Lemma 2.1(b) the condition that all c_i's are nonnegative is removed.

LEMMA 2.2. *For* $\boldsymbol{X}$, $\boldsymbol{Y}$ *and* $\boldsymbol{\delta}$ *defined in* (2.1), *we have*

$$(Y_{(1)}+\delta_{(1)},\cdots,Y_{(n)}+\delta_{(n)})\succ \boldsymbol{X}\succ(Y_{(1)}+\delta_{[1]},\cdots,Y_{(n)}+\delta_{[n]}) \tag{2.6}$$

for every point in the sample space.

PROOF. This lemma follows from the following result by induction: Let $\boldsymbol{z}=(z_1,\cdots,z_n)\in R^n$, $\boldsymbol{t}=(t_1,\cdots,t_n)\in R^n$ be such that $z_1\leqq\cdots\leqq z_n$ and $t_j\geqq t_k$ for $j<k$, where j, k are arbitrary but fixed. Let $\boldsymbol{t}^*$ be the vector $\boldsymbol{t}$ with t_j and t_k interchanged. Then we have $(\boldsymbol{z}+\boldsymbol{t}^*)\succ(\boldsymbol{z}+\boldsymbol{t})$. This result follows from the result given in Marshall and Olkin ([5], p. 21), where the value of λ in the T matrix is

$$\lambda=(z_k-z_j)/[(z_k-z_j)+(t_j-t_k)]$$

if $z_k>z_j$ or $t_j>t_k$, and one otherwise.

In Theorem 2.1 there are no assumption made on the distribution of $\boldsymbol{Y}$. Thus the bounds given in (2.5) are universal. If the density function of $\boldsymbol{Y}$ is Schur-concave, then we can prove a stochastic inequality which is stated below.

THEOREM 2.2. *Assume that* $\boldsymbol{Y}$ *in* (2.1) *has a density* $g(\boldsymbol{y})$, *and let* $\alpha(\boldsymbol{\delta})\equiv \mathrm{P}_{\boldsymbol{\delta}}\left[\sum_{i=1}^{n} c_i\phi_i(X_{(i)})\leqq t\right]$ *for* $t\in R^1$.
(a) *If* $g(\boldsymbol{y})$ *is a Schur-concave function of* $\boldsymbol{y}$, *Condition* 2.1 *is satisfied, and if* $0\leqq c_1\leqq\cdots\leqq c_n$, *then* $\alpha(\boldsymbol{\delta})$ *is a Schur-concave function of* $\boldsymbol{\delta}$ *for all* t.
(b) *If* $g(\boldsymbol{y})$ *is a Schur-concave function of* $\boldsymbol{y}$, $\phi_i(x)=ax+b$, $a\geqq 0$ *for* $i=1,\cdots,n$, *and if* $c_1\leqq\cdots\leqq c_n$, *then* $\alpha(\boldsymbol{\delta})$ *is a Schur-concave function of* $\boldsymbol{\delta}$ *for all* t.

PROOF. Assume that Condition 2.1 is satisfied; for arbitrary but fixed t let us consider the indicator function $\chi_A(\boldsymbol{x})$ of the set $A=\left\{\boldsymbol{x}\,\middle|\,\sum_{i=1}^{n} c_i\phi_i(x_{(i)})>t\right\}$. Since $\sum_{i=1}^{n} c_i\phi_i(x_{(i)})$ is a Schur-convex function of $\boldsymbol{x}$, and a nondecreasing function of a Schur-convex function is also Schur-convex (see, e.g., Nevius, Proschan and Sethuraman [7] or Tong ([11], p. 139)), it follows that $[1-\chi_A(\boldsymbol{x})]$ is a Schur-concave function of $\boldsymbol{x}$. By Theorem 3.J.1 in Marshall and Olkin ([5], p. 100) the probability

$$\mathrm{P}\,[(\boldsymbol{Y}-\boldsymbol{\delta})\notin A]=\mathrm{E}\,[1-\chi_A(\boldsymbol{X})]$$

is a Schur-concave function of $\boldsymbol{\delta}$. But $\boldsymbol{\delta}_1 \succ \boldsymbol{\delta}_2$ if and only if $-\boldsymbol{\delta}_1 \succ -\boldsymbol{\delta}_2$. Thus

$$\alpha(\boldsymbol{\delta}) = \mathrm{P}_{\boldsymbol{\delta}}\,[(\boldsymbol{Y}+\boldsymbol{\delta}) \notin A] = \mathrm{P}_{\boldsymbol{\delta}}\left[\sum_{i=1}^{n} c_i \phi_i(X_{(i)}) \leqq t\right]$$

is also a Schur-concave function of $\boldsymbol{\delta}$. This completes the proof of (a). The result in (b) is similar.

Note that the result in Theorem 2.2(b) was previously given in Mudholkar [6] with the additional condition that $c_i \geqq 0$ for $i=1,\cdots,n$.

When applying Theorems 2.1 and 2.2 to obtain bounds on the expectations of linear combinations of order statistics, they yield, respectively for $c_1 \leqq \cdots \leqq c_n$,

$$\sum_{i=1}^{n} c_i(\mathrm{E}\, Y_{(i)} + \delta_{(i)}) \geqq \sum_{i=1}^{n} c_i \,\mathrm{E}\, X_{(i)} \geqq \sum_{i=1}^{n} c_i(\mathrm{E}\, Y_{(i)} + \delta_{[i]}) \,, \tag{2.7}$$

$$\sum_{i=1}^{n} c_i \,\mathrm{E}\, X_{(i)} \geqq \sum_{i=1}^{n} c_i(\mathrm{E}\, Y_{(i)} + \bar{\delta}) \,. \tag{2.8}$$

It is easy to check that the bound in (2.8) is sharper. The improvement here is achieved with the additional condition that $g(\boldsymbol{y})$ is Schur-concave.

We now extend our results to a more general additive model given by

$$\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z} \tag{2.9}$$

where $\boldsymbol{Z} = (Z_1,\cdots,Z_n)$ is a random vector with ordered components

$$Z_{(1)} \leqq \cdots \leqq Z_{(n)},\quad Z_{[1]} \geqq \cdots \geqq Z_{[n]} \,. \tag{2.10}$$

Thus we give new results for the order statistics for sums of independent random vectors.

THEOREM 2.3. *If $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are independent, then the statement in Theorem 2.1 remains true after substituting the random vector $\boldsymbol{Z}$ for the real vector $\boldsymbol{\delta}$.*

PROOF. For every given $\boldsymbol{Z} = \boldsymbol{\delta}$ (2.5) holds. Thus Theorem 2.3 follows.

Our next theorem depends on the concept of stochastic majorization introduced recently by Nevius, Proschan and Sethuraman [7]. For n-dimensional random vectors $\boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{Z}_1, \boldsymbol{Z}_2$ we define $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z}$, $\boldsymbol{X}_j = \boldsymbol{Y} + \boldsymbol{Z}_j$, and use $X_{(1)}^{(j)} \leqq \cdots \leqq X_{(n)}^{(j)}$ to denote the ordered components of $\boldsymbol{X}_j$ $(j=1,2)$.

THEOREM 2.4. (a) *Let* $\phi_{(i)}: R^1 \to R^1$ *for* $i=1,\cdots,n$ *and let* $c_1,\cdots,c_n$ *be real numbers. If* (1) $\boldsymbol{Y}$ *and* $\boldsymbol{Z}$, $\boldsymbol{Z}_1$, $\boldsymbol{Z}_2$ *are independent and* $\boldsymbol{Z}_1 \overset{\text{st}}{\succ} \boldsymbol{Z}_2$, (2) *the density function* $g(\boldsymbol{y})$ *of* $\boldsymbol{Y}$ *is a Schur-concave function of* $\boldsymbol{y}$, *and* (3) *Condition* 2.1 *is satisfied and* $0 \leqq c_1 \leqq \cdots \leqq c_n$, *then the stochastic inequality*

$$\sum_{i=1}^{n} c_i \phi_i(X_{(i)}^{(1)}) \overset{\text{st}}{\geqq} \sum_{i=1}^{n} c_i \phi_i(X_{(i)}^{(2)}) \tag{2.11}$$

holds. Consequently we have, for $\bar{Z} = \frac{1}{n} \sum_{i=1}^{n} Z_i$,

$$\sum_{i=1}^{n} c_i \phi_i(X_{(i)}) \overset{\text{st}}{\geqq} \sum_{i=1}^{n} c_i \phi_i(Y_{(i)} + \bar{Z}) \; . \tag{2.12}$$

(b) *If* $\phi_i(x) = ax + b$, $a \geqq 0$ *for* $i=1,\cdots,n$, *then the condition that* $c_i \geqq 0$ *for* $i=1,\cdots,n$ *in* (a) *can be removed.*

Theorem 2.4 is a generalization of Theorem 2.2 and it reduces to Theorem 2.2 when $\boldsymbol{Z}_1$, $\boldsymbol{Z}_2$ are degenerate random vectors. It can still be partially extended using the concept of stochastic weak majorization (see Nevius, Proschan and Sethuraman [8] for definition and results). This is given below.

THEOREM 2.5. *In Theorem* 2.4(a), *if the condition that* $\boldsymbol{Z}_1 \overset{\text{st}}{\succ} \boldsymbol{Z}_2$ *is replaced by* $\boldsymbol{Z}_1 \overset{\text{st}}{\succ\succ} \boldsymbol{Z}_2$, *then the inequalities in* (2.11) *and* (2.12) *remain true.*

PROOFS OF THEOREMS 2.4 AND 2.5. The proofs are similar and they involve the following steps: (a) By Theorem 17.B.1 in Marshall and Olkin ([5], p. 483), $\boldsymbol{Z}_1 \overset{\text{st}}{\succ} \boldsymbol{Z}_2$ ($\boldsymbol{Z}_1 \overset{\text{st}}{\succ\succ} \boldsymbol{Z}_2$) implies that there exist random vectors $\tilde{\boldsymbol{Z}}_1$, $\tilde{\boldsymbol{Z}}_2$ such that $\boldsymbol{Z}_j$ and $\tilde{\boldsymbol{Z}}_j$ are identically distributed ($j=1,2$) and $\tilde{\boldsymbol{Z}}_1 \overset{\text{st}}{\succ} \tilde{\boldsymbol{Z}}_2$ a.s. ($\tilde{\boldsymbol{Z}}_1 \overset{\text{st}}{\succ\succ} \tilde{\boldsymbol{Z}}_2$ a.s.). Since $(\boldsymbol{Y}+\boldsymbol{Z}_j)$ and $(\boldsymbol{Y}+\tilde{\boldsymbol{Z}}_j)$ are identically distributed, we can consider the expectations of functions of $\boldsymbol{Y}+\tilde{\boldsymbol{Z}}_j$ ($j=1,2$) instead. (b) Applying Theorem 2.2, we can establish inequalities for the conditional probabilities for every given $(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = (\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)$ such that $\boldsymbol{\delta}_1 \succ \boldsymbol{\delta}_2$ ($\boldsymbol{\delta}_1 \succ\succ \boldsymbol{\delta}_2$). The proof is complete by taking expectations of the conditional probabilities.

We observe here the known fact that if $\boldsymbol{Z}_1 \overset{\text{st}}{\succ} \boldsymbol{Z}_2$ or $\boldsymbol{Z}_1 \overset{\text{st}}{\geqq} \boldsymbol{Z}_2$, then $\boldsymbol{Z} \overset{\text{st}}{\succ\succ} \boldsymbol{Z}_2$. Furthermore, we observe that if $\boldsymbol{Z}_1 = (U_1,\cdots,U_n)$, $\boldsymbol{Z}_2 = (V_1,\cdots,V_n)$ and $U_1,\cdots,U_n$ ($V_1,\cdots,V_n$) are i.i.d. random variables, then $\boldsymbol{Z}_1 \overset{\text{st}}{\succ\succ} \boldsymbol{Z}_2$ if and only if $U_1 \overset{\text{st}}{\geqq} V_1$. The proof of this result is left to the reader.

Remark. As a final remark, we note that the functions $\phi_i(x)$ are assumed differentiable everywhere in Conditions 2.1 and 2.2. We make this differentiability assumption there mainly to take advantage of a direct application of the result in Marshall and Olkin ([5], p. 92) for proving Lemma 2.1. But this assumption is not always satisfied, hence the following observation seems in order: If we modify part (b) in Condition 2.1 (Condition 2.2) to read "For all $x_1 > x_2 \in R^1[\phi_i(x_1)-\phi_i(x_2)]$ is monotonically nondecreasing (nonincreasing) in i", and if we further impose that the distribution of $\boldsymbol{Y}$ is absolutely continuous w.r.t. the Lebesgue measure (which is already satisfied in Theorems 2.2, 2.4 and 2.5), then the statements in Theorems 2.1–2.5 remain true. To see that, let

$$B \equiv \bigcup_{i=1}^{n} \{x \mid x \in R^1,\ \phi_i'(x) \text{ fails to exist}\} .$$

Then B is a countable set and $\phi_i'(x)$ is continuous and nondecreasing on $R^1 \backslash B$ for each i (see Roberts and Varberg ([10], pp. 5–7)). With the new condition stated above it can be verified similarly that the statement in Lemma 2.1 remains true except perhaps on a set of Lebesgue measure zero. Thus the statements of the theorems remain true when the distribution of $\boldsymbol{Y}$ is further assumed to be absolutely continuous.

3. Some applications

In this section we look at a number of situations in which the theorems of Section 2 apply. These applications are presented for the purpose of illustration; so, of course, this list is not complete.

Application 3.1 (Bounds for extreme order statistics and for the range). It was shown by Hartley and David [4] that if $Y_1, \cdots, Y_n$ are i.i.d. continuous random variables with means $\bar{\nu}$ and variances σ^2 (say), then $\mathrm{E}\, Y_{(n)} \leqq \bar{\nu} + \sigma(n-1)/(2n-1)^{1/2}$. Combining this with Theorem 2.1, we note that under the model (2.1) $\mu_{(n)} = \mathrm{E}\, X_{(n)}$ is bounded above by

$$\mathrm{E}\, X_{(n)} \leqq \bar{\nu} + \sigma(n-1)/(2n-1)^{1/2} + \delta_{(n)} . \tag{3.1}$$

Similarly after a sign change we have

$$\mu_{(1)} = \mathrm{E}\, X_{(1)} \geqq \bar{\nu} - \sigma(n-1)/(2n-1)^{1/2} + \delta_{(1)} . \tag{3.2}$$

Therefore the expected value of the range is bounded above by

$$\mu_{(n)} - \mu_{(1)} \leqq 2\sigma(n-1)/(2n-1)^{1/2} + (\delta_{(n)} - \delta_{(1)}) . \tag{3.3}$$

Application 3.2 (Location parameter families). Let $\boldsymbol{X}$ have a density $f(\boldsymbol{x}-\boldsymbol{\theta})$. Then $\boldsymbol{X}$ can be expressed as $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{\theta}$, where the density of $\boldsymbol{Y}$ is $g(\boldsymbol{x})=f(\boldsymbol{x}-\boldsymbol{0})$. Thus for all location parameter families Fact 2.1 and Theorem 2.1 apply with $\boldsymbol{\delta}$ replaced by $\boldsymbol{\theta}$. As a consequence, Theorem 2.1 implies that, for $c_1 \leqq \cdots \leqq c_n$,

$$\text{(3.4)} \qquad \mathrm{P}\left[\sum_{i=1}^{n} c_i Y_{(i)} \leqq t - \sum_{i=1}^{n} c_i \theta_{(i)}\right] \leqq \mathrm{P}_{\boldsymbol{\theta}}\left[\sum_{i=1}^{n} c_i X_{(i)} \leqq t\right] \leqq \mathrm{P}\left[\sum_{i=1}^{n} c_i Y_{(i)} \leqq t - \sum_{i=1}^{n} c_i \theta_{[i]}\right]$$

for all t. This includes the case when $X_1, \cdots, X_n$ are independent with densities $f^*(x_i-\theta_i)$, but independence is not essential.

If in addition $g(\boldsymbol{x})$ is a Schur-concave function of $\boldsymbol{x}$, then Theorem 2.2 also applies. Consequently, for $c_1 \leqq \cdots \leqq c_n$

$$\text{(3.5)} \qquad \mathrm{P}_{\boldsymbol{\theta}}\left[\sum_{i=1}^{n} c_i X_{(i)} \leqq t\right] \leqq \mathrm{P}\left[\sum_{i=1}^{n} c_i Y_{(i)} \leqq t - \bar{\theta} \sum_{i=1}^{n} c_i\right]$$

holds for all t. This includes the case in which $g(\boldsymbol{x})$ is permutation symmetric and unimodal. As a consequence, it includes the case in which $X_1, \cdots, X_n$ are independent with densities $f^*(x_i-\theta_i)$ such that $f^*(x)$ is a log-concave function of x.

Application 3.3 (Elliptically contoured distributions with a shift in location). Let the random vector $\boldsymbol{X}$ have a density function $f(\boldsymbol{x}, \boldsymbol{\theta})= h((\boldsymbol{x}-\boldsymbol{\theta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\theta})')$ where $h: R^1 \to [0, \infty)$ and $\boldsymbol{\Sigma}=(\sigma_{ij})$ is such that $\sigma_{ij}=\sigma^2$ for $i=j$, $\sigma_{ij}=\rho\sigma^2$ for $i \neq j$, $\sigma^2>0$, $\rho \in (-1/(n-1), 1)$. If h is nondecreasing, then $f(\boldsymbol{x}, \boldsymbol{0})$ is a Schur-concave function of $\boldsymbol{x}$ (see Marshall and Olkin ([5], p. 300)). Thus Fact 2.1 and Theorems 2.1, 2.2 apply to this family of distributions.

Application 3.4 (Multivariate normal distribution). Let $\boldsymbol{X} \sim N(\boldsymbol{0}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma}=(\sigma_{ij})$ then, by Application 3.2, Fact 2.1 and Theorem 2.1 apply with $\boldsymbol{\delta}=\boldsymbol{\theta}$. If in addition $\boldsymbol{\Sigma}$ is of the form described in Application 3.3, then Theorem 2.2 also applies. The latter case is of special importance, and two special applications are given below:

(a) For $c_1 \leqq \cdots \leqq c_n$, $\bar{c}=\frac{1}{n}\sum_{i=1}^{n} c_i$, $\bar{\theta}=\frac{1}{n}\sum_{i=1}^{n}\theta_i$, Theorem 2.2 yields

$$\text{(3.6)} \qquad \sum_{i=1}^{n} c_i \nu_{(i)} + \sum_{i=1}^{n} c_i(\theta_{(i)}-\bar{\theta}) \geqq \sum_{i=1}^{n} c_i(\mathrm{E}\, X_{(i)} - \mathrm{E}\, \bar{X}) \geqq \sum_{i=1}^{n} c_i \nu_{(i)}\,,$$

where $\nu_{(i)}$ is the expectation of $X_{(i)}$ when $\boldsymbol{\theta}=\boldsymbol{0}$. The middle term in (3.6) is a measure of the diversity of the means of order statistics, and the bounds in (3.6) can be calculated numerically from existing table values of $\nu_{(i)}$ when $\rho=0$ and a result in Owen and Steck [9]. Comparing (3.6) with the distribution-free inequality

$$(3.7)\qquad \left|\sum_{i=1}^{n} c_i(\mathrm{E}\, X_{(i)}-\mathrm{E}\,\bar{X})\right| \leqq \left[\left\{\sum_{i=1}^{n}(c_i-\bar{c})^2\right\}\left\{n\sigma^2+\sum_{i=1}^{n}(\theta_i-\bar{\theta})^2\right\}\right]^{1/2},$$

which follows from a result in Arnold and Groeneveld [1], the bounds in (3.6) are significantly better for most $\boldsymbol{\theta}$ values.
(b) There are existing statistical tables for the means of normal order statistics in the presence of an outlier for selected configurations $\boldsymbol{\theta}$. Obviously it is impossible to calculate table values for all such configurations, thus Theorem 2.2 becomes useful for obtaining bounds for a particular $\boldsymbol{\theta}$ vector based on existing table values. For the treatment of outlier in statistical applications, see, e.g. David ([2], pp. 170–195).

Application 3.5 (Power of Tukey's studentized range test). Tukey's studentized range test depends on a statistic $(t_{(n)}-t_{(1)})$, where $t_i=X_i/S$, $\boldsymbol{X}\sim N(\boldsymbol{\theta}, \boldsymbol{\Sigma})$, $\boldsymbol{\Sigma}$ is as in Application 3.3, $rS^2/\sigma^2\sim\chi^2(r)$ and S and $\boldsymbol{X}$ are independent. It is known that the distribution of $(t_1,\cdots,t_n)$ is elliptically contoured (see e.g. Tong ([11], p. 75)). Thus by Theorem 2.2 the power of the test $\pi(\boldsymbol{\theta})$ is a Schur-convex function of $\boldsymbol{\theta}$. In particular, $\pi(\boldsymbol{\theta})\geqq\pi(\bar{\boldsymbol{\theta}})$ holds for all $\boldsymbol{\theta}$.

Application 3.6 (Analysis-of-variance problems). Consider the one-way analysis-of-variance model

$$(3.8)\qquad Y_{ij}=\mu+\alpha_i+\varepsilon_{ij}\,,\qquad j=1,\cdots,J,\ i=1,\cdots,I,$$

where $\sum_{i=1}^{I}\alpha_i=0$; and consider the analysis of variance procedures which depend on the order statistics as described in David ([2], p. 158). It is easy to see that under the assumption of normality the power function $\pi(\boldsymbol{\alpha})$ is Schur-convex in $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_I)$. This remains true even without the assumption of normality as long as the density of $(\varepsilon_{11},\cdots,\varepsilon_{IJ})$ is Schur-concave and S^2 is independent of $(\bar{Y}_1,\cdots,\bar{Y}_I)$. Moreover, as noted in David ([2], p. 158) this type of argument also applies to random blocks and Latin square designs among others.

Application 3.7 (Censored data). In many statistical applications the estimation (or hypothesis-testing) of a location parameter may depend on censored data (see, e.g. David ([2], p. 109)). When a known number of k observations is missing at either end, we can choose $c_1=\cdots=c_k=0$, $c_{k+1}=\cdots=c_n=1$ or $c_1=\cdots=c_{n-k}=1$, $c_{n-k+1}=\cdots=c_n=0$. In this case the theorems in Section 2 apply.

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