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ESTIMATION OF FREQUENCY BY RANDOM SAMPLING

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Summary

We consider the estimation of frequency ω of a sinusoidal oscillation contaminated by a stationary noise under a random sampling scheme according to a stationary point process N. We prove the strong consistency and the asymptotic normality for a certain estimator of ω. Then we apply these results to the case where N is a stationary delayed renewal process.

1. Introduction

Let $Y=\{Y(t);\ t\in R\}$ be a stationary stochastic process, and $N=\{N(B);\ B$ is any bounded Borel set in $R\}$ be a stationary orderly point process which is independent of Y. Suppose that Y is sampled at epochs when events of N occur. Then, Brillinger [4] studied the estimation of mean of Y, and Brillinger [3] and Masry [11] studied the estimation of spectral density of Y. In this paper we investigate the estimation of the discrete spectrum of Y under the same random sampling scheme. More precisely, we consider the model $Y(t)=m(t)+X(t),\ t\in R$, where $m(t)$ is a nonrandom trigonometric polynomial with unknown parameters, and $X=\{X(t);\ t\in R\}$ a stationary stochastic process with zero mean. We shall propose certain estimators of these unknown parameters based on sample $\{Y(t_k);\ 1\leqq k\leqq K\}$, where $\{t_k;\ 1\leqq k\leqq K\}$ are epochs when events of N occur in a time interval $(0, T]$. And we shall investigate the strong consistency and the asymptotic normality of these estimators as T tends to infinity.

In what follows we study only the simplest case that $m(t)=a\cos\omega t+b\sin\omega t$, where ω, a, and b are unknown parameters such that $\omega>0$ and $a^2+b^2>0$. The general case that $m(t)$ contains more than one harmonic oscillations as well as a nonzero constant term can be studied in much the same way as the present case (see Walker [16], Hannan [8] and Ivanov [10]). Suppose that Y is sampled randomly according to N

Key words: Discrete spectrum, random sampling, stationary delayed renewal process.

in a time interval $(0, T]$. Then, following Walker [16] and Hannan [8], we introduce a periodogram

$$I_T(\lambda)=\left|\frac{1}{T}\int_0^T \exp(i\lambda t)Y(t)dN(t)\right|^2.$$

Suppose that we know that ω is contained in an interval $(0, \alpha)$, where α does not depend on N and is allowed to be arbitrarily large. Now we define an estimator of ω, $\hat{\omega}_T$, as the value λ which maximizes $I_T(\lambda)$ in the interval $[0, \alpha]$. Furthermore, assuming that the sampling rate, $\beta=\mathrm{E}(N(0,1])$, is known, we define estimators of a and b by

$$\hat{a}_T=\frac{2}{\beta T}\int_0^T \cos\hat{\omega}_T t\, Y(t)dN(t)$$

and

$$\hat{b}_T=\frac{2}{\beta T}\int_0^T \sin\hat{\omega}_T t\, Y(t)dN(t)$$

respectively (When β is unknown and N is ergodic, we can replace β by $\hat{\beta}_T=T^{-1}N(0, T]$ in the above definition without any change in the conclusion of this paper.).

In Section 2 we show the strong consistency of estimators $\hat{\omega}_T$, $\hat{a}_T$, and $\hat{b}_T$ under the assumption that both X and N are purely nondeterministic. In Section 3 we show the asymptotic normality of them under certain mixing conditions on X and N. In Section 4 we investigate the applicability of these results for the random sampling scheme according to a stationary delayed renewal process N. In particular, we show that, for the consistent estimation of frequency, that is, the discrete spectrum, it is sufficient to assume that the common probability distribution of time intervals of N is absolutely continuous and has a finite second moment. Thus we need not assume the condition that time intervals of arbitrarily small length appear in each realization of N. On the contrary, it is known that we need this condition for the consistent estimation of spectral density under the random sampling scheme (See Corollary 1.3 of Masry [11]. For the discrete time parameter versions of this Corollary, see Bloomfield [1] and Blum and Rosenblatt [2].).

Recently Vere Jones [15] studied a closely related problem to ours. He investigated the estimation of frequency of the periodic intensity function of a Poisson process. While we treat more general point processes than those in his paper, he discussed some interesting problems which we do not consider here.

In what follows, the stationarity always means that in the both strict and wide sense. As usual, $\boldsymbol{Z}$ and $\boldsymbol{R}$ stand for the set of integers and the set of real numbers respectively.

2. Strong consistency

In this section we prove the strong consistency of estimators $\hat{\omega}_T$, $\hat{a}_T$, and $\hat{b}_T$. Slightly generalizing the condition assumed by Hannan [8], we assume that X is purely nondeterministic. We need a similar assumption on N. In fact, consider a stationary deterministic point process with interval length d (see Daley and Vere Jones [5], p. 311). Then, using a realization of Y, we can not identify ω, even if X is identically zero. It is easily seen that this point process N has the purely atomic spectral measure $\sum_{m\in Z, m\neq 0} \delta_{2\pi m d^{-1}}$, where $\delta_{2\pi m d^{-1}}$ denotes the Dirac measure at point $2\pi m d^{-1}$. Considering this example, in Theorem 1, we assume that N is also purely nondeterministic.

THEOREM 1. *Assume that both X and N are purely nondeterministic. Then, as $T\to\infty$, it holds that $T(\hat{\omega}_T-\omega)\to 0$, $\hat{a}_T\to a$, and $\hat{b}_T\to b$ almost surely.*

In order to prove Theorem 1, we need several lemmas. The following lemma is a slight generalization of the lemma of Hannan [8].

LEMMA 1. *Let $\{x(n);\ n\in Z\}$ be a stationary stochastic process which is purely nondeterministic and of zero mean. We put*

$$I_n=\sup_{0\leq\lambda\leq\pi}\left|n^{-p-1}\sum_{k=1}^{n}k^p\exp(i\lambda k)x(k)\right|$$

for a nonnegative constant p. Then, $\lim_{n\to\infty} I_n=0$ almost surely (a.s.).

PROOF. We prove this lemma in the same way as in Hannan [8]. Let $\mathcal{F}_n$ be the σ-field generated by $\{x(m);\ m\leqq n\}$, and H_n be the Hilbert space of random variables measurable with respect to $\mathcal{F}_n$ and of finite second moment. Defining $u(n, j)$ to be the orthogonal projection of $x(n)$ on $H_j\ominus H_{j-1}$, we have $x(n)=\sum_{j=0}^{\infty}u(n, n-j)$. We put

$$v_r(n)=\sum_{j=r+1}^{\infty}u(n, n-j)\ , \qquad \xi_j(n)=u(n, n-j)I(|u(n, n-j)|\leqq C)\ ,$$

$$\varepsilon_j(n)=\xi_j(n)-\mathrm{E}\,(\xi_j(n)\,|\,\mathcal{F}_{n-j-1})\ , \qquad \eta_j(n)=u(n, n-j)-\varepsilon_j(n)\ ,$$

and

$$I_{n,j}=\sup_{0\leq\lambda\leq\pi}\left|n^{-p-1}\sum_{k=1}^{n}k^p\exp(i\lambda k)\varepsilon_j(k)\right|$$

for $j=0, 1,\cdots, r$, where $I(\ )$ denotes the indicator function of a set, and C is a positive constant. It holds that

$$I_n \leqq \left(n^{-1}\sum_{k=1}^{n} v_r(k)^2\right)^{1/2} + \sum_{j=0}^{r}\left(I_{n,j} + n^{-1}\sum_{k=1}^{n}|\eta_j(k)|\right).$$

Then, by the ergodic theorem, we have

$$\limsup_{n\to\infty} I_n \leqq \mathrm{E}\,(v_r(0)^2|\mathcal{G})^{1/2} + \sum_{j=0}^{r}(\limsup_{n\to\infty} I_{n,j} + \mathrm{E}\,(|\eta_j(0)||\mathcal{G})\,,$$

where $\mathcal{G}$ is the σ-field of sets invariant with respect to the measure preserving transformation defined by $\{x(n);\ n\in Z\}$. Since, for each j, $\{\varepsilon_j(n);\ n\in Z\}$ is a stationary bounded martingale difference, we have $\lim_{n\to\infty} I_{n,j}=0$ a.s. as is shown in Hannan [8]. Since $\mathcal{G}\subset\mathcal{F}_{-\infty}=\bigcap_{n\in Z}\mathcal{F}_n$ by Lemma 4.6.1 of Rozanov [12], it holds that

$$\mathrm{E}\,(v_r(0)^2|\mathcal{G}) = \mathrm{E}\,(\mathrm{E}\,(v_r(0)^2|\mathcal{F}_{-r-2})|\mathcal{G}) \geqq \mathrm{E}\,(v_{r+1}(0)^2|\mathcal{G})\,.$$

Then, noting that $\lim_{r\to\infty}\mathrm{E}\,(v_r(0)^2)=0$, we have $\lim_{r\to\infty}\mathrm{E}\,(v_r(0)^2|\mathcal{G})=0$ a.s. Similarly, from the fact that $|\eta_j(0)|\leqq 2|u(0,-j)|I(|u(0,-j)|>C)$, we can show $\lim_{C\to\infty}\mathrm{E}\,(|\eta_j(0)||\mathcal{G})=0$ a.s. Accordingly we obtain $\lim_{n\to\infty} I_n=0$ a.s. Thus we complete the proof of the lemma.

We define a stationary random measure Z by $dZ=XdN$, that is, $Z(B)=\int_B X(t)dN(t)$ for any bounded Borel set B. We denote spectral densities of X and N by g and f respectively. From the formula (5.16) of Daley and Vere Jones [5], the random measure Z has a spectral density $h(\lambda)=g*f(\lambda)+\beta^2 g(\lambda)$, where $*$ denotes the convolution operation. We put $Z_\delta(n)=Z(n\delta,(n+1)\delta]$ for a positive constant δ.

LEMMA 2. *Assume that g is bounded on R. Then, a stationary stochastic process $Z_\delta=\{Z_\delta(n);\ n\in Z\}$ has a spectral density*

$$h_\delta(\lambda)=4\sin^2(2^{-1}\delta\lambda)\sum_{n\in Z}(\lambda+2\pi n\delta^{-1})^{-2}h(\lambda+2\pi n\delta^{-1})\,,\qquad |\lambda|<\pi\delta^{-1}\,.$$

PROOF. We define a stationary process $\{Z(t);\ t\in R\}$ by $Z(t)=\int_R I_{[0,\delta)}(t-u)dZ(u)$, where $I_{[0,\delta)}$ denotes the indicator function of an interval $[0,\delta)$. We approximate $I_{[0,\delta)}$ by a sequence of infinitely differentiable functions ϕ_n such that the support of ϕ_n is $[-n^{-1},\delta]$, $0\leqq\phi_n\leqq 1$ on R, $\phi_n=1$ identically on $[0,\delta-n^{-1}]$, and $|\hat{\phi}_n(\lambda)|\leqq C(1+|\lambda|)^{-1}$ for all $\lambda\in R$, where C is a positive constant and $\hat{\phi}_n$ is the Fourier transform of ϕ_n. We define a stationary process $\{Z_n(t);\ t\in R\}$ by $Z_n(t)=\int_R\phi_n(t-u)\cdot dZ(u)$. Then we can easily show that $\lim_{n\to\infty}\mathrm{E}\,(Z_n(t)-Z(t))^2=0$. Now, from Theorem 2 of Vere Jones [14], $\{Z_n(t);\ t\in R\}$ has a spectral density $|\hat{\phi}_n(\lambda)|^2 h(\lambda)$. Accordingly, $\{Z(t);\ t\in \boldsymbol{R}\}$ has a spectral density $|\hat{I}_{[0,\delta)}|^2 \cdot h(\lambda)$, where $\hat{I}_{[0,\delta)}$ denotes the Fourier transform of $I_{[0,\delta)}$. Hence we can derive a spectral density of Z_δ immediately. Thus the proof is completed.

LEMMA 3. *Under the same assumption as in Theorem* 1,

$$\lim_{T\to\infty} \sup_{0\leq\lambda\leq\alpha} \left| T^{-p-1}\int_0^T t^p \exp(i\lambda t)(Y(t)dN(t)-\beta m(t)dt)\right| =0$$

a.s. for any constants $p\geqq 0$ *and* $\alpha>0$.

PROOF. Obviously the problem can be reduced to proving that

$$(1) \qquad \lim_{T\to\infty} \sup_{0\leq\lambda\leq\alpha} \left| T^{-p-1}\int_0^T t^p \exp(i\lambda t)d\dot{Z}(t)\right| =0 \text{ a.s.}$$

and

$$(2) \qquad \lim_{T\to\infty} \sup_{0\leq\lambda\leq\alpha} \left| T^{-p-1}\int_0^T t^p \exp(i\lambda t)m(t)(dN(t)-\beta dt)\right| =0 \text{ a.s.}$$

We shall prove only (1), since (2) can be proved similarly. As is easily seen, it suffices to prove (1) when T tends to infinity through $\{n\delta;\ n=1,2,\cdots\}$. Now, defining a random measure $|Z|$ by $d|Z|(t)=|X(t)|\cdot dN(t)$, we have

$$(3) \qquad \left|(n\delta)^{-p-1}\int_0^{n\delta} t^p \exp(i\lambda t)dZ(t)-(n\delta)^{-p-1}\sum_{j=0}^{n-1}(j\delta)^p \exp(i\lambda j\delta)Z_\delta(j)\right|$$
$$\leqq (p+\alpha)n^{-1}\sum_{j=0}^{n-1}|Z|(j\delta,(j+1)\delta]\ .$$

By the ergodic theorem, the right hand side of (3) converges to $(p+\alpha)\,\mathrm{E}\,(|Z|(0,\delta]\,|\,\mathcal{F})$ a.s. as $n\to\infty$, where $\mathcal{F}$ is the σ-field of sets invariant with respect to the measure preserving transformation defined by both X and N. Since $\lim_{\delta\to 0}\mathrm{E}\,(|Z|(0,\delta]\,|\,\mathcal{F})=0$ a.s., it suffices to show that

$$(4) \qquad \lim_{n\to\infty} \sup_{0\leq\lambda\leq\alpha} \left| n^{-p-1}\sum_{j=0}^{n-1} j^p \exp(i\lambda j\delta)Z_\delta(j)\right| =0 \text{ a.s.}$$

Since both X and N are purely nondeterministic, so is Z_δ. Accordingly, (4) follows immediately from Lemma 1. Thus the proof is completed.

PROOF OF THEOREM 1. We can prove the theorem in the same way as in Walker [16] or Hannan [8]. Thus we give only a sketch of the proof. First we note that

$$\limsup_{T\to\infty} \sup_{|\lambda-\omega|\geq\eta T^{-1}} \left| T^{-1}\int_0^T m(t)\exp(i\lambda t)dt\right|^2$$

$$<4^{-1}(a^2+b^2)=\lim_{T\to\infty}\left|T^{-1}\int_0^T m(t)\exp(i\lambda t)dt\right|^2,$$

where η is a positive constant. Then, using Lemma 3 with $p=0$, and recalling the definition of $\hat{\omega}_T$, we have $|\hat{\omega}_T-\omega|<\eta T^{-1}$ for all sufficiently large T a.s. Since η can be made arbitrarily small, we obtain $\lim_{T\to\infty} T(\hat{\omega}_T-\omega)=0$ a.s. Using this result, we can prove the strong consistency of $\hat{a}_T$ and $\hat{b}_T$. Thus the proof is completed.

3. Asymptotic normality

In this section we prove the asymptotic normality of estimators $\hat{\omega}_T$, $\hat{a}_T$, and $\hat{b}_T$. Let $\mathcal{M}_t$ and $\mathcal{N}_t$ be the σ-fields generated by $\{X(s);\ s\leqq t\}$ and $\{N(B);\ B$ is any bounded Borel set contained in $(-\infty, t]\}$ respectively. We put $N_\delta(k)=\mathrm{E}(N(k\delta, (k+1)\delta]\,|\,\mathcal{N}_0)-\beta\delta$ for a positive constant δ. Furthermore, we put

$$\phi_\delta(n, k)=\sup\{|\mathrm{E}(X(t)\,\mathrm{E}(X(s)\,|\,\mathcal{M}_0))|;\ n\delta\leqq s\leqq(n+1)\delta,\ k\delta\leqq t\leqq(k+1)\delta\}$$

and

$$\psi_\delta(n, k)=|\mathrm{E}(N_\delta(k)\,\mathrm{E}(N_\delta(n)\,|\,\mathcal{N}_0))|\,.$$

In Theorem 2, we assume the following conditions;

$$\lim_{n\to\infty}\sum_{k=n+1}^{\infty}\phi_\delta(n, k)=0 \tag{5}$$

and

$$\lim_{n\to\infty}\sum_{k=n+1}^{\infty}\psi_\delta(n, k)=0 \tag{6}$$

for any $\delta>0$ (see Hall and Heyde [7], Chapter 5).

THEOREM 2. *Assume that both X and N are purely nondeterministic and weakly mixing, and satisfy the conditions* (5) *and* (6) *respectively, and moreover, assume that the covariance function of X is integrable on R and the reduced covariance measure of N is totally finite on R. Then, as $T\to\infty$, the joint distribution of $(T^{3/2}(\hat{\omega}_T-\omega), T^{1/2}(\hat{a}_T-a), T^{1/2}(\hat{b}_T-b))$ converges to the normal distribution with zero mean and covariance matrix*

$$(4\pi(a^2+b^2)^{-1}g(\omega)+4\pi\beta^{-2}(a^2+b^2)^{-1}(g*f)(\omega)+\pi\beta^{-2}f(2\omega))\Sigma_1+2\pi\beta^{-2}f(0)\Sigma_2\,,$$

where

$$\Sigma_1=\begin{pmatrix}12 & -6b & 6a\\ -6b & a^2+4b^2 & -3ab\\ 6a & -3ab & 4a^2+b^2\end{pmatrix}\quad \textit{and}\quad \Sigma_2=\begin{pmatrix}0&0&0\\0&a^2&ab\\0&ab&b^2\end{pmatrix}.$$

In order to prove Theorem 2, we need several lemmas. The following lemma is a slight modification of Theorem 1 of Hannan [9].

LEMMA 4. *Let U be a measure preserving transformation, $\mathcal{F}_0$ be a σ-field such that $\mathcal{F}_0 \subset U^{-1}\mathcal{F}_0$, and $x(0)$ be a random variable measurable with respect to $\mathcal{F}_0$ and with zero mean and finite variance. We use the same notation for the unitary transformation which is defined by U in the Hilbert space of random variables with finite second moment. We put $x(n)=U^n x(0)$, and $c(n)=n^p \cos \lambda n$ or $n^p \sin \lambda n$, where p is a non-negative constant and λ is a number such that $0<\lambda<\pi$. Now we assume that the stationary process $\{x(n);\ n \in \boldsymbol{Z}\}$ is purely nondeterministic and weakly mixing, and satisfies the condition*

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} |\mathrm{E}\,(x(k)\,\mathrm{E}\,(x(n)\,|\,\mathcal{F}_0))|=0\ .$$

Then, the distribution of $n^{-p-2^{-1}} \sum_{k=1}^{n} c(k)x(k)$ converges to the normal distribution with zero mean and variance $\pi(2p+1)^{-1}s(\lambda)$, where s denotes a spectral density of $\{x(n);\ n \in \boldsymbol{Z}\}$.

PROOF. We put

$$d(n)^2=\sum_{k=1}^{n} c(k)^2\ , \qquad y(n)=d(n)^{-1}\sum_{k=1}^{n} c(k)x(k)\ ,$$

and

$$y_r(n)=d(n)^{-1}\sum_{k=1}^{n} c(k)(x(k)-\mathrm{E}\,(x(k)\,|\,\mathcal{F}_{k-r}))\ ,$$

where $\mathcal{F}_n=U^{-n}\mathcal{F}_0$. Then we have easily

$$\mathrm{E}\,(y(n)-y_r(n))^2 \leqq \mathrm{E}\,(\mathrm{E}\,(x(0)\,|\,\mathcal{F}_{-r})^2)+2\sum_{k=r+1}^{\infty} |\mathrm{E}\,(x(k)\,\mathrm{E}\,(x(r)\,|\,\mathcal{F}_0))|\ ,$$

which can be made arbitrarily small by letting r large. Now, the central limit theorem (c.l.t) for $\{y_r(n);\ n \in \boldsymbol{Z}\}$ is established in the proof of Theorem 1 of Hannan [9]. Therefore we complete the proof.

LEMMA 5. *Under the same assumption as in Theorem 2, the stationary process Z_δ is purely nondeterministic and weakly mixing, and satisfies the condition*

$$(7) \qquad \lim_{n\to\infty} \sum_{k=n+1}^{\infty} |\mathrm{E}\,(Z_\delta(k)\,\mathrm{E}\,(Z_\delta(n)\,|\,\mathcal{M}_0 \vee \mathcal{N}_0))|=0\ .$$

PROOF. First we prove (7). For brevity, we write $\mathrm{E}_l\,(\ )$ for the conditional expectation given $\mathcal{N}_0$ and $\{N(n\delta,(n+1)\delta]=l\}$ simultaneously. We put $\mathcal{N}=\bigvee_{t \in R} \mathcal{N}_t$. Then we have

$$\begin{aligned}&\mathrm{E}\,(Z_\delta(k)\,\mathrm{E}\,(Z_\delta(n)\,|\,\mathcal{M}_0\vee\mathcal{N}_0)\,|\,\mathcal{N})\\&=\sum_{i=1}^{m}\sum_{l=0}^{\infty}\mathrm{E}\left\{\mathrm{P}\,(N(n\delta,(n+1)\delta]=l\,|\,\mathfrak{N}_0)\,\mathrm{E}_l\sum_{j=1}^{l}\mathrm{E}\,(X(t_i)\,\mathrm{E}\,(X(s_j)\,|\,\mathfrak{M}_0))\right\},\end{aligned}$$

where $\{t_i;\ 1\leqq i\leqq m\}$ and $\{s_j;\ 1\leqq j\leqq l\}$ denote epochs when events of N occuring in intervals $(k\delta, (k+1)\delta]$ and $(n\delta, (n+1)\delta]$ respectively. Hence we can deduce

$$\begin{aligned}&|\mathrm{E}\,(Z_\delta(k)\,\mathrm{E}\,(Z_\delta(n)\,|\,\mathcal{M}_0\vee\mathcal{N}_0))|\\&\quad\leqq\phi_\delta(n,k)\,\mathrm{E}\,(N(k\delta,(k+1)\delta]\,\mathrm{E}\,(N(n\delta,(n+1)\delta]\,|\,\mathcal{N}_0))\\&\quad\leqq\phi_\delta(n,k)\,\mathrm{E}\,(N(0,\delta]^2)\,.\end{aligned}$$

Accordingly, the assumption (5) implies (7).

Finally, noting the mutual independence of X and N, we can prove easily that Z_δ is weakly mixing as in the lemma of Hannan [9]. Thus we complete the proof.

Now we establish the c.l.t. for

$$U_T=T^{-1/2}\int_0^T\cos\omega t\,dW(t)\,,\qquad V_T=T^{-1/2}\int_0^T\sin\omega t\,dW(t)\,,$$

$$R_T=T^{-3/2}\int_0^T t\cos\omega t\,dW(t)\,,\qquad S_T=T^{-3/2}\int_0^T t\sin\omega t\,dW(t)\,.$$

where $dW(t)=Y(t)dN(t)-\beta m(t)dt=dZ(t)+m(t)(dN(t)-\beta dt)$.

LEMMA 6. *Under the same assumption as in Theorem 2, as $T\to\infty$, the joint distribution of (U_T, V_T, R_T, S_T) converges to the normal distribution with zero mean and covariance matrix*

$$(\pi h(\omega)+4^{-1}\pi(a^2+b^2)f(2\omega))\begin{pmatrix}I & 2^{-1}I\\ 2^{-1}I & 3^{-1}I\end{pmatrix}+2^{-1}\pi f(0)\begin{pmatrix}K & 2^{-1}K\\ 2^{-1}K & 3^{-1}K\end{pmatrix},$$

where I is the 2×2-identity matrix, and

$$K=\begin{pmatrix}a^2 & ab\\ ab & b^2\end{pmatrix}.$$

PROOF. It is sufficient to prove the c.l.t. for any linear combinations of U_T, V_T, R_T, and S_T. Taking $Q_T=T^{-1/2}\int_0^T\cos\omega t\,dZ(t)$ and $P_T=T^{-1/2}\int_0^T\cos 2\omega t(dN(t)-\beta dt)$ as typical terms in these linear combinations, we shall give the detailed proof only for these terms. As is easily seen, it suffices to show the c.l.t. when T tends to infinity through $\{n\delta;\ n=1,2,\cdots\}$. We put $Q_\delta(n)=n^{-1/2}\sum_{k=0}^{n-1}Z_\delta(k)\cos\omega k\delta$. Then we have

$$\mathrm{E}\,(Q_{n\delta}-\delta^{-1/2}Q_\delta(n))^2\leqq \delta\omega^2\int_R |dC_Z(u)|\ ,$$

where $dC_Z(u)$ denotes the reduced covariance measure of Z. Denoting the covariance function of X by $R(u)$ and reduced covariance measure of N by $dC_N(u)$, we have $dC_Z(u)=R(u)dC_N(u)+\beta^2R(u)du$ (see Daley and Vere Jones [5], p. 340). Then, from the assumptions on $R(u)$ and $dC_N(u)$, it follows that $dC_Z(u)$ is totally finite on R. Accordingly, letting δ small, we can approximate $Q_{n\delta}$ arbitrarily by $\delta^{-1/2}Q_\delta(n)$ uniformly in n. Now Lemmas 4 and 5 show that, for $\delta<\pi\omega^{-1}$, the distribution of $Q_\delta(n)$ converges to the normal distribution with zero mean and variance $\pi\delta^{-1}h_\delta(\omega)$ as $n\to\infty$. Furthermore, using Lemma 2, we can easily see that $\lim_{\delta\to 0}\delta^{-2}h_\delta(\omega)=h(\omega)$. Accordingly, the distribution of Q_T converges to the normal distribution with zero mean and variance $\pi h(\omega)$ as $T\to\infty$. Similarly, using the assumption (6), we can prove the c.l.t. for P_T. Therefore, with the aid of the Cramér-Wold technique, we complete the proof.

PROOF OF THEOREM 2. We can prove the theorem in the same way as in Hannan [8]. First we note that $T^{-1/2}I_T'(\omega)=T^{3/2}(\hat\omega_T-\omega)\cdot(-T^{-2}I_T''(\tilde\omega_T))$, where I_T' and I_T'' denote the first and the second derivatives of I_T respectively, and $\tilde\omega_T$ is a number between ω and $\hat\omega_T$. Using Lemma 3, we can prove that

$$\lim_{T\to\infty} T^{-2}I_T''(\tilde\omega_T)=-(24)^{-1}\beta^2(a^2+b^2)\ \text{a.s.}$$

Moreover, we can show that

$$T^{-1/2}I_T'(\omega)=\beta(-2^{-1}bU_T+2^{-1}aV_T+bR_T-aS_T)+\varepsilon_T\ ,$$

where ε_T denotes the term which converges to zero in probability. Accordingly, Lemma 6 implies the c.l.t. for $T^{3/2}(\hat\omega_T-\omega)$. Furthermore, we can show that

$$T^{1/2}(\hat a_T-a)=2\beta^{-1}U_T-2^{-1}bT^{3/2}(\hat\omega_T-\omega)+\xi_T$$

and

$$T^{1/2}(\hat b_T-b)=2\beta^{-1}V_T+2^{-1}aT^{3/2}(\hat\omega_T-\omega)+\eta_T\ ,$$

where ξ_T and η_T denote the terms which converge to zero in probability. Hence follows the c.l.t. for $T^{1/2}(\hat a_T-a)$ and $T^{1/2}(\hat b_T-b)$. Thus the proof is completed.

Remark 1. As will be expected, when the sampling rate β tends to infinity, the asymptotic covariance matrix in Theorem 2 reduces to the same form $4\pi(a^2+b^2)^{-1}g(\omega)\Sigma_1$ as was obtained by Walker [15], Hannan

[8], and Ivanov [10] in the case of nonrandom complete sampling of Y.

4. Random sampling by stationary delayed renewal processes

In this section we study the random sampling scheme according to a stationary delayed renewal process N. Following Daley and Vere Jones [5], p. 310, we define N by specifying its points as $\{\xi_0, \xi_{-1}, \xi_0+\sum_{k=1}^{n}\eta_k, -\xi_{-1}-\sum_{k=1}^{n}\eta_{-k}\ (n=1,2,\cdots)\}$, where $\{\eta_n;\ n\in Z,\ n\neq 0\}$ is a sequence of independent and identically distributed nonnegative random variables which have a common distribution function A with a finite first moment β^{-1}, and ξ_0 and ξ_{-1} are nonnegative random variables which are independent of $\{\eta_n;\ n\in Z,\ n\neq 0\}$ and have the joint distribution $\mathrm{P}(\xi_0 \leqq u,\ \xi_{-1}\leqq v)=\beta\int_0^u (A(w+v)-A(w))dw$. We define the renewal function H by $H=\sum_{n=1}^{\infty}A^{n*}$, where A^{n*} denotes the n-fold convolution of A with itself. Then, the reduced covariance measure of N is given by $dC(u) =\beta d\delta_0(u)+\beta(dH(|u|)-\beta du)$, where δ_0 denotes the Dirac measure at the origin (see Daley and Vere Jones [5], p. 323).

THEOREM 3. *Assume that A^{n*} has a non-trivial a.c. component for some n. Then, if A has a finite second moment, N satisfies the assumption in Theorem* 1. *Furthermore, if A has a finite fourth moment, N satisfies the assumption in Theorem* 2.

PROOF. If A has a finite second moment, then, by a theorem of Smith (see Stone [13]), we have $\int_0^{\infty}|dH(u)-\beta du|<\infty$. Accordingly, $dC(u)$ is totally finite. Thus N has a bounded continuous spectral density.

Now we prove the condition (6) introduced in the Section 3. For this purpose, it suffies to show that

$$\sum_{k=0}^{\infty}\mathrm{var}\,(\mathrm{E}\,(N_\delta(k)\,|\,\mathcal{N}_0))^{1/2}<\infty \tag{8}$$

for any $\delta>0$. For brevity, we put $\delta=1$. Denote the conditional distribution function of ξ_0 given ξ_{-1} by $dA_0(u\,|\,\xi_{-1})$. Then we have

$$\begin{aligned}(9)\qquad &|\mathrm{E}\,(N(k,k+1]|\mathcal{N}_0)-\beta|\\ &\leqq\int_0^{2^{-1}k}|\mathrm{E}\,(N(k,k+1]|\xi_0=u,\ \xi_{-1})-\beta\,|\,dA_0(u\,|\,\xi_{-1})\\ &\quad+\int_{2^{-1}k}^{\infty}(\mathrm{E}\,(N(k,k+1]|\xi_0=u,\ \xi_{-1})+\beta\,|\,dA_0(u\,|\,\xi_{-1})\,.\end{aligned}$$

Note that $\mathrm{E}\,(N(k,k+1]\,|\,\xi_0=u,\ \xi_{-1})=H(k-u,\ k-u+1]$ for $0\leqq u<k$, and

$\sup_{u\geq 0} \mathrm{E}(N(k, k+1] \mid \xi_0=u, \xi_{-1}) \leqq \sup_{u\geq 0} H(u, u+1] < \infty$. Then, from (9), we can deduce

$$(10)\qquad \mathrm{var}\,(\mathrm{E}\,(N_\delta(k) \mid \mathcal{N}_0))^{1/2} \leqq \mathrm{var}\,(\varepsilon(2^{-1}k)+C_1\,\mathrm{P}\,(\xi_0>2^{-1}k \mid \xi_{-1}))^{1/2} \leqq \varepsilon(2^{-1}k)+C_1\,\mathrm{P}\,(\xi_0>2^{-1}k)^{1/2}\,,$$

where $\varepsilon(2^{-1}k)=\sup_{u\geq 2^{-1}k} |H(u, u+1]-\beta|$, and C_1 is a positive constant. Using a theorem of Stone [13], we have $\varepsilon(2^{-1}k)\leqq C_2k^{-2}$, and moreover, noting that $\xi_0+\xi_{-1}$ has the distribution function $\beta u dA(u)$, we have $\mathrm{P}\,(\xi_0>2^{-1}k) \leqq C_3k^{-3}$, where C_2 and C_3 are positive constants. Then (8) follows from (10) immediately.

Next we show that N is purely nondeterministic. We put $\mathcal{N}_{-\infty}=\bigcap_{t\in R} \mathcal{N}_t$. By the martingale convergence theorem and the fact (8), we have

$$\begin{aligned}\mathrm{var}\,(\mathrm{E}\,(N(k\delta, (k+1)\delta] \mid \mathcal{N}_{-\infty})) &= \lim_{n\to\infty} \mathrm{var}\,(\mathrm{E}\,(N(k\delta, (k+1)\delta] \mid \mathcal{N}_{-n\delta}))\\ &= \lim_{n\to\infty} \mathrm{var}\,(\mathrm{E}\,(N((k+n)\delta, (k+n+1)\delta] \mid \mathcal{N}_0))\\ &=0\,.\end{aligned}$$

Thus, $\mathrm{E}\,(N(k\delta, (k+1)\delta] \mid \mathcal{N}_{-\infty})$ is a constant a.s. for any $k\in \boldsymbol{Z}$ and $\delta>0$. Then we can easily show that $\mathrm{E}\,(N(B) \mid \mathcal{N}_{-\infty})$ is a constant a.s. for any bounded Borel set B. Thus N is purely nondeterministic.

Finally we prove that N is mixing, a fortiori, weakly mixing. As is easily seen, for this purpose, it is sufficient to show that

$$(11)\qquad \lim_{t\to\infty} \mathrm{E}\,|\mathrm{P}\,(M(t)\in S \mid \mathcal{N}_0)-\mathrm{P}\,(M(t)\in S)|=0\,,$$

for any subset S on $\boldsymbol{Z}^n$ and for $M(t)=(N(B_1+t), N(B_2+t),\cdots, N(B_n+t))$, where B_j $(j=1, 2,\cdots, n)$ are any bounded Borel sets contained in $(0, \infty)$, and $B_j+t=\{u+t;\ u\in B_j\}$. Denote the distribution function of ξ_0 by A_0. For any $t>0$, we have

$$(12)\qquad \left|\mathrm{P}\,(M(t)\in S \mid \mathcal{N}_0)-\int_0^{2^{-1}t} \mathrm{P}\,(M(t)\in S \mid \xi_0=u)dA_0(u \mid \xi_{-1})\right| \leqq \mathrm{P}\,(\xi_0>2^{-1}t \mid \xi_{-1})\,,$$

and

$$(13)\qquad \left|\mathrm{P}\,(M(t)\in S)-\int_0^{2^{-1}t} \mathrm{P}\,(M(t)\in S \mid \xi_0=u)dA_0(u)\right| \leqq \mathrm{P}\,(\xi_0>2^{-1}t)\,.$$

Now, a renewal theorem shows that $\lim_{t\to\infty} \mathrm{P}\,(M(t)\in S \mid \xi_0=0)$ exists (see Feller [6], p. 379). We denote this limit by p. Then, noting that $\mathrm{P}\,(M(t)\in S \mid \xi_0=u)=\mathrm{P}\,(M(t-u)\in S \mid \xi_0=0)$ for $0\leqq u<t$ and using (12) and

(13), we can deduce

$$|P(M(t)\in S|\mathcal{N}_0)-P(M(t)\in S)| \leqq \sup_{s\geqq 2^{-1}t}|P(M(s)\in S|\xi_0=0)-p|+2(P(\xi_0>2^{-1}t|\xi_{-1})+P(\xi_0>2^{-1}t)).$$

Hence (11) follows immediately. Thus we complete the proof of the theorem.

Remark 2. Suppose that A is a uniform distribution on an interval $(d-\varepsilon, d+\varepsilon)$, where d and ε are constants such that $d>\varepsilon>0$. Then, even if ε is arbitrarily small, that is, N is nearly a deterministic point process with interval length d, N satisfies the assumptions in Theorem 3. Another interesting point process which satisfies these assumptions is a Poisson process.

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