Ann. Inst. Statist. Math.
35 (1983), Part A, 193-199

# AN ADMISSIBLE ESTIMATOR IN THE ONE-PARAMETER EXPONENTIAL FAMILY WITH AMBIGUOUS INFORMATION

YASUSHI NAGATA

(Received June 23, 1982; revised Sept. 30, 1982)

## Summary

Let $X$ be a random variable from the one-parameter exponential family with the probability element $\beta(\theta)\exp(\theta x)dm(x)$ for which an ambiguous prior information is available to the effect that $\theta$ is likely to be larger than or equal to a known constant. The information is represented by a fuzzy set with the membership function $\chi(\theta)$. Then it is shown that $X+\int_{-\infty}^{\infty}\chi'(\theta)\beta(\theta)\exp(\theta X)d\theta\Big/\int_{-\infty}^{\infty}\chi(\theta)\beta(\theta)\exp(\theta X)d\theta$ is an admissible estimator for $\mathrm{E}_\theta(X)$ under the quadratic loss function.

## 1. Introduction

Let $X$ be a random sample from the one-parameter exponential family with the probability element $\beta(\theta)\exp(\theta x)dm(x)$, where $\theta\in\Omega=(-\infty,\infty)$. We consider the estimation for $h(\theta)=\mathrm{E}_\theta(X)$ on the basis of $X$ under the quadratic loss function. Karlin [4] gave a sufficient condition for $\gamma X$ with a constant $\gamma$ to be an admissible estimator for $h(\theta)$. Katz [5] showed that if the parameter space $\Omega$ is truncated so that $\theta$ is known to be $\geqq a$ for a fixed $a$, then $X+\beta(a)\exp(aX)\Big/\int_a^{\infty}\beta(\theta)\exp(\theta X)d\theta$ is admissible for $h(\theta)$.

In this paper we suppose that the parameter space is loosely truncated so that $\theta$ is likely to be $\geqq a$ for a fixed $a$. This ambiguous information may be represented by a membership function $\chi(\cdot)$, following the theory of fuzzy sets by Zadeh [6]. The membership function $\chi(\cdot)$ is a generalization of the characteristic function of an ordinary set, mapping the parameter space $\Omega$ into the interval $[0, 1]$. Specifically, $\chi(\theta)$ is one for $\theta$ being sufficiently larger than $a$, is zero for $\theta$ being sufficiently smaller than $a$ and satisfies $0<\chi(\theta)<1$ near $a$. The value of $\chi(\theta)$ indicates the grade for each point $\theta$ to belong to the fuzzy set '$\theta$ is likely to be $\geqq a$.' Then we shall show that

Key words: Admissibility; fuzzy set; membership function.

$X+\int_{-\infty}^{\infty}\chi'(\theta)\beta(\theta)\exp(\theta X)d\theta\Big/\int_{-\infty}^{\infty}\chi(\theta)\beta(\theta)\exp(\theta X)d\theta$ is an admissible estimator for $h(\theta)$, extending the estimators by Karlin [4] and Katz [5].

We note that our estimator is a special case of a generalized Bayes estimator. Some proofs of admissibility for such a general problem have been given (e.g. See Farrel [2], Theorem 3.1). In particular, since elements of a exponential family are absolutely continuous each other, the assumption of Theorem 2.1 (Zidek [7]) is satisfied and the admissibility of a estimator is implied by its almost admissibility. And sufficient conditions for the almost admissibility were given by James and Stein [3] (Theorem 3.1) and Zidek [7] (Theorem 2.2). But generally these conditions are difficult to check. In this paper we can prove the admissibility of our estimator comparatively simply using a property of a membership function.

## 2. Results

In this section we give some assumptions, the main theorem and some remarks, leaving the proof of the theorem to the next section.

ASSUMPTION 1. $\chi(\theta)$ is differentiable.

ASSUMPTION 2. There exists a positive number $M$ $(-M\leqq a\leqq M)$ such that $\chi(\theta)=0$ for each $\theta\leqq -M$ and $\chi(\theta)=1$ for each $\theta\geqq M$.

Since if $\chi(\theta)$ is not differentiable $\chi(\theta)$ can be smoothed without a significant influence to the result, Assumption 1 is not too severe. And Assumption 2 is quite reasonable as we stated in Section 1. Furthermore we consider the exponential family which satisfies the following assumptions.

ASSUMPTION 3. For each $x$, $\beta(\theta)\exp(\theta x)\to 0$ as $\theta\to\pm\infty$.

Next we define

$$A(x)=\int_{-\infty}^{\infty}\chi(\theta)\beta(\theta)\exp(\theta x)d\theta\,,\qquad B(x)=\int_{-\infty}^{\infty}\chi'(\theta)\beta(\theta)\exp(\theta x)d\theta\,,$$

and give another assumption.

ASSUMPTION 4. There exist $m$-integrable functions $f$ and $g$ such that for each $\sigma>0$ and for each $x$,

$$[B(x)A(x-1/\sigma)-B(x-1/\sigma)A(x)]^2/[A^2(x)A(x-1/\sigma)]<f(x)\,,$$

and

$$|B(x)A(x-1/\sigma)/A(x)|<g(x)\,.$$

From above assumptions, we obtain the next theorem.

THEOREM. *If Assumptions* 1–4 *are satisfied, then*

$$(1)\qquad d(X)=X+\int_{-\infty}^{\infty}\chi'(\theta)\beta(\theta)\exp(\theta X)d\theta\Big/\int_{-\infty}^{\infty}\chi(\theta)\beta(\theta)\exp(\theta X)d\theta$$
$$=X+B(X)/A(X)$$

*is an admissible estimator for $h(\theta)$ under the quadratic loss function.*

*Remark* 1. Since the sum of $n$ random variables from an exponential family is sufficient and also belongs to an exponential family, there is no loss of generality in restricting ourselves to a single random variable.

*Remark* 2. Define $u=\inf\{\theta;\chi(\theta)>0\}$. Since $h(\theta)$ is an increasing function of $\theta$ and $d(X)$ is the limit of $d_\sigma(X)=\int_{-\infty}^{\infty}h(\theta)\chi(\theta)\beta(\theta)\exp(\theta(X-1/\sigma))d\theta\Big/\int_{-\infty}^{\infty}\chi(\theta)\beta(\theta)\exp(\theta(X-1/\sigma))d\theta$ as $\sigma\to\infty$ (see Section 3), the inequality $d_\sigma(X)\geqq h(u)$ implies that $d(X)\geqq h(u)$. Therefore we can state that $X$, which is the maximal likelihood estimator for $h(\theta)$ in the no-restriction case, is modified by the second term of the right-hand-side in (1) with our ambiguous information.

*Remark* 3. (i) When there is no restriction on the parameter space $\Omega$, we take the characteristic function of $\Omega=(-\infty,\infty)$ as $\chi(\cdot)$. Then the second term of the right-hand-side in (1) vanishes and we obtain $d(X)=X$, which is an admissible estimator of $h(\theta)$ by Karlin [4].

(ii) When the parameter space $\Omega$ is truncated to $[a,\infty)$ exactly as in Katz' formulation [5], we take the characteristic function of $[a,\infty)$ as $\chi(\cdot)$ and consider $\chi'(\cdot)$ as a $\delta$-function which has total mass at $a$. Then (1) becomes $d(X)=X+\beta(a)\exp(aX)\Big/\int_a^{\infty}\beta(\theta)\exp(\theta X)d\theta$, which is Katz' admissible estimator for $h(\theta)$. Therefore our estimator (1) for $h(\theta)$ is an extension of both Karlin's and Katz' estimators.

*Example.* Let $X$ follow the normal distribution $N(\theta,1)$. Since $h(\theta)=\mathrm{E}_\theta(X)=\theta$,

$$d(X)=X+\int_{-\infty}^{\infty}\chi'(\theta)\exp(-(X-\theta)^2/2)d\theta\Big/\int_{-\infty}^{\infty}\chi(\theta)\exp(-(X-\theta)^2/2)d\theta$$

is an admissible estimator for $\theta$ when we have an ambiguous restriction on the parameter space.

## 3. Proof of the theorem

In this section we shall state some lemmas and then give a proof of the main theorem.

LEMMA 1.

$$h(\theta)=\mathrm{E}_\theta(X)=-\beta'(\theta)/\beta(\theta), \tag{2}$$

*and $h'(\theta)$ is the variance of $X$. Therefore $h(\theta)$ is an increasing function of $\theta$.*

LEMMA 2. *If we take as a priori distribution*

$$g_\sigma(\theta)=c\chi(\theta)\exp(-\theta/\sigma), \qquad -\infty<\theta<\infty,$$

*where $\sigma>0$ and $c=1\Big/\int_{-\infty}^{\infty}\chi(\theta)\exp(-\theta/\sigma)d\theta$, then*

$$\sigma\exp(-M/\sigma)\leqq 1/c\leqq\sigma\exp(M/\sigma) \tag{3}$$

*and the Bayes estimator for $h(\theta)$ with respect to $g_\sigma$ is given by*

$$\begin{aligned} d_\sigma(X)&=\int_{-\infty}^{\infty}h(\theta)\chi(\theta)\beta(\theta)\exp(\theta(X-1/\sigma))d\theta\Big/\int_{-\infty}^{\infty}\chi(\theta)\beta(\theta)\exp(\theta(X-1/\sigma))d\theta \\ &=X-1/\sigma+\int_{-\infty}^{\infty}\chi'(\theta)\beta(\theta)\exp(\theta(X-1/\sigma))d\theta\Big/\int_{-\infty}^{\infty}\chi(\theta)\beta(\theta)\exp(\theta(X-1/\sigma))d\theta \\ &=X-1/\sigma+B(X-1/\sigma)/A(X-1/\sigma). \end{aligned} \tag{4}$$

PROOF. Using Assumption 2, we obtain $\int_{M}^{\infty}\exp(-\theta/\sigma)d\theta\leqq 1/c\leqq\int_{-M}^{\infty}\exp(-\theta/\sigma)d\theta$, from which (3) follows. The second equality in (4) is obtained by integration by parts and Assumption 3 after using (2).

Note that $d_\sigma$ converges to $d(X)=X+B(X)/A(X)$ as $\sigma\to\infty$.

LEMMA 3. *The risk functions of the estimators $d_\sigma$ and $d$ are given respectively by*

$$\begin{aligned} R(\theta,d_\sigma)&=\mathrm{E}_\theta[d_\sigma(X)-h(\theta)]^2 \\ &=h'(\theta)+1/\sigma^2+2\,\mathrm{E}_\theta[XB(X-1/\sigma)/A(X-1/\sigma)] \\ &\quad-2h(\theta)\,\mathrm{E}_\theta[B(X-1/\sigma)/A(X-1/\sigma)] \\ &\quad+\mathrm{E}_\theta[B(X-1/\sigma)/A(X-1/\sigma)]^2 \\ &\quad-(2/\sigma)\,\mathrm{E}_\theta[B(X-1/\sigma)/A(X-1/\sigma)], \end{aligned}$$

*and*

$$R(\theta, d)=h'(\theta)+2\,\mathrm{E}_\theta\,[XB(X)/A(X)]$$
$$-2h(\theta)\,\mathrm{E}_\theta\,[B(X)/A(X)]+\mathrm{E}_\theta\,[B(X)/A(X)]^2\,.$$

*The Bayes risks of* $d_\sigma$ *and* $d$ *with respect to* $g_\sigma$ *are given respectively by*

$$(5)\qquad r(g_\sigma, d_\sigma)=\int_{-\infty}^{\infty} R(\theta, d_\sigma)c\chi(\theta)\exp(-\theta/\sigma)d\theta$$
$$=\int_{-\infty}^{\infty} h'(\theta)c\chi(\theta)\exp(-\theta/\sigma)d\theta+1/\sigma^2$$
$$-c\int B^2(x-1/\sigma)/A(x-1/\sigma)dm(x)\,,$$

*and*

$$(6)\qquad r(g_\sigma, d)=\int_{-\infty}^{\infty} h'(\theta)c\chi(\theta)\exp(-\theta/\sigma)d\theta$$
$$-2c\int B(x)B(x-1/\sigma)/A(x)dm(x)$$
$$+(2c/\sigma)\int B(x)A(x-1/\sigma)/A(x)dm(x)$$
$$+c\int B^2(x)A(x-1/\sigma)/A^2(x)dm(x)\,.$$

The proof of this lemma is straightforward.

PROOF OF THE THEOREM. Now we shall show that $d(X)=X+B(X)/A(X)$ is admissible. The method of the proof is due to Blyth [1]. Suppose that $d$ is not admissible. Then there exists an estimator $d^*$ such that

$$(7)\qquad R(\theta, d^*)\leqq R(\theta, d)\qquad \text{for all } \theta\,,$$

and

$$R(\theta_0, d^*)<R(\theta_0, d)\qquad \text{for at least one } \theta_0\,.$$

We may assume that $\theta_0$ is an interior point of the support of $\chi(\cdot)$. For we can interpret that $\theta$ with $\chi(\theta)=0$ is not realizable. Since $R(\theta, d^*)$ is continuous in $\theta$, there exist some positive number $\varepsilon$ and some interval $(\underline{\theta}, \bar{\theta})$ which is included in the interior points of the support of $\chi(\cdot)$ such that for each $\theta\in(\underline{\theta}, \bar{\theta})$,

$$(8)\qquad R(\theta, d^*)<R(\theta, d)-\varepsilon\,.$$

It suffices to show that for sufficiently large $\sigma(>0)$,

$$(9)\qquad [r(g_\sigma, d)-r(g_\sigma, d^*)]/[r(g_\sigma, d)-r(g_\sigma, d_\sigma)]>1\,,$$

since this implies $r(g_\sigma, d^*) < r(g_\sigma, d_\sigma)$ (the denominator of ( 8 ) is clearly nonnegative), contradicting the fact that $d_\sigma$ is a Bayes estimator with respect to $g_\sigma$.

Now, by ( 5 ) and ( 6 ) the denominator of ( 9 ) is written as

$$
\begin{aligned}
(10)\quad & r(g_\sigma, d) - r(g_\sigma, d_\sigma) \\
&= -2c\int B(x)B(x-1/\sigma)/A(x)dm(x) \\
&\quad + (2c/\sigma)\int B(x)A(x-1/\sigma)/A(x)dm(x) \\
&\quad + c\int B^2(x)A(x-1/\sigma)/A^2(x)dm(x) \\
&\quad + c\int B^2(x-1/\sigma)/A(x-1/\sigma)dm(x) - 1/\sigma^2 \\
&= c\int [B(x)A(x-1/\sigma) - B(x-1/\sigma)A(x)]^2/[A^2(x)A(x-1/\sigma)]dm(x) \\
&\quad + (2c/\sigma)\int B(x)A(x-1/\sigma)/A(x)dm(x) - 1/\sigma^2 .
\end{aligned}
$$

On the other hand by integrating ( 7 ) by $g_\sigma$ and by taking ( 8 ) into account, the numerator of ( 9 ) becomes

$$
(11)\qquad r(g_\sigma, d) - r(g_\sigma, d^*) > \varepsilon c\int_{\underline{\theta}}^{\bar{\theta}} \chi(\theta)\exp(-\theta/\sigma)d\theta = cK ,
$$

where $K$ is a positive constant. Then from (10) and (11) it follows that

$$
\begin{aligned}
(12)\quad & [r(g_\sigma, d) - r(g_\sigma, d^*)]/[r(g_\sigma, d) - r(g_\sigma, d_\sigma)] \\
&> K\Big/\Big[\int [B(x)A(x-1/\sigma) - B(x-1/\sigma)A(x)]^2/[A^2(x)A(x-1/\sigma)]dm(x) \\
&\qquad + (2/\sigma)\int B(x)A(x-1/\sigma)/A(x)dm(x) - 1/(c\sigma^2)\Big] .
\end{aligned}
$$

The first term and the second term of the denominator of the right-hand-side in (12) converges to zero by Assumption 4 and Lebesgue dominated convergence theorem, whereas the third term converges to zero by ( 3 ). Therefore the relation ( 9 ) holds for sufficiently large $\sigma$.

OSAKA UNIVERSITY

## REFERENCES

[1] Blyth, C. R. (1951). On minimax statistical decision procedures and their admissibility, *Ann. Math. Statist.*, **22**, 22-42.

[2] Farrel, R. H. (1966). Weak limits of sequences of Bayes procedures in estimation theory, *Proc. 5th Berkeley Symp. Math. Statist. Prob.*, **1**, 83-111.

[3] James, W. and Stein, C. (1961). Estimation with quadratic loss, *Proc. 4th Berkeley Symp. Math. Statist. Prob.*, **1**, 361-379.

[4] Karlin, S. (1958). Admissibility for estimation with quadratic loss, *Ann. Math. Statist.*, **29**, 406-436.

[5] Katz, M. W. (1961). Admissible and minimax estimates of parameters in truncated spaces, *Ann. Math. Statist.*, **32**, 136-142.

[6] Zadeh, L. A. (1965). Fuzzy sets. *Inf. Control*, **8**, 338-353.

[7] Zidek, J. V. (1970). Sufficient conditions for the admissibility under squared errors loss of formal Bayes estimators, *Ann. Math. Statist.*, **41**, 446-456.