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A BAYESIAN APPROACH TO BINARY RESPONSE CURVE ESTIMATION

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Summary

The purpose of the present paper is to propose a practical procedure for the estimation of the binary response curve. The procedure is based on a model which approximates the response curve by a finely segmented piecewise constant function. To obtain a stable estimate we assume a prior distribution of the parameters of the model. The prior distribution has several parameters (hyper-parameters) which are chosen to minimize an information criterion ABIC. The procedure is applicable to data consisting of observations of a binary response variable and a single explanatory variable. The practical utility of the procedure is demonstrated by examples of applications to the dose response curve estimation, to the intensity function estimation of a point process and to the analysis of social survey data. The application of the procedure to the discriminant analysis is also briefly discussed.

1. Introduction

The most important problem of the analysis of binary data is to study how the probability of occurrence of a certain phenomenon depends on explanatory variables. Various methods to treat this problem are studied by many statisticians (for example, see [3]). A new Bayesian approach to a specific problem of this field, that is, cohort analysis problem, is recently developed by Nakamura [10]. This Bayesian approach is originally introduced by Akaike [1] as a tool to deal with a regression analysis problem where the number of the parameters to be estimated is large compared with the sample size and already found a wide range of applications [1], [2], [4], [5], [6], [10].

The purpose of the present paper is to show that this approach can be applied to the problem of the estimation of the conditional probability of occurrence of a specific phenomenon given a value of an explanatory variable. The basic assumptions are that the values of the explanatory variable are ordered and that the conditional probabilities of the occurrence are changing smoothly with order. Our method is applicable not only to those cases where the explanatory variable has ordered classifications but also to cases where it takes continuous values. The method easily realizes the estimation of a series of probabilities even if they vary in a complicated fashion as far as the change is smooth.

A review of Akaike's Bayesian procedure is briefly given in Section 2. Our Bayesian response curve model is proposed in Section 3 and a numerical procedure is set out in Section 4. The practical utility of the present procedure is demonstrated in Section 5 by a wide range of applications: the dose response curve fitting, the estimation of the intensity function in the point process analysis and the analysis of public opinion poll data. A new approach to the discriminant analysis is also demonstrated by applying it to a set of artificial data. In Section 6 we discuss the stability and the accuracy of the procedure and the relation to classical procedures.

2. Akaike's Bayesian procedure

In this section we will briefly review the Bayesian technique to estimate regression coefficients proposed by Akaike [1].

Assume that the relation between a variable y and a vector of independent variable $\boldsymbol{x}$ is expressed by

$$y=\boldsymbol{x}^t\boldsymbol{\theta}+\varepsilon \ , \tag{1}$$

where ε is a random variable which is normally distributed with mean 0 and unknown variance σ^2 and $\boldsymbol{\theta}$ is an unknown coefficient vector. When observations $(y_i, \boldsymbol{x}_i)$ $i=1,\cdots,n$ are given, the maximum likelihood estimate of $\boldsymbol{\theta}$ is obtained by minimizing

$$\sum_{i=1}^{n}|y_i-\boldsymbol{x}_i^t\boldsymbol{\theta}|^2=|\boldsymbol{y}-X\boldsymbol{\theta}|^2 \ , \tag{2}$$

where $\boldsymbol{y}=(y_1,\cdots,y_n)^t$ and $X=(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n)^t$. This estimate, however, is unstable when n is small compared with the dimension of $\boldsymbol{\theta}$.

If it is known a priori that $\boldsymbol{\theta}$ is close to a known value $\boldsymbol{\theta}_0$, or $|D(\boldsymbol{\theta}-\boldsymbol{\theta}_0)|^2$ is small for some fixed matrix D, it is reasonable to estimate $\boldsymbol{\theta}$ by minimizing

$$|\boldsymbol{y}-X\boldsymbol{\theta}|^2+w^2|D(\boldsymbol{\theta}-\boldsymbol{\theta}_0)|^2 \ . \tag{3}$$

The result, however, depends on the choice of the weight w^2. Akaike argued that the minimization of (3) is the maximization of

$$\exp\{-|\boldsymbol{y}-X\boldsymbol{\theta}|^2/2\sigma^2\}\cdot\exp\{-|D(\boldsymbol{\theta}-\boldsymbol{\theta}_0)|^2w^2/2\sigma^2\} \ . \tag{4}$$

The first term of (4) is proportional to the likelihood $L(\boldsymbol{y}|\boldsymbol{\theta}, \sigma^2)$ of $\boldsymbol{\theta}$ and σ^2 for a given set of observations. When normalizd, the second term can be regarded as a density function $\pi(\boldsymbol{\theta}|w^2, \sigma^2)$ of $\boldsymbol{\theta}$. Thus the estimate of $\boldsymbol{\theta}$ obtained by minimizing (3) is regarded as the mode of the posterior distribution of $\boldsymbol{\theta}$.

In this context the choice of w^2 is interpreted as that of a parameter of a prior distribution of $\boldsymbol{\theta}$. From these considerations Akaike proposed the use of the marginal likelihood given by

$$(5) \qquad \int L(\boldsymbol{y}|\boldsymbol{\theta}, \sigma^2)\pi(\boldsymbol{\theta}|w^2, \sigma^2)d\boldsymbol{\theta}$$

as a criterion for the choice of w^2 and σ^2. Those values are to be chosen so that (5) is maximized. Considering the relation to the statistic AIC (Akaike Information Criterion), Akaike defined the statistic ABIC by

$$(6) \qquad \text{ABIC} = -2 \log \int L(\boldsymbol{y}|\boldsymbol{\theta}, \sigma^2)\pi(\boldsymbol{\theta}|w^2, \sigma^2)d\boldsymbol{\theta} \ .$$

3. Bayesian response curve model

Our purpose is to estimate the conditional probability $p^*(x)$ of occurrence of a certain phenomenon given a value of an explanatory variable x, which may take either continuous or discrete value. In the following, however, we will assume that it takes continuous value. The treatment of the discrete value case will be apparent. We assume that $p^*(x)$ can be approximated by a piece-wise constant function $p(x)$ defined by

$$(7) \qquad p(x) = p_j \qquad \text{if } a_{j-1} < x \leqq a_j \ ,$$

where $a_0 < a_1 < a_2 < \cdots < a_c$ are suitably chosen segment points. Note that this assumption is not too severe if we can take $|a_j - a_{j-1}|$'s sufficiently small.

Suppose that a set of data (h_i, x_i), $i=1, 2, \cdots, n$ are given, where h_i takes 1 or 0 according to the occurrence or un-occurrence of the phenomenon and x_i denotes a value of the explanatory variable.

Without loss of generality we assume that $\{x_i\}$ satisfies $a_0 < x_i \leqq a_c$ for $i=1, \cdots, n$. Then the likelihood of the parameter vector $\boldsymbol{p} = (p_1, \cdots, p_c)^t$ of the model (7) is given by

$$(8) \qquad L(\boldsymbol{p}) = \prod_{i=1}^{n} p(x_i)^{h_i}\{1-p(x_i)\}^{(1-h_i)} = \prod_{j=1}^{c} p_j^{n(1,j)}(1-p_j)^{n(0,j)} \ ,$$

where $n(k, j)$ denotes the number of data which satisfy both $h_i = k$ and $a_{j-1} < x_i \leqq a_j$.

Let $n(j)$ be defined by

$$n(j)=n(1, j)+n(0, j) . \tag{9}$$

If $n(j)>0$ for $j=1,\cdots, c$, the maximum likelihood estimator of $p(x)$ is given by

$$\tilde{p}(x)=\frac{n(1, j)}{n(j)} \quad \text{if } a_{j-1}<x\leqq a_j . \tag{10}$$

It is well known that $\tilde{p}(x)$ is unstable or unreliable when $n(j)$ is small. Moreover, if $n(j)$ is equal to zero for some j, $\tilde{p}(x)$ remains undefined for $a_{j-1}<x\leqq a_j$.

Our purpose is to develop a procedure which is applicable even if $n(j)$, $j=1,\cdots, c$ are close or equal to zero. To avoid the above stated difficulty, we develop a Bayesian modeling method, which is an analogous to Akaike's method reviewed in the preceding section.

As a preparation, we introduce here the logit transformation $\{q_j\}$ of the parameters $\{p_j\}$ of our model, which are defined by

$$q_j=\log \frac{p_j}{1-p_j} \quad (j=1,\cdots, c) , \tag{11}$$

where and hereafter log refers to the natural logarithm. Then, the inverse transformation is given by

$$p_j=\frac{\exp (q_j)}{1+\exp (q_j)} \quad (j=1,\cdots, c) , \tag{12}$$

and the likelihood of our model as the function of q_j's is given by

$$\begin{aligned} L(\boldsymbol{q}) &= \prod_{j=1}^{c} \left\{\frac{\exp (q_j)}{1+\exp (q_j)}\right\}^{n(1,j)} \left\{\frac{1}{1+\exp (q_j)}\right\}^{n(0,j)} \\ &= \prod_{j=1}^{c} \frac{\{\exp (q_j)\}^{n(1,j)}}{\{1+\exp (q_j)\}^{n(j)}} . \end{aligned} \tag{13}$$

We assume that $p^*(x)$ is a continuous function of x, then it is natural to expect that the second order differences of $\{q_j\}$

$$q_j-2q_{j-1}+q_{j-2}$$

are close to zero for $j=1,\cdots, c$, when q_0 and q_{-1} are suitably chosen. To formalize this expectation we assume that the parameter vector $\boldsymbol{q}=(q_1, q_2,\cdots, q_c)^t$ obeys to the prior distribution defined by

$$\pi(\boldsymbol{q}\,|\,v^2)=\left(\frac{1}{\sqrt{2\pi}\, v}\right)^c \exp \left\{-\frac{1}{2v^2}|D\boldsymbol{q}+r|^2\right\} , \tag{14}$$

where D is a matrix defined by

$$(15) \qquad D=\begin{bmatrix} 1 & & & & & 0 \\ -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ 0 & & & 1 & -2 & 1 \end{bmatrix}$$

and r is a vector defined by

$$(16) \qquad r=\begin{bmatrix} 1 & -2 \\ 0 & 1 \\ \multicolumn{2}{c}{0} \end{bmatrix}\begin{pmatrix} q_{-1} \\ q_0 \end{pmatrix} .$$

If the value of v^2 is specified, it is reasonable to define the estimate of $\boldsymbol{q}$ by the mode $\hat{\boldsymbol{q}}$ of the posterior distribution of $\boldsymbol{q}$ which is proportional to $L(\boldsymbol{q})\pi(\boldsymbol{q}\,|\,v^2)$.

As a criterion for the choice of v^2 we adopt ABIC in the preceding section, which is defined by

$$(17) \qquad \mathrm{ABIC}=-2\log\int L(\boldsymbol{q})\pi(\boldsymbol{q}\,|\,v^2)d\boldsymbol{q} .$$

The value of v^2 is to be chosen so that it minimizes ABIC.

4. Numerical consideration

The main difficulty with our approach lies in the maximization and integration of $L(\boldsymbol{q})\pi(\boldsymbol{q}\,|\,v^2)$.

The maximization can be numerically carried out adopting Newton's procedure, which is efficient for the present situation because of the negative definitness of the Hessian of the log-likelihood.

It is also observed that the higher order derivatives of the log-likelihood are bounded for entire space of parameters (see Appendix). This indicates that when v^2 is sufficiently small, i.e. $\pi(\boldsymbol{q}\,|\,v^2)$ is sharp compared with $L(\boldsymbol{q})$, the integral can be well approximated by

$$(18) \qquad \int \exp\{T(\boldsymbol{q})\}d\boldsymbol{q} ,$$

where $T(\boldsymbol{q})$ denotes the Taylor expansion of $\log\{L(\boldsymbol{q})\pi(\boldsymbol{q}\,|\,v^2)\}$ up to the second order term around the maximizing point $\hat{\boldsymbol{q}}$.

Since $T(\boldsymbol{q})$ attains its maximum at $\hat{\boldsymbol{q}}$, the first order term vanishes and $T(\boldsymbol{q})$ has the form

$$(19) \qquad T(\boldsymbol{q})=T(\hat{\boldsymbol{q}})-\frac{1}{2}(\boldsymbol{q}-\hat{\boldsymbol{q}})^t H(\hat{\boldsymbol{q}})(\boldsymbol{q}-\hat{\boldsymbol{q}}) ,$$

where $T(\hat{q})$ is defined by

$$T(\hat{q})=\log L(\hat{q})\pi(\hat{q}\,|\,v^2) \tag{20}$$

and $H(\hat{q})$ can be decomposed into the form

$$H(\hat{q})=\hat{G}+\frac{1}{v^2}D^tD\ . \tag{21}$$

Here D is the matrix defined by (15) and $\hat{G}$ is a diagonal matrix whose (j,j) element is given by

$$\hat{G}_{jj}=n(j)\hat{p}_j(1-\hat{p}_j)\ , \tag{22}$$

where $\hat{p}_j$ is defined by

$$\hat{p}_j=\frac{\exp(\hat{q}_j)}{1+\exp(\hat{q}_j)}\ . \tag{23}$$

Though the exact evaluation of ABIC (17) for fixed q_{-1}, q_0 and v^2 is difficult, an approximate value is given by

$$\mathrm{ABIC}=-2T(\hat{q})+\log\{\det H(\hat{q})\}-c\log 2\pi\ . \tag{24}$$

Our proposal is that q_{-1}, q_0 and v^2 should be chosen so that they minimize (24) instead of (17). Note that q_{-1} and q_0 affect (24) only through the first term and can be determined by maximizing $L(\boldsymbol{q})\pi(\boldsymbol{q}\,|\,v^2)$ simultaneously with $\boldsymbol{q}$. The optimization of (24) with respect to v^2 is nonlinear, so we adopt the grid search technique here.

5. Numerical examples

We first apply the procedure to the data shown in Table 1 which were introduced by Mays and Lloyed [8], [9] and were analyzed by Noda and Murakami [11]. The table shows the number of mice with sarcoma in each of eleven treatment groups. The object is the estimation of dose-response curve for carcinogenesis on the basis of the data. Noda and Murakami dealt with the data as the problem of the maximum likelihood estimation of two-phase segmented line and have got the result shown in Figure 1.

In our procedure we must choose the number of classes c. Let $x_{(1)}$ and $x_{(n)}$ be the smallest and the largest respectively in the values taken by a given explanatory variable. When the precision of the observation is d, we set $a_0=x_{(1)}-d/2$, $a_c=x_{(n)}+d/2$ and divide the interval $(a_0, a_c]$ into c equal intervals. Here we may properly choose the value of c so as not to lose the information of the data by discretization. It is convenient to adopt $\min\{2n, R/d, 100\}$ as the value of c unless other-

Table 1. ^{226}Ra-injected mice

Dose	Number of mice		Inc.
(Rads. before death 140 days)	at 150 days postinjection	with sarcoma	%
6,420 (3.81)	44	38	86.4
3,640 (3.56)	43	34	79.1
2,040 (3.31)	45	28	62.2
1,190 (3.08)	104	45	43.3
614 (2.79)	104	22	21.2
480 (2.68)	239	56	23.4
383 (2.58)	504	94	18.7
244 (2.39)	683	80	11.7
109 (2.04)	247	19	7.7
62 (1.79)	252	5	2.0
26 (1.41)	254	11	4.3

From Table 1 in [11]. Numerical values in the parentheses represent the common logarithm of doses.

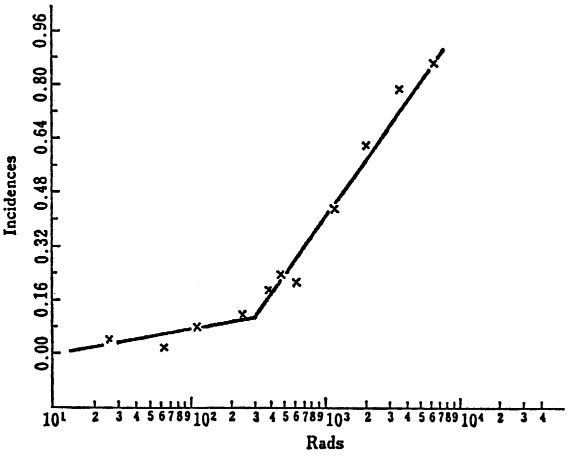
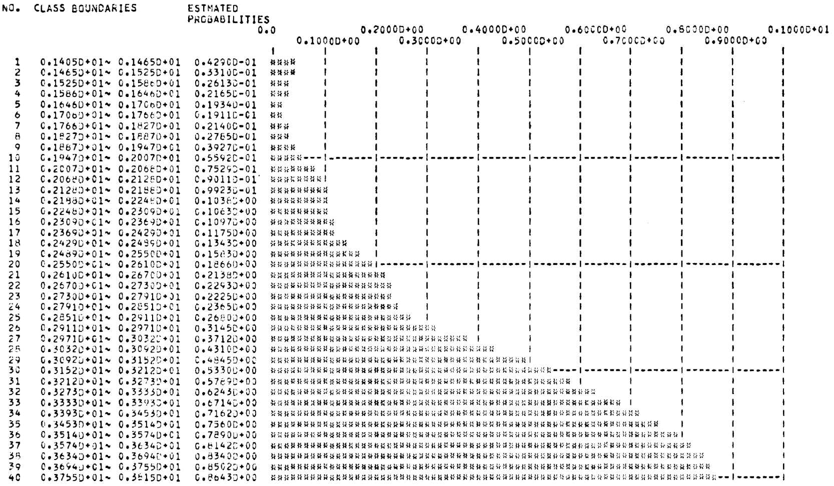


Fig. 1. From Fig. 1 in [11].

wise designated, where $R=a_c-a_0$. Although we may choose the value of c according to this rule for the above data set, we put $c=40$ to serve the convenience of being compared with the result by Noda and Murakami.

Some results for the above data are shown in Figures 2.a–e each of which corresponds to a value of the hyper-parameter v. Each figure consists of class boundaries and the estimates of response probability in each class. In Figures 2.a–e log dose is adopted. Since ABIC has its minimum 1919.86 for $v=0.0078125$, the minimum ABIC procedure gives the smoothly increasing curve shown in Figure 2.d as the final estimate.



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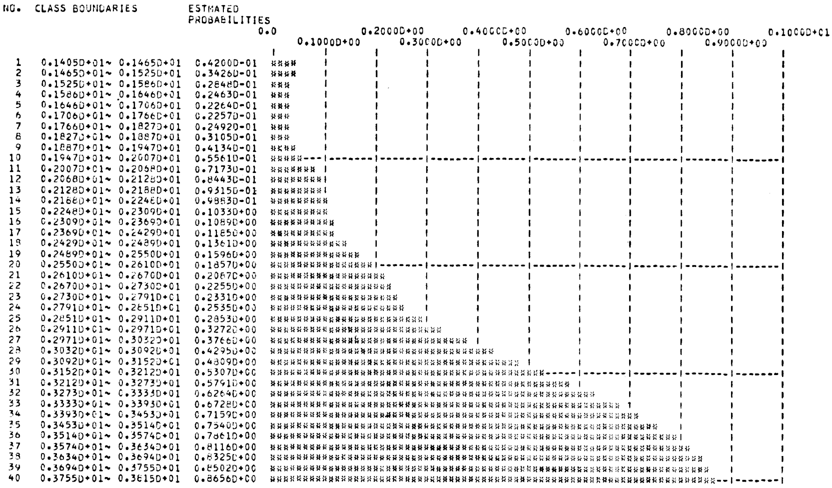
                               PROBABILITIES
  1  0.1405D+01~ 0.1465D+01  0.4290D-01
  2  0.1465D+01~ 0.1525D+01  0.3310D-01
  3  0.1525D+01~ 0.1586D+01  0.2613D-01
  4  0.1586D+01~ 0.1646D+01  0.2165D-01
  5  0.1646D+01~ 0.1706D+01  0.1934D-01
  6  0.1706D+01~ 0.1766D+01  0.1911D-01
  7  0.1766D+01~ 0.1827D+01  0.2140D-01
  8  0.1827D+01~ 0.1887D+01  0.2785D-01
  9  0.1887D+01~ 0.1947D+01  0.3927D-01
 10  0.1947D+01~ 0.2007D+01  0.5592D-01
 11  0.2007D+01~ 0.2068D+01  0.7529D-01
 12  0.2068D+01~ 0.2128D+01  0.9011D-01
 13  0.2128D+01~ 0.2188D+01  0.9923D-01
 14  0.2188D+01~ 0.2248D+01  0.1038D+00
 15  0.2248D+01~ 0.2309D+01  0.1063D+00
 16  0.2309D+01~ 0.2369D+01  0.1097D+00
 17  0.2369D+01~ 0.2429D+01  0.1175D+00
 18  0.2429D+01~ 0.2489D+01  0.1343D+00
 19  0.2489D+01~ 0.2550D+01  0.1583D+00
 20  0.2550D+01~ 0.2610D+01  0.1866D+00
 21  0.2610D+01~ 0.2670D+01  0.2138D+00
 22  0.2670D+01~ 0.2730D+01  0.2293D+00
 23  0.2730D+01~ 0.2791D+01  0.2225D+00
 24  0.2791D+01~ 0.2851D+01  0.2365D+00
 25  0.2851D+01~ 0.2911D+01  0.2680D+00
 26  0.2911D+01~ 0.2971D+01  0.3145D+00
 27  0.2971D+01~ 0.3032D+01  0.3712D+00
 28  0.3032D+01~ 0.3092D+01  0.4310D+00
 29  0.3092D+01~ 0.3152D+01  0.4845D+00
 30  0.3152D+01~ 0.3212D+01  0.5330D+00
 31  0.3212D+01~ 0.3273D+01  0.5789D+00
 32  0.3273D+01~ 0.3333D+01  0.6243D+00
 33  0.3333D+01~ 0.3393D+01  0.6714D+00
 34  0.3393D+01~ 0.3453D+01  0.7162D+00
 35  0.3453D+01~ 0.3514D+01  0.7560D+00
 36  0.3514D+01~ 0.3574D+01  0.7890D+00
 37  0.3574D+01~ 0.3634D+01  0.8142D+00
 38  0.3634D+01~ 0.3694D+01  0.8340D+00
 39  0.3694D+01~ 0.3755D+01  0.8502D+00
 40  0.3755D+01~ 0.3815D+01  0.8643D+00
```

a) $v=0.25$ ABIC=1941.25



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NO.  CLASS BOUNDARIES          ESTMATED
                               PROBABILITIES
  1  0.1405D+01~ 0.1465D+01  0.4200D-01
  2  0.1465D+01~ 0.1525D+01  0.3426D-01
  3  0.1525D+01~ 0.1586D+01  0.2848D-01
  4  0.1586D+01~ 0.1646D+01  0.2463D-01
  5  0.1646D+01~ 0.1706D+01  0.2264D-01
  6  0.1706D+01~ 0.1766D+01  0.2257D-01
  7  0.1766D+01~ 0.1827D+01  0.2492D-01
  8  0.1827D+01~ 0.1887D+01  0.3105D-01
  9  0.1887D+01~ 0.1947D+01  0.4134D-01
 10  0.1947D+01~ 0.2007D+01  0.5561D-01
 11  0.2007D+01~ 0.2068D+01  0.7173D-01
 12  0.2068D+01~ 0.2128D+01  0.8443D-01
 13  0.2128D+01~ 0.2188D+01  0.9315D-01
 14  0.2188D+01~ 0.2248D+01  0.9883D-01
 15  0.2248D+01~ 0.2309D+01  0.1033D+00
 16  0.2309D+01~ 0.2369D+01  0.1089D+00
 17  0.2369D+01~ 0.2429D+01  0.1185D+00
 18  0.2429D+01~ 0.2489D+01  0.1361D+00
 19  0.2489D+01~ 0.2550D+01  0.1596D+00
 20  0.2550D+01~ 0.2610D+01  0.1857D+00
 21  0.2610D+01~ 0.2670D+01  0.2067D+00
 22  0.2670D+01~ 0.2730D+01  0.2255D+00
 23  0.2730D+01~ 0.2791D+01  0.2331D+00
 24  0.2791D+01~ 0.2851D+01  0.2535D+00
 25  0.2851D+01~ 0.2911D+01  0.2853D+00
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 27  0.2971D+01~ 0.3032D+01  0.3766D+00
 28  0.3032D+01~ 0.3092D+01  0.4295D+00
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 32  0.3273D+01~ 0.3333D+01  0.6264D+00
 33  0.3333D+01~ 0.3393D+01  0.6728D+00
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 35  0.3453D+01~ 0.3514D+01  0.7540D+00
 36  0.3514D+01~ 0.3574D+01  0.7861D+00
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 38  0.3634D+01~ 0.3694D+01  0.8325D+00
 39  0.3694D+01~ 0.3755D+01  0.8502D+00
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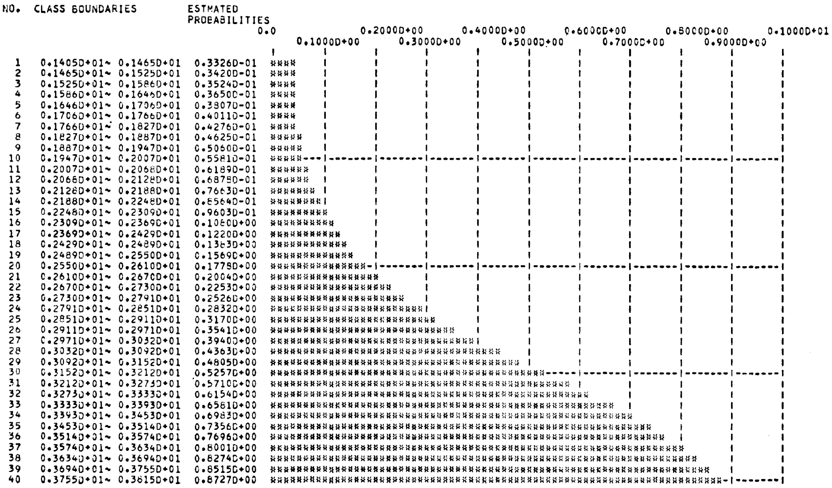
b) $v=0.125$ ABIC=1932.84



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NO.  CLASS BOUNDARIES          ESTMATED
                               PROBABILITIES
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  3  0.1525D+01~ 0.1586D+01  0.3524D-01
  4  0.1586D+01~ 0.1646D+01  0.3650D-01
  5  0.1646D+01~ 0.1706D+01  0.3807D-01
  6  0.1706D+01~ 0.1766D+01  0.4011D-01
  7  0.1766D+01~ 0.1827D+01  0.4276D-01
  8  0.1827D+01~ 0.1887D+01  0.4625D-01
  9  0.1887D+01~ 0.1947D+01  0.5060D-01
 10  0.1947D+01~ 0.2007D+01  0.5581D-01
 11  0.2007D+01~ 0.2068D+01  0.6189D-01
 12  0.2068D+01~ 0.2128D+01  0.6875D-01
 13  0.2128D+01~ 0.2188D+01  0.7663D-01
 14  0.2188D+01~ 0.2248D+01  0.8564D-01
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 17  0.2369D+01~ 0.2429D+01  0.1220D+00
 18  0.2429D+01~ 0.2489D+01  0.1363D+00
 19  0.2489D+01~ 0.2550D+01  0.1569D+00
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 21  0.2610D+01~ 0.2670D+01  0.2004D+00
 22  0.2670D+01~ 0.2730D+01  0.2253D+00
 23  0.2730D+01~ 0.2791D+01  0.2526D+00
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 39  0.3694D+01~ 0.3755D+01  0.8515D+00
 40  0.3755D+01~ 0.3815D+01  0.8727D+00
```
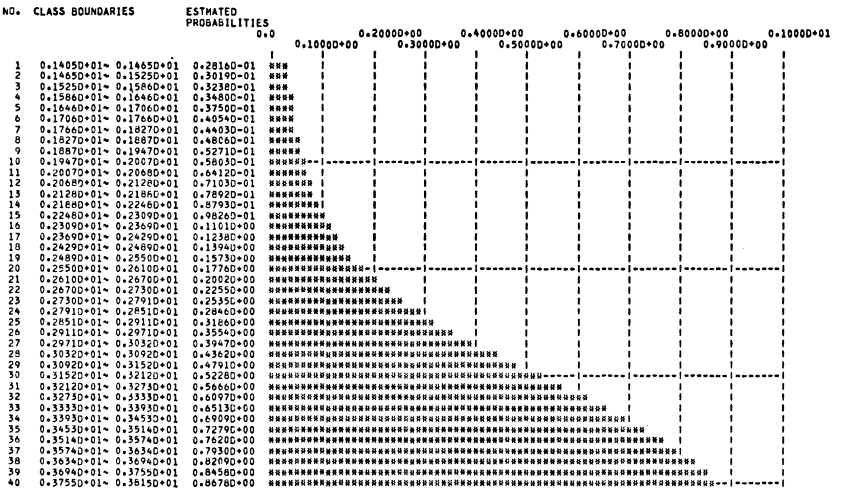

c) $v=0.015625$ ABIC=1920.09

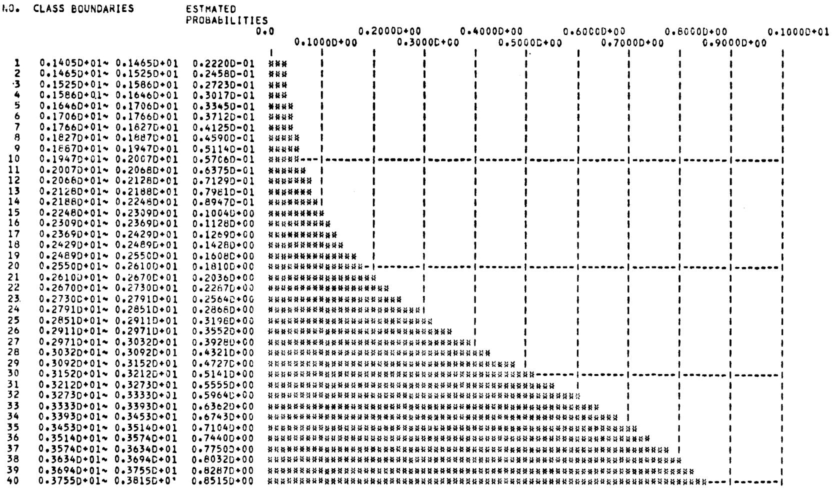
Fig. 2.a-c

d) $v=0.0078125$ ABIC=1919.86



```
NO.  CLASS BOUNDARIES          ESTMATED
                               PROBABILITIES
                                         0.0         0.2000D+00        0.4000D+00        0.6000D+00        0.8000D+00        0.1000D+01
                                                0.1000D+00        0.3000D+00        0.5000D+00        0.7000D+00        0.9000D+00
 1   0.1405D+01~ 0.1465D+01   0.2816D-01
 2   0.1465D+01~ 0.1525D+01   0.3019D-01
 3   0.1525D+01~ 0.1586D+01   0.3238D-01
 4   0.1586D+01~ 0.1646D+01   0.3480D-01
 5   0.1646D+01~ 0.1706D+01   0.3750D-01
 6   0.1706D+01~ 0.1766D+01   0.4054D-01
 7   0.1766D+01~ 0.1827D+01   0.4403D-01
 8   0.1827D+01~ 0.1887D+01   0.4806D-01
 9   0.1887D+01~ 0.1947D+01   0.5271D-01
10   0.1947D+01~ 0.2007D+01   0.5803D-01
11   0.2007D+01~ 0.2068D+01   0.6412D-01
12   0.2068D+01~ 0.2128D+01   0.7103D-01
13   0.2128D+01~ 0.2186D+01   0.7892D-01
14   0.2188D+01~ 0.2246D+01   0.8793D-01
15   0.2248D+01~ 0.2309D+01   0.9826D-01
16   0.2309D+01~ 0.2369D+01   0.1101D+00
17   0.2369D+01~ 0.2429D+01   0.1238D+00
18   0.2429D+01~ 0.2489D+01   0.1394D+00
19   0.2489D+01~ 0.2550D+01   0.1573D+00
20   0.2550D+01~ 0.2610D+01   0.1776D+00
21   0.2610D+01~ 0.2670D+01   0.2002D+00
22   0.2670D+01~ 0.2730D+01   0.2255D+00
23   0.2730D+01~ 0.2791D+01   0.2535D+00
24   0.2791D+01~ 0.2851D+01   0.2846D+00
25   0.2851D+01~ 0.2911D+01   0.3186D+00
26   0.2911D+01~ 0.2971D+01   0.3554D+00
27   0.2971D+01~ 0.3032D+01   0.3947D+00
28   0.3032D+01~ 0.3092D+01   0.4362D+00
29   0.3092D+01~ 0.3152D+01   0.4791D+00
30   0.3152D+01~ 0.3212D+01   0.5228D+00
31   0.3212D+01~ 0.3273D+01   0.5666D+00
32   0.3273D+01~ 0.3333D+01   0.6097D+00
33   0.3333D+01~ 0.3393D+01   0.6513D+00
34   0.3393D+01~ 0.3453D+01   0.6909D+00
35   0.3453D+01~ 0.3514D+01   0.7279D+00
36   0.3514D+01~ 0.3574D+01   0.7620D+00
37   0.3574D+01~ 0.3634D+01   0.7930D+00
38   0.3634D+01~ 0.3694D+01   0.8209D+00
39   0.3694D+01~ 0.3755D+01   0.8458D+00
40   0.3755D+01~ 0.3815D+01   0.8678D+00
```

e) $v=0.0039062$ ABIC=1920.57



```
NO.  CLASS BOUNDARIES          ESTMATED
                               PROBABILITIES
                                         0.0         0.2000D+00        0.4000D+00        0.6000D+00        0.8000D+00        0.1000D+01
                                                0.1000D+00        0.3000D+00        0.5000D+00        0.7000D+00        0.9000D+00
 1   0.1405D+01~ 0.1465D+01   0.2220D-01
 2   0.1465D+01~ 0.1525D+01   0.2458D-01
 3   0.1525D+01~ 0.1586D+01   0.2723D-01
 4   0.1586D+01~ 0.1646D+01   0.3017D-01
 5   0.1646D+01~ 0.1706D+01   0.3345D-01
 6   0.1706D+01~ 0.1766D+01   0.3712D-01
 7   0.1766D+01~ 0.1827D+01   0.4125D-01
 8   0.1827D+01~ 0.1887D+01   0.4590D-01
 9   0.1887D+01~ 0.1947D+01   0.5114D-01
10   0.1947D+01~ 0.2007D+01   0.5706D-01
11   0.2007D+01~ 0.2068D+01   0.6375D-01
12   0.2068D+01~ 0.2128D+01   0.7129D-01
13   0.2128D+01~ 0.2188D+01   0.7981D-01
14   0.2188D+01~ 0.2246D+01   0.8947D-01
15   0.2248D+01~ 0.2309D+01   0.1004D+00
16   0.2309D+01~ 0.2369D+01   0.1128D+00
17   0.2369D+01~ 0.2429D+01   0.1269D+00
18   0.2429D+01~ 0.2489D+01   0.1428D+00
19   0.2489D+01~ 0.2550D+01   0.1608D+00
20   0.2550D+01~ 0.2610D+01   0.1810D+00
21   0.2610D+01~ 0.2670D+01   0.2036D+00
22   0.2670D+01~ 0.2730D+01   0.2267D+00
23   0.2730D+01~ 0.2791D+01   0.2564D+00
24   0.2791D+01~ 0.2851D+01   0.2868D+00
25   0.2851D+01~ 0.2911D+01   0.3198D+00
26   0.2911D+01~ 0.2971D+01   0.3552D+00
27   0.2971D+01~ 0.3032D+01   0.3928D+00
28   0.3032D+01~ 0.3092D+01   0.4321D+00
29   0.3092D+01~ 0.3152D+01   0.4727D+00
30   0.3152D+01~ 0.3212D+01   0.5141D+00
31   0.3212D+01~ 0.3273D+01   0.5555D+00
32   0.3273D+01~ 0.3333D+01   0.5964D+00
33   0.3333D+01~ 0.3393D+01   0.6362D+00
34   0.3393D+01~ 0.3453D+01   0.6743D+00
35   0.3453D+01~ 0.3514D+01   0.7104D+00
36   0.3514D+01~ 0.3574D+01   0.7440D+00
37   0.3574D+01~ 0.3634D+01   0.7750D+00
38   0.3634D+01~ 0.3694D+01   0.8032D+00
39   0.3694D+01~ 0.3755D+01   0.8287D+00
40   0.3755D+01~ 0.3815D+0'   0.8515D+00
```

Fig. 2.d-e

Our procedure is applicable to cases where the response curve has more complex structure. The next example is a binary time series data pertaining to the sequence of the dates of rainfall observed in the basin of the Kanna River (1. 1. 1956–12. 31. 1956) and is analyzed earlier by Ozaki [12] as a point process analysis problem.

When x and $n(a, b]$ denote the time and the number of occurrences of a specific event during the time interval $a<x\leqq b$, the point process analysis concerns with the estimation of the intensity function $\lambda(x)$ defined by

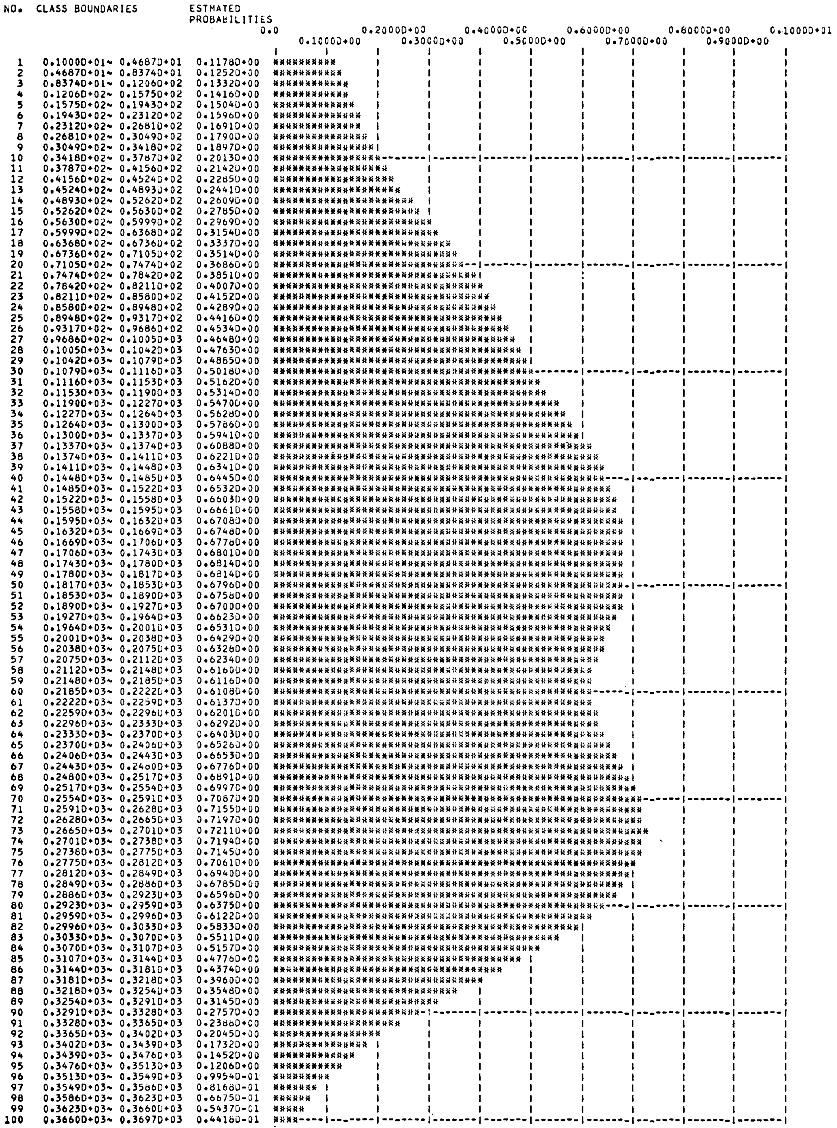
$$\Pr\{n(x, x+\Delta x]=1\}=\lambda(x)\Delta x+o(\Delta x) .$$

If $o(\Delta x)$ is negligible uniformly for some finite Δx, compared with $\lambda(x)\Delta x$, we can approximate $\Pr\{n(x, x+\Delta x]=1\}$ by a conditional probability function $p(x)$ defined by

$$p(x)=\lambda(x)\Delta x\ .$$

$p(x)$ is approximately the probability of the occurrence of a specific event during time x to $x+\Delta x$.

When we set Δt equal to one day and apply the present procedure to the series of events of rainfall occurrences, we get Figure 3 as the final estimation of the conditional probability function. In the figure seasonal variation of the intensity function is clearly seen: the two



```
NO.  CLASS BOUNDARIES          ESTMATED
                               PROBABILITIES
  1  0.1000D+01~ 0.4687D+01    0.1178D+00
  2  0.4687D+01~ 0.8374D+01    0.1252D+00
  3  0.8374D+01~ 0.1206D+02    0.1332D+00
  4  0.1206D+02~ 0.1575D+02    0.1416D+00
  5  0.1575D+02~ 0.1943D+02    0.1504D+00
  6  0.1943D+02~ 0.2312D+02    0.1596D+00
  7  0.2312D+02~ 0.2681D+02    0.1691D+00
  8  0.2681D+02~ 0.3049D+02    0.1790D+00
  9  0.3049D+02~ 0.3418D+02    0.1897D+00
 10  0.3418D+02~ 0.3787D+02    0.2013D+00
 11  0.3787D+02~ 0.4156D+02    0.2142D+00
 12  0.4156D+02~ 0.4524D+02    0.2285D+00
 13  0.4524D+02~ 0.4893D+02    0.2441D+00
 14  0.4893D+02~ 0.5262D+02    0.2609D+00
 15  0.5262D+02~ 0.5630D+02    0.2785D+00
 16  0.5630D+02~ 0.5999D+02    0.2969D+00
 17  0.5999D+02~ 0.6368D+02    0.3154D+00
 18  0.6368D+02~ 0.6736D+02    0.3337D+00
 19  0.6736D+02~ 0.7105D+02    0.3514D+00
 20  0.7105D+02~ 0.7474D+02    0.3686D+00
 21  0.7474D+02~ 0.7842D+02    0.3851D+00
 22  0.7842D+02~ 0.8211D+02    0.4007D+00
 23  0.8211D+02~ 0.8580D+02    0.4152D+00
 24  0.8580D+02~ 0.8948D+02    0.4289D+00
 25  0.8948D+02~ 0.9317D+02    0.4416D+00
 26  0.9317D+02~ 0.9686D+02    0.4534D+00
 27  0.9686D+02~ 0.1005D+03    0.4648D+00
 28  0.1005D+03~ 0.1042D+03    0.4763D+00
 29  0.1042D+03~ 0.1079D+03    0.4865D+00
 30  0.1079D+03~ 0.1116D+03    0.5018D+00
 31  0.1116D+03~ 0.1153D+03    0.5162D+00
 32  0.1153D+03~ 0.1190D+03    0.5314D+00
 33  0.1190D+03~ 0.1227D+03    0.5470D+00
 34  0.1227D+03~ 0.1264D+03    0.5628D+00
 35  0.1264D+03~ 0.1300D+03    0.5786D+00
 36  0.1300D+03~ 0.1337D+03    0.5941D+00
 37  0.1337D+03~ 0.1374D+03    0.6088D+00
 38  0.1374D+03~ 0.1411D+03    0.6221D+00
 39  0.1411D+03~ 0.1448D+03    0.6341D+00
 40  0.1448D+03~ 0.1485D+03    0.6445D+00
 41  0.1485D+03~ 0.1522D+03    0.6532D+00
 42  0.1522D+03~ 0.1558D+03    0.6603D+00
 43  0.1558D+03~ 0.1595D+03    0.6661D+00
 44  0.1595D+03~ 0.1632D+03    0.6708D+00
 45  0.1632D+03~ 0.1669D+03    0.6748D+00
 46  0.1669D+03~ 0.1706D+03    0.6776D+00
 47  0.1706D+03~ 0.1743D+03    0.6801D+00
 48  0.1743D+03~ 0.1780D+03    0.6814D+00
 49  0.1780D+03~ 0.1817D+03    0.6814D+00
 50  0.1817D+03~ 0.1853D+03    0.6796D+00
 51  0.1853D+03~ 0.1890D+03    0.6758D+00
 52  0.1890D+03~ 0.1927D+03    0.6700D+00
 53  0.1927D+03~ 0.1964D+03    0.6623D+00
 54  0.1964D+03~ 0.2001D+03    0.6531D+00
 55  0.2001D+03~ 0.2038D+03    0.6429D+00
 56  0.2038D+03~ 0.2075D+03    0.6328D+00
 57  0.2075D+03~ 0.2112D+03    0.6234D+00
 58  0.2112D+03~ 0.2148D+03    0.6160D+00
 59  0.2148D+03~ 0.2185D+03    0.6116D+00
 60  0.2185D+03~ 0.2222D+03    0.6108D+00
 61  0.2222D+03~ 0.2259D+03    0.6137D+00
 62  0.2259D+03~ 0.2296D+03    0.6201D+00
 63  0.2296D+03~ 0.2333D+03    0.6292D+00
 64  0.2333D+03~ 0.2370D+03    0.6403D+00
 65  0.2370D+03~ 0.2406D+03    0.6526D+00
 66  0.2406D+03~ 0.2443D+03    0.6653D+00
 67  0.2443D+03~ 0.2480D+03    0.6776D+00
 68  0.2480D+03~ 0.2517D+03    0.6891D+00
 69  0.2517D+03~ 0.2554D+03    0.6997D+00
 70  0.2554D+03~ 0.2591D+03    0.7087D+00
 71  0.2591D+03~ 0.2628D+03    0.7155D+00
 72  0.2628D+03~ 0.2665D+03    0.7197D+00
 73  0.2665D+03~ 0.2701D+03    0.7211D+00
 74  0.2701D+03~ 0.2738D+03    0.7194D+00
 75  0.2738D+03~ 0.2775D+03    0.7145D+00
 76  0.2775D+03~ 0.2812D+03    0.7061D+00
 77  0.2812D+03~ 0.2849D+03    0.6940D+00
 78  0.2849D+03~ 0.2886D+03    0.6785D+00
 79  0.2886D+03~ 0.2923D+03    0.6596D+00
 80  0.2923D+03~ 0.2959D+03    0.6375D+00
 81  0.2959D+03~ 0.2996D+03    0.6122D+00
 82  0.2996D+03~ 0.3033D+03    0.5833D+00
 83  0.3033D+03~ 0.3070D+03    0.5511D+00
 84  0.3070D+03~ 0.3107D+03    0.5157D+00
 85  0.3107D+03~ 0.3144D+03    0.4776D+00
 86  0.3144D+03~ 0.3181D+03    0.4374D+00
 87  0.3181D+03~ 0.3218D+03    0.3960D+00
 88  0.3218D+03~ 0.3254D+03    0.3548D+00
 89  0.3254D+03~ 0.3291D+03    0.3145D+00
 90  0.3291D+03~ 0.3328D+03    0.2757D+00
 91  0.3328D+03~ 0.3365D+03    0.2386D+00
 92  0.3365D+03~ 0.3402D+03    0.2045D+00
 93  0.3402D+03~ 0.3439D+03    0.1732D+00
 94  0.3439D+03~ 0.3476D+03    0.1452D+00
 95  0.3476D+03~ 0.3513D+03    0.1206D+00
 96  0.3513D+03~ 0.3549D+03    0.9954D-01
 97  0.3549D+03~ 0.3586D+03    0.8168D-01
 98  0.3586D+03~ 0.3623D+03    0.6675D-01
 99  0.3623D+03~ 0.3660D+03    0.5437D-01
100  0.3660D+03~ 0.3697D+03    0.4418D-01
```

Fig. 3

peaks corresponds to the two rainy season of late June and September; the one hump corresponds to the rainy spell at rape-plant blossoming time in late March and early April. The shape of the estimated intensity function is almost the same as that by Ozaki shown in Figure 4.

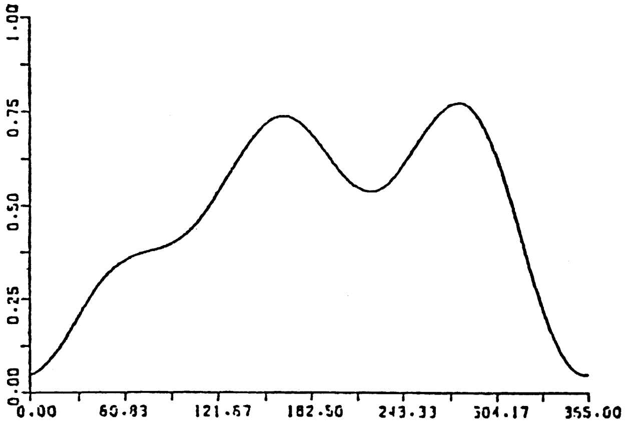


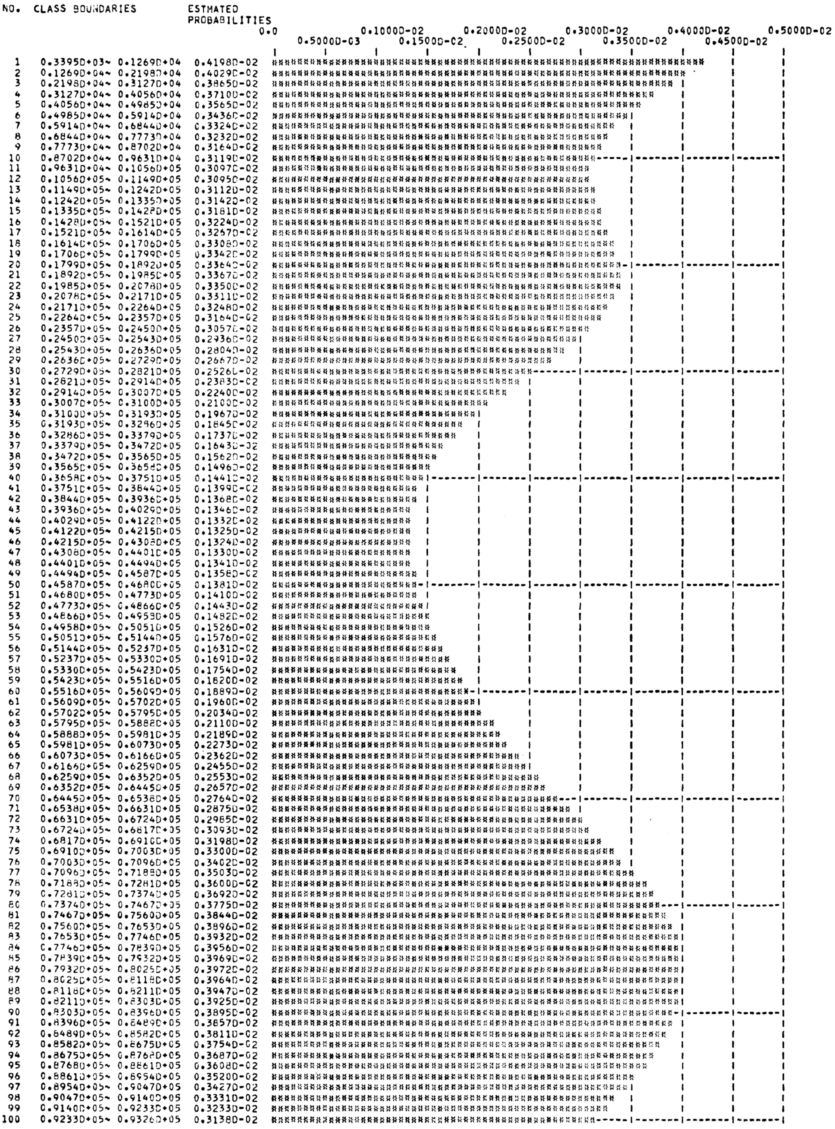
Fig. 4. From Fig. 13 in [12].

In the point process analysis the probabilities to be estimated are in general very close to zero. Our procedure remains valid to such cases. Fig. 5.a shows the final estimation of the intensity function for the time series data of computer failures (Machine 3) reported by Lewis [7]. Readers should note that the rate of occurrence of failures in very small and that a particular trend in the rate is clearly found. No significant trends were found for the data of Machine 1 and 2 of the above paper (Only the result for Machine 1 is shown in Figure 5.b).

The present procedure is also useful for the analysis of social survey data. The data shown in Table 2 were obtained in a public opinion poll conducted by the Institute of Statistical Mathematics in Tokyo in December 1981. They asked whether a respondent prospects as the existing one-party cabinet will endure or not. Table 2 is the two-way table of responses to this question by age. Our purpose is to estimate the probability supporting the first response category "will endure" in each age group. The final result is shown in Figure 6 which shows that the probability of the first opinion takes its maximum value 0.7143 at the age of 42. This seems to mean that a man of middle age has the most conservative view. Thus the present procedure is applicable even to the data with some empty cells as seen in Table 2.

We shall finally add that the present procedure leads to a new look at the discriminant analysis.

The object of discriminant analysis has long been believed to set up, using the information conveyed by observed data, a rule to allocate



```
NO.  CLASS BOUNDARIES          ESTMATED
                               PROBABILITIES
  1  0.3395D+03~ 0.1269D+04  0.4198D-02
  2  0.1269D+04~ 0.2198D+04  0.4029D-02
  3  0.2198D+04~ 0.3127D+04  0.3865D-02
  4  0.3127D+04~ 0.4056D+04  0.3710D-02
  5  0.4056D+04~ 0.4985D+04  0.3565D-02
  6  0.4985D+04~ 0.5914D+04  0.3436D-02
  7  0.5914D+04~ 0.6844D+04  0.3324D-02
  8  0.6844D+04~ 0.7773D+04  0.3232D-02
  9  0.7773D+04~ 0.8702D+04  0.3164D-02
 10  0.8702D+04~ 0.9631D+04  0.3119D-02
 11  0.9631D+04~ 0.1056D+05  0.3097D-02
 12  0.1056D+05~ 0.1149D+05  0.3095D-02
 13  0.1149D+05~ 0.1242D+05  0.3112D-02
 14  0.1242D+05~ 0.1335D+05  0.3142D-02
 15  0.1335D+05~ 0.1428D+05  0.3181D-02
 16  0.1428D+05~ 0.1521D+05  0.3224D-02
 17  0.1521D+05~ 0.1614D+05  0.3267D-02
 18  0.1614D+05~ 0.1706D+05  0.3308D-02
 19  0.1706D+05~ 0.1799D+05  0.3342D-02
 20  0.1799D+05~ 0.1892D+05  0.3364D-02
 21  0.1892D+05~ 0.1985D+05  0.3367D-02
 22  0.1985D+05~ 0.2078D+05  0.3350D-02
 23  0.2078D+05~ 0.2171D+05  0.3311D-02
 24  0.2171D+05~ 0.2264D+05  0.3248D-02
 25  0.2264D+05~ 0.2357D+05  0.3164D-02
 26  0.2357D+05~ 0.2450D+05  0.3057D-02
 27  0.2450D+05~ 0.2543D+05  0.2936D-02
 28  0.2543D+05~ 0.2636D+05  0.2804D-02
 29  0.2636D+05~ 0.2729D+05  0.2667D-02
 30  0.2729D+05~ 0.2821D+05  0.2526D-02
 31  0.2821D+05~ 0.2914D+05  0.2383D-02
 32  0.2914D+05~ 0.3007D+05  0.2240D-02
 33  0.3007D+05~ 0.3100D+05  0.2100D-02
 34  0.3100D+05~ 0.3193D+05  0.1967D-02
 35  0.3193D+05~ 0.3286D+05  0.1845D-02
 36  0.3286D+05~ 0.3379D+05  0.1737D-02
 37  0.3379D+05~ 0.3472D+05  0.1643D-02
 38  0.3472D+05~ 0.3565D+05  0.1562D-02
 39  0.3565D+05~ 0.3658D+05  0.1496D-02
 40  0.3658D+05~ 0.3751D+05  0.1441D-02
 41  0.3751D+05~ 0.3844D+05  0.1399D-02
 42  0.3844D+05~ 0.3936D+05  0.1368D-02
 43  0.3936D+05~ 0.4029D+05  0.1346D-02
 44  0.4029D+05~ 0.4122D+05  0.1332D-02
 45  0.4122D+05~ 0.4215D+05  0.1325D-02
 46  0.4215D+05~ 0.4308D+05  0.1324D-02
 47  0.4308D+05~ 0.4401D+05  0.1330D-02
 48  0.4401D+05~ 0.4494D+05  0.1341D-02
 49  0.4494D+05~ 0.4587D+05  0.1358D-02
 50  0.4587D+05~ 0.4680D+05  0.1381D-02
 51  0.4680D+05~ 0.4773D+05  0.1410D-02
 52  0.4773D+05~ 0.4866D+05  0.1443D-02
 53  0.4866D+05~ 0.4958D+05  0.1482D-02
 54  0.4958D+05~ 0.5051D+05  0.1526D-02
 55  0.5051D+05~ 0.5144D+05  0.1576D-02
 56  0.5144D+05~ 0.5237D+05  0.1631D-02
 57  0.5237D+05~ 0.5330D+05  0.1691D-02
 58  0.5330D+05~ 0.5423D+05  0.1754D-02
 59  0.5423D+05~ 0.5516D+05  0.1820D-02
 60  0.5516D+05~ 0.5609D+05  0.1889D-02
 61  0.5609D+05~ 0.5702D+05  0.1960D-02
 62  0.5702D+05~ 0.5795D+05  0.2034D-02
 63  0.5795D+05~ 0.5888D+05  0.2110D-02
 64  0.5888D+05~ 0.5981D+05  0.2189D-02
 65  0.5981D+05~ 0.6073D+05  0.2273D-02
 66  0.6073D+05~ 0.6166D+05  0.2362D-02
 67  0.6166D+05~ 0.6259D+05  0.2455D-02
 68  0.6259D+05~ 0.6352D+05  0.2553D-02
 69  0.6352D+05~ 0.6445D+05  0.2657D-02
 70  0.6445D+05~ 0.6538D+05  0.2764D-02
 71  0.6538D+05~ 0.6631D+05  0.2875D-02
 72  0.6631D+05~ 0.6724D+05  0.2985D-02
 73  0.6724D+05~ 0.6817D+05  0.3093D-02
 74  0.6817D+05~ 0.6910D+05  0.3198D-02
 75  0.6910D+05~ 0.7003D+05  0.3300D-02
 76  0.7003D+05~ 0.7096D+05  0.3402D-02
 77  0.7096D+05~ 0.7189D+05  0.3503D-02
 78  0.7188D+05~ 0.7281D+05  0.3600D-02
 79  0.7281D+05~ 0.7374D+05  0.3692D-02
 80  0.7374D+05~ 0.7467D+05  0.3775D-02
 81  0.7467D+05~ 0.7560D+05  0.3844D-02
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 83  0.7653D+05~ 0.7746D+05  0.3932D-02
 84  0.7746D+05~ 0.7839D+05  0.3956D-02
 85  0.7839D+05~ 0.7932D+05  0.3969D-02
 86  0.7932D+05~ 0.8025D+05  0.3972D-02
 87  0.8025D+05~ 0.8118D+05  0.3964D-02
 88  0.8118D+05~ 0.8211D+05  0.3947D-02
 89  0.8211D+05~ 0.8303D+05  0.3925D-02
 90  0.8303D+05~ 0.8396D+05  0.3895D-02
 91  0.8396D+05~ 0.8489D+05  0.3857D-02
 92  0.8489D+05~ 0.8582D+05  0.3811D-02
 93  0.8582D+05~ 0.8675D+05  0.3754D-02
 94  0.8675D+05~ 0.8768D+05  0.3687D-02
 95  0.8768D+05~ 0.8861D+05  0.3608D-02
 96  0.8861D+05~ 0.8954D+05  0.3520D-02
 97  0.8954D+05~ 0.9047D+05  0.3427D-02
 98  0.9047D+05~ 0.9140D+05  0.3331D-02
 99  0.9140D+05~ 0.9233D+05  0.3233D-02
100  0.9233D+05~ 0.9326D+05  0.3138D-02
```
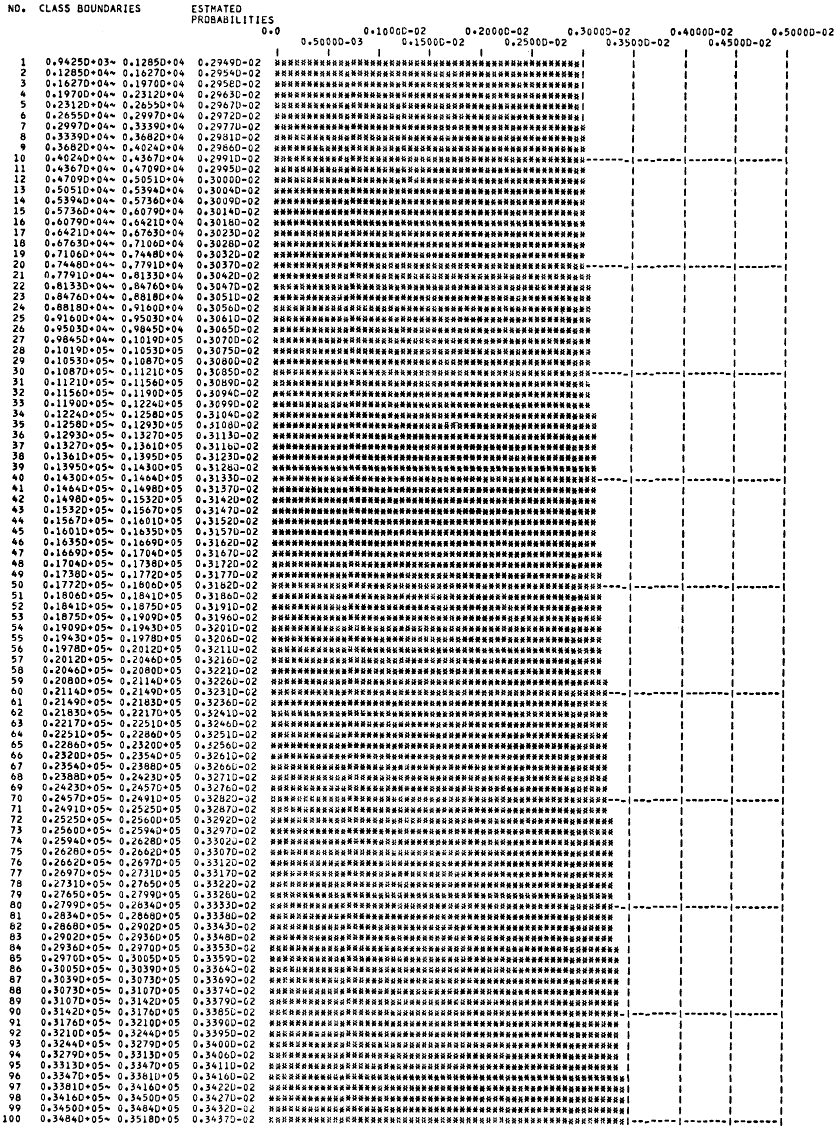

Fig. 5.a

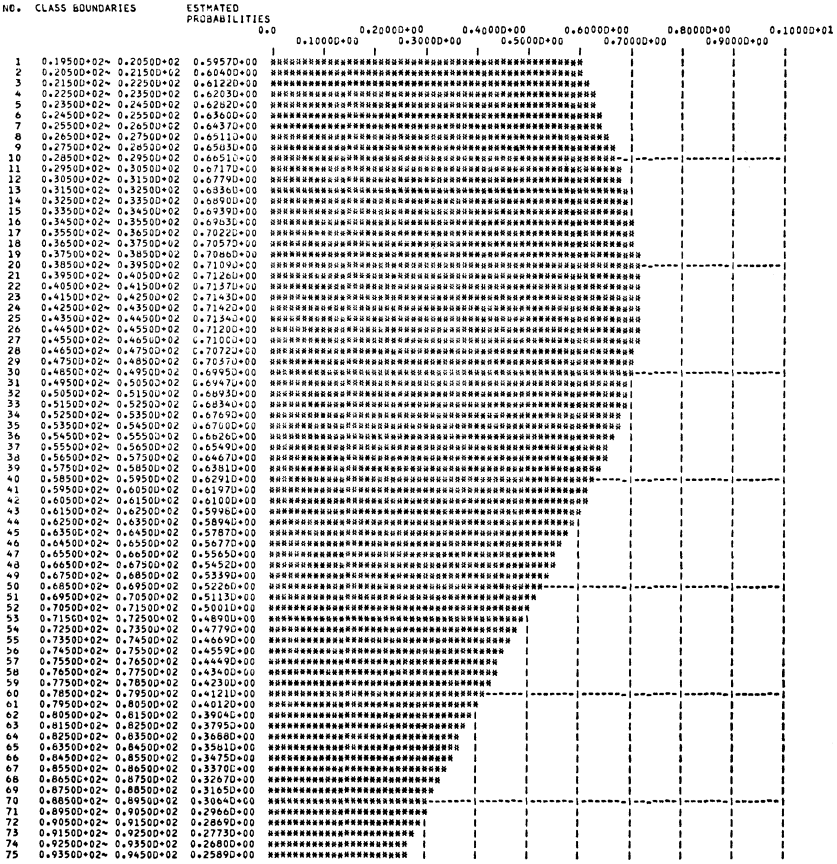


```
NO.  CLASS BOUNDARIES         ESTMATED
                              PROBABILITIES
                                        0.0           0.1000D-02      0.2000D-02      0.3000D-02      0.4000D-02      0.5000D-02
                                               0.5000D-03      0.1500D-02      0.2500D-02      0.3500D-02      0.4500D-02
  1   0.9425D+03~ 0.1285D+04  0.2949D-02
  2   0.1285D+04~ 0.1627D+04  0.2954D-02
  3   0.1627D+04~ 0.1970D+04  0.2958D-02
  4   0.1970D+04~ 0.2312D+04  0.2963D-02
  5   0.2312D+04~ 0.2655D+04  0.2967D-02
  6   0.2655D+04~ 0.2997D+04  0.2972D-02
  7   0.2997D+04~ 0.3339D+04  0.2977D-02
  8   0.3339D+04~ 0.3682D+04  0.2981D-02
  9   0.3682D+04~ 0.4024D+04  0.2986D-02
 10   0.4024D+04~ 0.4367D+04  0.2991D-02
 11   0.4367D+04~ 0.4709D+04  0.2995D-02
 12   0.4709D+04~ 0.5051D+04  0.3000D-02
 13   0.5051D+04~ 0.5394D+04  0.3004D-02
 14   0.5394D+04~ 0.5736D+04  0.3009D-02
 15   0.5736D+04~ 0.6079D+04  0.3014D-02
 16   0.6079D+04~ 0.6421D+04  0.3018D-02
 17   0.6421D+04~ 0.6763D+04  0.3023D-02
 18   0.6763D+04~ 0.7106D+04  0.3028D-02
 19   0.7106D+04~ 0.7448D+04  0.3032D-02
 20   0.7448D+04~ 0.7791D+04  0.3037D-02
 21   0.7791D+04~ 0.8133D+04  0.3042D-02
 22   0.8133D+04~ 0.8476D+04  0.3047D-02
 23   0.8476D+04~ 0.8818D+04  0.3051D-02
 24   0.8818D+04~ 0.9160D+04  0.3056D-02
 25   0.9160D+04~ 0.9503D+04  0.3061D-02
 26   0.9503D+04~ 0.9845D+04  0.3065D-02
 27   0.9845D+04~ 0.1019D+05  0.3070D-02
 28   0.1019D+05~ 0.1053D+05  0.3075D-02
 29   0.1053D+05~ 0.1087D+05  0.3080D-02
 30   0.1087D+05~ 0.1121D+05  0.3085D-02
 31   0.1121D+05~ 0.1156D+05  0.3089D-02
 32   0.1156D+05~ 0.1190D+05  0.3094D-02
 33   0.1190D+05~ 0.1224D+05  0.3099D-02
 34   0.1224D+05~ 0.1258D+05  0.3104D-02
 35   0.1258D+05~ 0.1293D+05  0.3108D-02
 36   0.1293D+05~ 0.1327D+05  0.3113D-02
 37   0.1327D+05~ 0.1361D+05  0.3116D-02
 38   0.1361D+05~ 0.1395D+05  0.3123D-02
 39   0.1395D+05~ 0.1430D+05  0.3128D-02
 40   0.1430D+05~ 0.1464D+05  0.3133D-02
 41   0.1464D+05~ 0.1498D+05  0.3137D-02
 42   0.1498D+05~ 0.1532D+05  0.3142D-02
 43   0.1532D+05~ 0.1567D+05  0.3147D-02
 44   0.1567D+05~ 0.1601D+05  0.3152D-02
 45   0.1601D+05~ 0.1635D+05  0.3157D-02
 46   0.1635D+05~ 0.1669D+05  0.3162D-02
 47   0.1669D+05~ 0.1704D+05  0.3167D-02
 48   0.1704D+05~ 0.1738D+05  0.3172D-02
 49   0.1738D+05~ 0.1772D+05  0.3177D-02
 50   0.1772D+05~ 0.1806D+05  0.3182D-02
 51   0.1806D+05~ 0.1841D+05  0.3186D-02
 52   0.1841D+05~ 0.1875D+05  0.3191D-02
 53   0.1875D+05~ 0.1909D+05  0.3196D-02
 54   0.1909D+05~ 0.1943D+05  0.3201D-02
 55   0.1943D+05~ 0.1978D+05  0.3206D-02
 56   0.1978D+05~ 0.2012D+05  0.3211D-02
 57   0.2012D+05~ 0.2046D+05  0.3216D-02
 58   0.2046D+05~ 0.2080D+05  0.3221D-02
 59   0.2080D+05~ 0.2114D+05  0.3226D-02
 60   0.2114D+05~ 0.2149D+05  0.3231D-02
 61   0.2149D+05~ 0.2183D+05  0.3236D-02
 62   0.2183D+05~ 0.2217D+05  0.3241D-02
 63   0.2217D+05~ 0.2251D+05  0.3246D-02
 64   0.2251D+05~ 0.2286D+05  0.3251D-02
 65   0.2286D+05~ 0.2320D+05  0.3256D-02
 66   0.2320D+05~ 0.2354D+05  0.3261D-02
 67   0.2354D+05~ 0.2388D+05  0.3266D-02
 68   0.2388D+05~ 0.2423D+05  0.3271D-02
 69   0.2423D+05~ 0.2457D+05  0.3276D-02
 70   0.2457D+05~ 0.2491D+05  0.3282D-02
 71   0.2491D+05~ 0.2525D+05  0.3287D-02
 72   0.2525D+05~ 0.2560D+05  0.3292D-02
 73   0.2560D+05~ 0.2594D+05  0.3297D-02
 74   0.2594D+05~ 0.2628D+05  0.3302D-02
 75   0.2628D+05~ 0.2662D+05  0.3307D-02
 76   0.2662D+05~ 0.2697D+05  0.3312D-02
 77   0.2697D+05~ 0.2731D+05  0.3317D-02
 78   0.2731D+05~ 0.2765D+05  0.3322D-02
 79   0.2765D+05~ 0.2799D+05  0.3328D-02
 80   0.2799D+05~ 0.2834D+05  0.3333D-02
 81   0.2834D+05~ 0.2868D+05  0.3338D-02
 82   0.2868D+05~ 0.2902D+05  0.3343D-02
 83   0.2902D+05~ 0.2936D+05  0.3348D-02
 84   0.2936D+05~ 0.2970D+05  0.3353D-02
 85   0.2970D+05~ 0.3005D+05  0.3359D-02
 86   0.3005D+05~ 0.3039D+05  0.3364D-02
 87   0.3039D+05~ 0.3073D+05  0.3369D-02
 88   0.3073D+05~ 0.3107D+05  0.3374D-02
 89   0.3107D+05~ 0.3142D+05  0.3379D-02
 90   0.3142D+05~ 0.3176D+05  0.3385D-02
 91   0.3176D+05~ 0.3210D+05  0.3390D-02
 92   0.3210D+05~ 0.3244D+05  0.3395D-02
 93   0.3244D+05~ 0.3279D+05  0.3400D-02
 94   0.3279D+05~ 0.3313D+05  0.3406D-02
 95   0.3313D+05~ 0.3347D+05  0.3411D-02
 96   0.3347D+05~ 0.3381D+05  0.3416D-02
 97   0.3381D+05~ 0.3416D+05  0.3422D-02
 98   0.3416D+05~ 0.3450D+05  0.3427D-02
 99   0.3450D+05~ 0.3484D+05  0.3432D-02
100   0.3484D+05~ 0.3518D+05  0.3437D-02
```

Fig. 5.b

Table 2

Age (yrs.)	The existing cabinet		Total
	‘Will endure’	‘Will not’	
20	3	4	7
21	5	5	10
22	10	3	13
23	5	6	11
24	6	4	10
25	6	2	8
26	5	7	12
27	10	3	13
28	9	5	14
29	6	6	12
30	12	4	16
31	10	2	12
32	11	6	17
33	14	5	19
34	12	7	19
35	6	4	10
36	6	1	7
37	7	5	12
38	14	3	17
39	13	3	16
40	7	4	11
41	6	3	9
42	5	2	7
43	9	3	12
44	10	3	13
45	7	5	12
46	13	1	14
47	11	3	14
48	8	0	8
49	7	1	8
50	4	5	9
51	12	3	15
52	9	7	16
53	6	4	10
54	7	5	12
55	6	2	8
56	5	5	10
57	5	4	9
58	9	3	12
59	5	3	8
60	4	2	6
61	7	3	10
62	5	5	10
63	5	1	6
64	7	2	9
65	1	2	3
66	1	3	4
67	3	1	4
68	0	3	3
69	1	4	5
70	0	1	1
71	3	6	9
72	2	3	5
73	3	1	4
74	0	0	0
75	1	2	3
76	1	0	1
77	1	1	2
78	2	2	4
79	0	0	0
80	2	1	3
81	0	0	0
82	1	1	2
83	0	0	0
84	0	0	0
85	0	1	1
86	0	0	0
87	0	0	0
88	0	0	0
89	0	0	0
90	0	0	0
91	0	0	0
92	0	0	0
93	0	0	0
94	0	1	1
Total	371	197	568



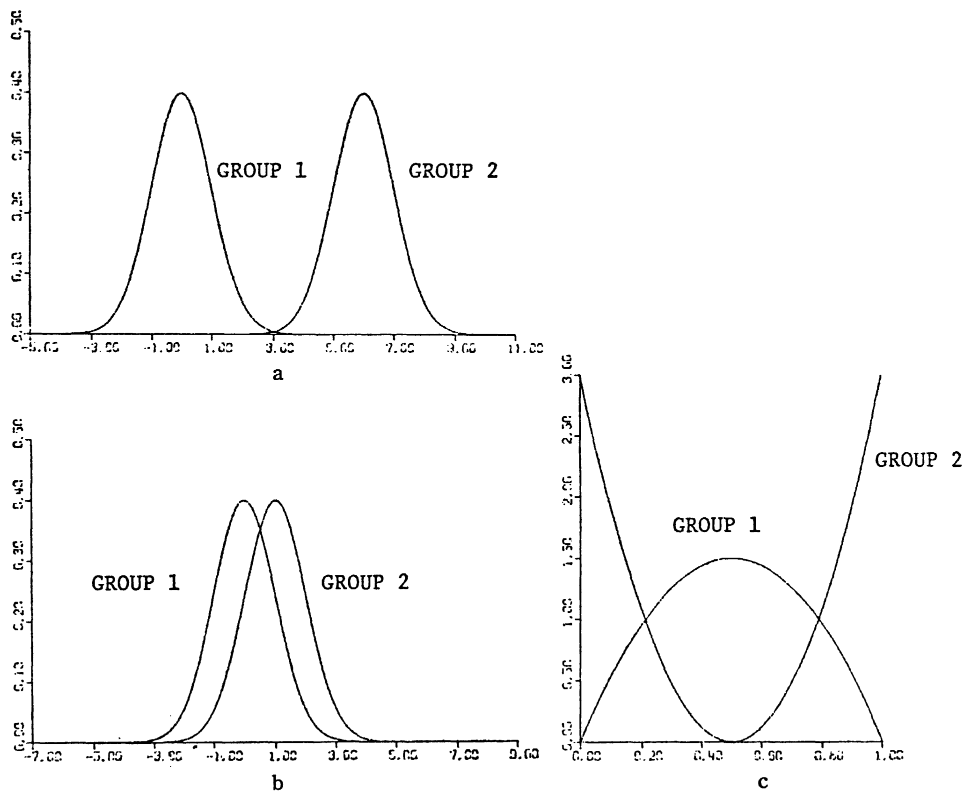
```
NO.  CLASS BOUNDARIES            ESTMATED
                                 PROBABILITIES
                                        0.0        0.2000D+00   0.4000D+00   0.6000D+00   0.8000D+00   0.1000D+01
                                              0.1000D+00   0.3000D+00   0.5000D+00   0.7000D+00   0.9000D+00
  1   0.1950D+02~ 0.2050D+02   0.5957D+00
  2   0.2050D+02~ 0.2150D+02   0.6040D+00
  3   0.2150D+02~ 0.2250D+02   0.6122D+00
  4   0.2250D+02~ 0.2350D+02   0.6203D+00
  5   0.2350D+02~ 0.2450D+02   0.6282D+00
  6   0.2450D+02~ 0.2550D+02   0.6360D+00
  7   0.2550D+02~ 0.2650D+02   0.6437D+00
  8   0.2650D+02~ 0.2750D+02   0.6511D+00
  9   0.2750D+02~ 0.2850D+02   0.6583D+00
 10   0.2850D+02~ 0.2950D+02   0.6651D+00
 11   0.2950D+02~ 0.3050D+02   0.6717D+00
 12   0.3050D+02~ 0.3150D+02   0.6779D+00
 13   0.3150D+02~ 0.3250D+02   0.6836D+00
 14   0.3250D+02~ 0.3350D+02   0.6890D+00
 15   0.3350D+02~ 0.3450D+02   0.6939D+00
 16   0.3450D+02~ 0.3550D+02   0.6983D+00
 17   0.3550D+02~ 0.3650D+02   0.7022D+00
 18   0.3650D+02~ 0.3750D+02   0.7057D+00
 19   0.3750D+02~ 0.3850D+02   0.7086D+00
 20   0.3850D+02~ 0.3950D+02   0.7109D+00
 21   0.3950D+02~ 0.4050D+02   0.7126D+00
 22   0.4050D+02~ 0.4150D+02   0.7137D+00
 23   0.4150D+02~ 0.4250D+02   0.7143D+00
 24   0.4250D+02~ 0.4350D+02   0.7142D+00
 25   0.4350D+02~ 0.4450D+02   0.7134D+00
 26   0.4450D+02~ 0.4550D+02   0.7120D+00
 27   0.4550D+02~ 0.4650D+02   0.7100D+00
 28   0.4650D+02~ 0.4750D+02   0.7072D+00
 29   0.4750D+02~ 0.4850D+02   0.7037D+00
 30   0.4850D+02~ 0.4950D+02   0.6995D+00
 31   0.4950D+02~ 0.5050D+02   0.6947D+00
 32   0.5050D+02~ 0.5150D+02   0.6893D+00
 33   0.5150D+02~ 0.5250D+02   0.6834D+00
 34   0.5250D+02~ 0.5350D+02   0.6769D+00
 35   0.5350D+02~ 0.5450D+02   0.6700D+00
 36   0.5450D+02~ 0.5550D+02   0.6626D+00
 37   0.5550D+02~ 0.5650D+02   0.6549D+00
 38   0.5650D+02~ 0.5750D+02   0.6467D+00
 39   0.5750D+02~ 0.5850D+02   0.6381D+00
 40   0.5850D+02~ 0.5950D+02   0.6291D+00
 41   0.5950D+02~ 0.6050D+02   0.6197D+00
 42   0.6050D+02~ 0.6150D+02   0.6100D+00
 43   0.6150D+02~ 0.6250D+02   0.5998D+00
 44   0.6250D+02~ 0.6350D+02   0.5894D+00
 45   0.6350D+02~ 0.6450D+02   0.5787D+00
 46   0.6450D+02~ 0.6550D+02   0.5677D+00
 47   0.6550D+02~ 0.6650D+02   0.5565D+00
 48   0.6650D+02~ 0.6750D+02   0.5452D+00
 49   0.6750D+02~ 0.6850D+02   0.5339D+00
 50   0.6850D+02~ 0.6950D+02   0.5226D+00
 51   0.6950D+02~ 0.7050D+02   0.5113D+00
 52   0.7050D+02~ 0.7150D+02   0.5001D+00
 53   0.7150D+02~ 0.7250D+02   0.4890D+00
 54   0.7250D+02~ 0.7350D+02   0.4779D+00
 55   0.7350D+02~ 0.7450D+02   0.4669D+00
 56   0.7450D+02~ 0.7550D+02   0.4559D+00
 57   0.7550D+02~ 0.7650D+02   0.4449D+00
 58   0.7650D+02~ 0.7750D+02   0.4340D+00
 59   0.7750D+02~ 0.7850D+02   0.4230D+00
 60   0.7850D+02~ 0.7950D+02   0.4121D+00
 61   0.7950D+02~ 0.8050D+02   0.4012D+00
 62   0.8050D+02~ 0.8150D+02   0.3904D+00
 63   0.8150D+02~ 0.8250D+02   0.3795D+00
 64   0.8250D+02~ 0.8350D+02   0.3688D+00
 65   0.8350D+02~ 0.8450D+02   0.3581D+00
 66   0.8450D+02~ 0.8550D+02   0.3475D+00
 67   0.8550D+02~ 0.8650D+02   0.3370D+00
 68   0.8650D+02~ 0.8750D+02   0.3267D+00
 69   0.8750D+02~ 0.8850D+02   0.3165D+00
 70   0.8850D+02~ 0.8950D+02   0.3064D+00
 71   0.8950D+02~ 0.9050D+02   0.2966D+00
 72   0.9050D+02~ 0.9150D+02   0.2869D+00
 73   0.9150D+02~ 0.9250D+02   0.2773D+00
 74   0.9250D+02~ 0.9350D+02   0.2680D+00
 75   0.9350D+02~ 0.9450D+02   0.2589D+00
```

Fig. 6

a new individual to the correct population. In almost every works in the literature it is taken for granted that each population is of normal distribution and discrimination can be realized by dividing the sample space into the corresponding regions. Such procedures may be useful if the approximation by normal distribution is natural and they are well separated as illustrated in Figure 7.a. In practical situations, however, it is not rare that the populations are entangled as shown in Figure 7.b. In this case it is dangerous to insist on the one-or-another type procedures. Moreover, we can seldom believe in the assumption of the normality. For example, given the following data set drawn from each of the two populations shown in Figure 7.c, any procedure based on the normality assumption has to fail.

GROUP 1	0.10	0.21	0.26	0.29	0.52	0.57	0.64	0.73	0.80	0.81
GROUP 2	0.01	0.04	0.21	0.29	0.79	0.90	0.94	0.96	0.97	1.00

These observations strongly suggest that we should not try to set

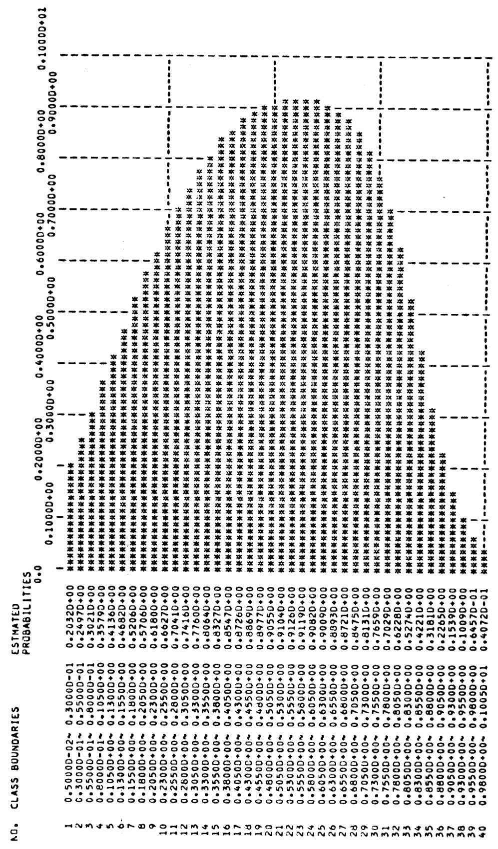


a b c

Fig. 7.a-c

up a rule to allocate an individual to one or another population but should modestly try only to estimate the probability of obtaining the individual from a specific population. Thus the problem reduces to the estimation of the probability of an individual's belonging to a specific population given a value of the explanatory variable.

When we set $c=40$ and apply the procedure to the above artificial data, we get the final result shown in Figure 8 which shows the series of estimates of the probability belonging to GROUP 1 in each class. The truth is that these two groups of data have randomly drawn from the two populations having the probability density functions $f_1(x)=-6x^2+6x$ and $f_2(x)=12x^2-12x+3$ $(0\leqq x\leqq 1)$, respectively. Thus the true function to be estimated is $p^*(x)=f_1(x)/\{f_1(x)+f_2(x)\}=-1+3/(6x^2-6x+3)$ as shown in Figure 9. The final estimate by our procedure resembles closely the true function in shape. These observations show that the present procedure reconstructs the true function fairly well. As is illustrated by the practical data in Table 2, it will be very often convenient to adopt our formulation.



```
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                              PROBABILITIES
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                                             0.1000D+00    0.3000D+00    0.5000D+00    0.7000D+00    0.9000D+00
  1  0.5000D-02~ 0.3000D-01  0.2032D+00
  2  0.3000D-01~ 0.5500D-01  0.2497D+00
  3  0.5500D-01~ 0.8000D-01  0.3021D+00
  4  0.8000D-01~ 0.1050D+00  0.3579D+00
  5  0.1050D+00~ 0.1300D+00  0.4136D+00
  6  0.1300D+00~ 0.1550D+00  0.4682D+00
  7  0.1550D+00~ 0.1800D+00  0.5206D+00
  8  0.1800D+00~ 0.2050D+00  0.5706D+00
  9  0.2050D+00~ 0.2300D+00  0.6180D+00
 10  0.2300D+00~ 0.2550D+00  0.6627D+00
 11  0.2550D+00~ 0.2800D+00  0.7041D+00
 12  0.2600D+00~ 0.3050D+00  0.7416D+00
 13  0.3050D+00~ 0.3300D+00  0.7760D+00
 14  0.3300D+00~ 0.3550D+00  0.8064D+00
 15  0.3550D+00~ 0.3800D+00  0.8327D+00
 16  0.3800D+00~ 0.4050D+00  0.8547D+00
 17  0.4050D+00~ 0.4300D+00  0.8726D+00
 18  0.4300D+00~ 0.4550D+00  0.8869D+00
 19  0.4550D+00~ 0.4800D+00  0.8977D+00
 20  0.4800D+00~ 0.5050D+00  0.9055D+00
 21  0.5050D+00~ 0.5300D+00  0.9104D+00
 22  0.5300D+00~ 0.5550D+00  0.9126D+00
 23  0.5550D+00~ 0.5800D+00  0.9119D+00
 24  0.5800D+00~ 0.6050D+00  0.9082D+00
 25  0.6050D+00~ 0.6300D+00  0.9009D+00
 26  0.6300D+00~ 0.6550D+00  0.8893D+00
 27  0.6550D+00~ 0.6800D+00  0.8721D+00
 28  0.6800D+00~ 0.7050D+00  0.8475D+00
 29  0.7050D+00~ 0.7300D+00  0.8131D+00
 30  0.7300D+00~ 0.7550D+00  0.7659D+00
 31  0.7550D+00~ 0.7800D+00  0.7029D+00
 32  0.7800D+00~ 0.8050D+00  0.6228D+00
 33  0.8050D+00~ 0.8300D+00  0.5274D+00
 34  0.8300D+00~ 0.8550D+00  0.4221D+00
 35  0.8550D+00~ 0.8800D+00  0.3181D+00
 36  0.8800D+00~ 0.9050D+00  0.2265D+00
 37  0.9050D+00~ 0.9300D+00  0.1539D+00
 38  0.9300D+00~ 0.9550D+00  0.1009D+00
 39  0.9550D+00~ 0.9800D+00  0.6457D-01
 40  0.9800D+00~ 0.1005D+01  0.4072D-01
```

Fig. 8

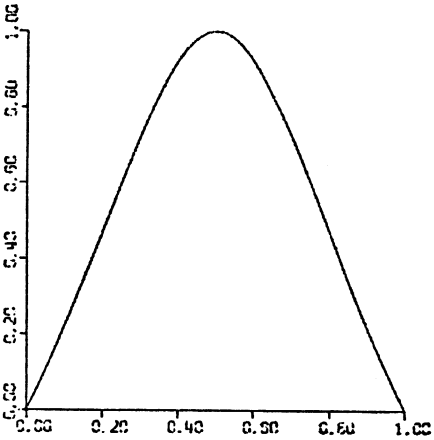
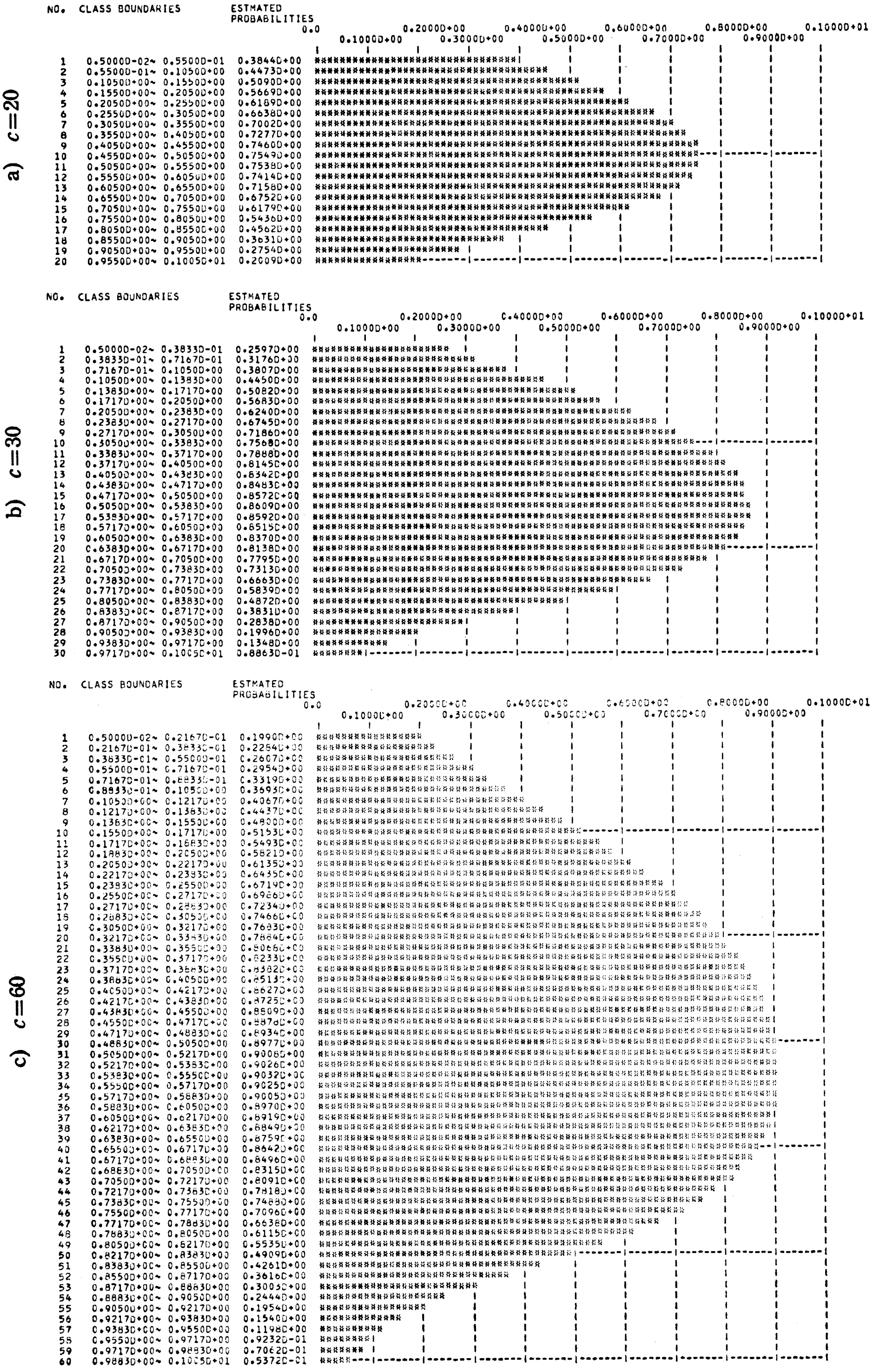


Fig. 9

6. Discussions

In our procedure the value of an explanatory variable is discretized into c classes. The reader may therefore fear that the result will seriously be affected by the choice of c. In this section we shall examine the effect of the discretization on the stability and the accuracy of the estimation.

Figures 10.a–c are the final estimates for the sample of 20 numbers in the preceding section in cases where we choose the number of initial categories c as 20, 30 and 60, respectively. We can see that these figures reconstruct the true function shown in Figure 9 in shape when c is sufficiently large. We can estimate the true function even in the case where c exceeds the sample size n and that the final estimate is fairly stable for a sufficiently large value of c. Only a result for the case of $n=20$ was shown here but the accuracy of the estimation, in general, improves as n increases.



a) $c=20$

```
NO.  CLASS BOUNDARIES          ESTMATED
                               PROBABILITIES
                                          0.0          0.2000D+00     0.4000D+00     0.6000D+00     0.8000D+00     0.1000D+01
                                                 0.1000D+00     0.3000D+00     0.5000D+00     0.7000D+00     0.9000D+00
  1   0.5000D-02~ 0.5500D-01   0.3844D+00
  2   0.5500D-01~ 0.1050D+00   0.4473D+00
  3   0.1050D+00~ 0.1550D+00   0.5090D+00
  4   0.1550D+00~ 0.2050D+00   0.5669D+00
  5   0.2050D+00~ 0.2550D+00   0.6189D+00
  6   0.2550D+00~ 0.3050D+00   0.6638D+00
  7   0.3050D+00~ 0.3550D+00   0.7002D+00
  8   0.3550D+00~ 0.4050D+00   0.7277D+00
  9   0.4050D+00~ 0.4550D+00   0.7460D+00
 10   0.4550D+00~ 0.5050D+00   0.7549D+00
 11   0.5050D+00~ 0.5550D+00   0.7538D+00
 12   0.5550D+00~ 0.6050D+00   0.7414D+00
 13   0.6050D+00~ 0.6550D+00   0.7158D+00
 14   0.6550D+00~ 0.7050D+00   0.6752D+00
 15   0.7050D+00~ 0.7550D+00   0.6179D+00
 16   0.7550D+00~ 0.8050D+00   0.5436D+00
 17   0.8050D+00~ 0.8550D+00   0.4562D+00
 18   0.8550D+00~ 0.9050D+00   0.3631D+00
 19   0.9050D+00~ 0.9550D+00   0.2754D+00
 20   0.9550D+00~ 0.1005D+01   0.2009D+00
```

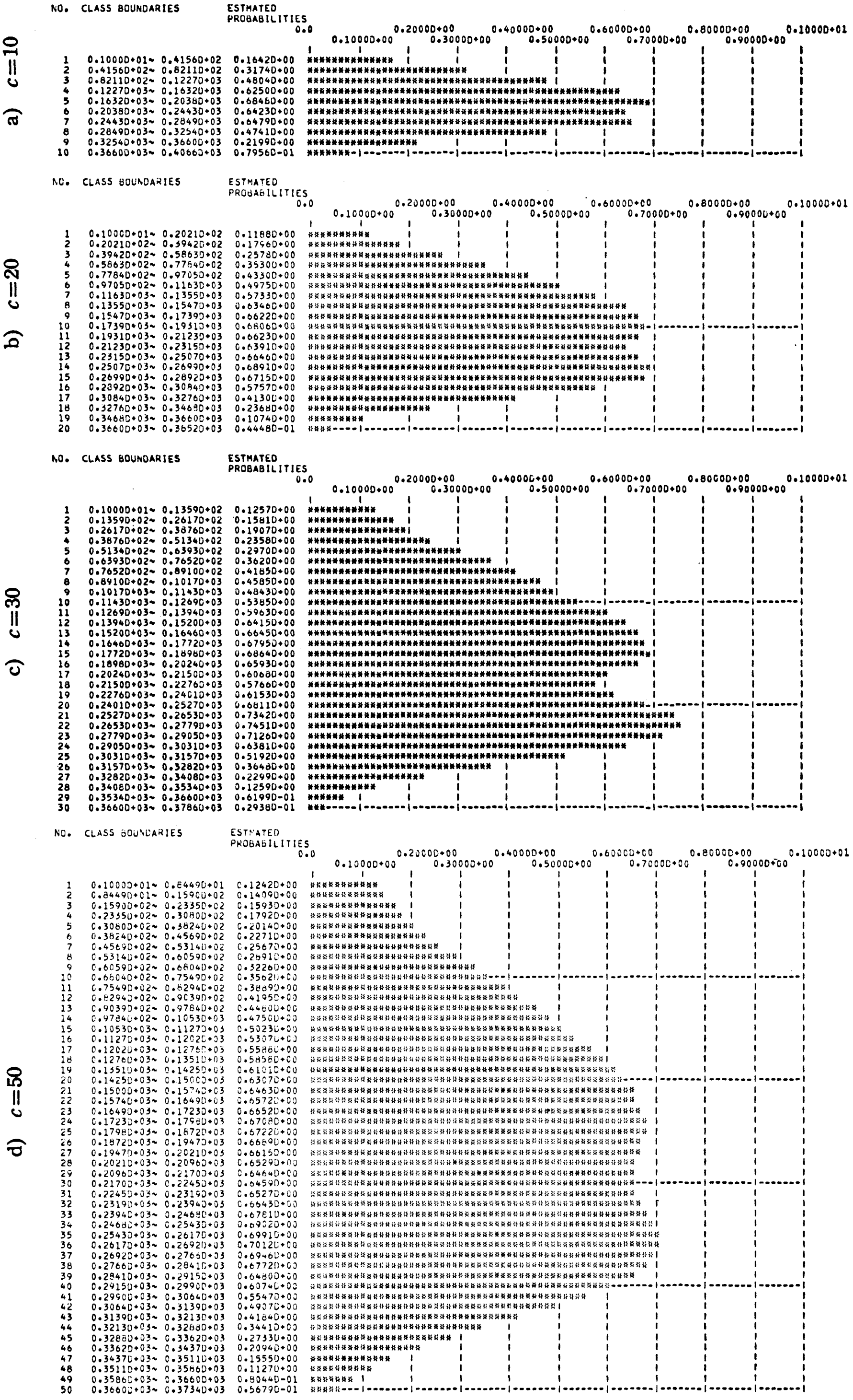
b) $c=30$

```
NO.  CLASS BOUNDARIES          ESTMATED
                               PROBABILITIES
                                          0.0          0.2000D+00     C.4000D+00     0.6000D+00     0.8000D+00     0.1000D+01
                                                 0.1000D+00     0.3000D+00     0.5000D+00     0.7000D+00     0.9000D+00
  1   0.5000D-02~ 0.3833D-01   0.2597D+00
  2   0.3833D-01~ 0.7167D-01   0.3176D+00
  3   0.7167D-01~ 0.1050D+00   0.3807D+00
  4   0.1050D+00~ 0.1383D+00   0.4450D+00
  5   0.1383D+00~ 0.1717D+00   0.5082D+00
  6   0.1717D+00~ 0.2050D+00   0.5683D+00
  7   0.2050D+00~ 0.2383D+00   0.6240D+00
  8   0.2383D+00~ 0.2717D+00   0.6745D+00
  9   0.2717D+00~ 0.3050D+00   0.7186D+00
 10   0.3050D+00~ 0.3383D+00   0.7568D+00
 11   0.3383D+00~ 0.3717D+00   0.7888D+00
 12   0.3717D+00~ 0.4050D+00   0.8145D+00
 13   0.4050D+00~ 0.4383D+00   0.8342D+00
 14   0.4383D+00~ 0.4717D+00   0.8483D+00
 15   0.4717D+00~ 0.5050D+00   0.8572D+00
 16   0.5050D+00~ 0.5383D+00   0.8609D+00
 17   0.5383D+00~ 0.5717D+00   0.8592D+00
 18   0.5717D+00~ 0.6050D+00   0.8515D+00
 19   0.6050D+00~ 0.6383D+00   0.8370D+00
 20   C.6383D+00~ 0.6717D+00   0.8138D+00
 21   0.6717D+00~ 0.7050D+00   0.7795D+00
 22   0.7050D+00~ 0.7383D+00   0.7313D+00
 23   0.7383D+00~ 0.7717D+00   0.6663D+00
 24   0.7717D+00~ 0.8050D+00   0.5839D+00
 25   0.8050D+00~ 0.8383D+00   0.4872D+00
 26   0.8383D+00~ 0.8717D+00   0.3831D+00
 27   0.8717D+00~ 0.9050D+00   0.2838D+00
 28   0.9050D+00~ 0.9383D+00   0.1996D+00
 29   0.9383D+00~ 0.9717D+00   0.1348D+00
 30   0.9717D+00~ 0.1005D+01   0.8863D-01
```

c) $c=60$

```
NO.  CLASS BOUNDARIES          ESTMATED
                               PROBABILITIES
                                          0.0          0.2000D+00     0.4000D+00     0.6000D+00     0.8000D+00     0.1000D+01
                                                 0.1000D+00     0.3000D+00     0.5000D+00     0.7000D+00     0.9000D+00
  1   0.5000D-02~ 0.2167D-01   0.1990D+00
  2   0.2167D-01~ 0.3833D-01   0.2284D+00
  3   0.3833D-01~ 0.5500D-01   0.2607D+00
  4   0.5500D-01~ 0.7167D-01   0.2954D+00
  5   0.7167D-01~ 0.8833D-01   0.3319D+00
  6   0.8833D-01~ 0.1050D+00   0.3693D+00
  7   0.1050D+00~ 0.1217D+00   0.4067D+00
  8   0.1217D+00~ 0.1383D+00   0.4437D+00
  9   0.1383D+00~ 0.1550D+00   0.4800D+00
 10   0.1550D+00~ 0.1717D+00   0.5153D+00
 11   0.1717D+00~ 0.1883D+00   0.5493D+00
 12   0.1883D+00~ 0.2050D+00   0.5821D+00
 13   0.2050D+00~ 0.2217D+00   0.6135D+00
 14   0.2217D+00~ 0.2383D+00   0.6435D+00
 15   0.2383D+00~ 0.2550D+00   0.6719D+00
 16   0.2550D+00~ 0.2717D+00   0.6986D+00
 17   0.2717D+00~ 0.2883D+00   0.7234D+00
 18   0.2883D+00~ 0.3050D+00   0.7466D+00
 19   0.3050D+00~ 0.3217D+00   0.7683D+00
 20   0.3217D+00~ 0.3383D+00   0.7864D+00
 21   0.3383D+00~ 0.3550D+00   0.8066D+00
 22   0.3550D+00~ 0.3717D+00   0.8233D+00
 23   0.3717D+00~ 0.3883D+00   0.8382D+00
 24   0.3883D+00~ 0.4050D+00   0.8513D+00
 25   0.4050D+00~ 0.4217D+00   0.8627D+00
 26   0.4217D+00~ 0.4383D+00   0.8725D+00
 27   0.4383D+00~ 0.4550D+00   0.8809D+00
 28   0.4550D+00~ 0.4717D+00   0.8876D+00
 29   0.4717D+00~ 0.4883D+00   0.8934D+00
 30   0.4883D+00~ 0.5050D+00   0.8977D+00
 31   0.5050D+00~ 0.5217D+00   0.9006D+00
 32   0.5217D+00~ 0.5383D+00   0.9026D+00
 33   0.5383D+00~ 0.5550D+00   0.9032D+00
 34   0.5550D+00~ 0.5717D+00   0.9025D+00
 35   0.5717D+00~ 0.5883D+00   0.9005D+00
 36   0.5883D+00~ 0.6050D+00   0.8970D+00
 37   0.6050D+00~ 0.6217D+00   0.8919D+00
 38   0.6217D+00~ 0.6383D+00   0.8849D+00
 39   0.6383D+00~ 0.6550D+00   0.8759D+00
 40   0.6550D+00~ 0.6717D+00   0.8642D+00
 41   0.6717D+00~ 0.6883D+00   0.8496D+00
 42   0.6883D+00~ 0.7050D+00   0.8315D+00
 43   0.7050D+00~ 0.7217D+00   0.8091D+00
 44   0.7217D+00~ 0.7383D+00   0.7818D+00
 45   0.7383D+00~ 0.7550D+00   0.7488D+00
 46   0.7550D+00~ 0.7717D+00   0.7096D+00
 47   0.7717D+00~ 0.7883D+00   0.6638D+00
 48   0.7883D+00~ 0.8050D+00   0.6115D+00
 49   0.8050D+00~ 0.8217D+00   0.5535D+00
 50   0.8217D+00~ 0.8383D+00   0.4909D+00
 51   0.8383D+00~ 0.8550D+00   0.4261D+00
 52   0.8550D+00~ 0.8717D+00   0.3616D+00
 53   0.8717D+00~ 0.8883D+00   0.3003D+00
 54   0.8883D+00~ 0.9050D+00   0.2444D+00
 55   0.9050D+00~ 0.9217D+00   0.1954D+00
 56   0.9217D+00~ 0.9383D+00   0.1540D+00
 57   0.9383D+00~ 0.9550D+00   0.1198D+00
 58   0.9550D+00~ 0.9717D+00   0.9232D-01
 59   0.9717D+00~ 0.9883D+00   0.7062D-01
 60   0.9883D+00~ 0.1005D+01   0.5372D-01
```

Fig. 10.a-c



a) $c=10$

```
NO.  CLASS BOUNDARIES           ESTMATED
                                PROBABILITIES
                                         0.0          0.2000D+00       0.4000D+00       0.6000D+00       0.8000D+00       0.1000D+01
                                              0.1000D+00       0.3000D+00       0.5000D+00       0.7000D+00       0.9000D+00
  1   0.1000D+01~ 0.4156D+02  0.1642D+00
  2   0.4156D+02~ 0.8211D+02  0.3174D+00
  3   0.8211D+02~ 0.1227D+03  0.4804D+00
  4   0.1227D+03~ 0.1632D+03  0.6250D+00
  5   0.1632D+03~ 0.2038D+03  0.6846D+00
  6   0.2038D+03~ 0.2443D+03  0.6423D+00
  7   0.2443D+03~ 0.2849D+03  0.6479D+00
  8   0.2849D+03~ 0.3254D+03  0.4741D+00
  9   0.3254D+03~ 0.3660D+03  0.2199D+00
 10   0.3660D+03~ 0.4066D+03  0.7956D-01
```

b) $c=20$

```
NO.  CLASS BOUNDARIES           ESTMATED
                                PROBABILITIES
                                         0.0          0.2000D+00       0.4000D+00       0.6000D+00       0.8000D+00       0.1000D+01
                                              0.1000D+00       0.3000D+00       0.5000D+00       0.7000D+00       0.9000D+00
  1   0.1000D+01~ 0.2021D+02  0.1188D+00
  2   0.2021D+02~ 0.3942D+02  0.1796D+00
  3   0.3942D+02~ 0.5863D+02  0.2578D+00
  4   0.5863D+02~ 0.7784D+02  0.3530D+00
  5   0.7784D+02~ 0.9705D+02  0.4330D+00
  6   0.9705D+02~ 0.1163D+03  0.4975D+00
  7   0.1163D+03~ 0.1355D+03  0.5733D+00
  8   0.1355D+03~ 0.1547D+03  0.6346D+00
  9   0.1547D+03~ 0.1739D+03  0.6622D+00
 10   0.1739D+03~ 0.1931D+03  0.6806D+00
 11   0.1931D+03~ 0.2123D+03  0.6623D+00
 12   0.2123D+03~ 0.2315D+03  0.6391D+00
 13   0.2315D+03~ 0.2507D+03  0.6646D+00
 14   0.2507D+03~ 0.2699D+03  0.6891D+00
 15   0.2699D+03~ 0.2892D+03  0.6715D+00
 16   0.2892D+03~ 0.3084D+03  0.5757D+00
 17   0.3084D+03~ 0.3276D+03  0.4130D+00
 18   0.3276D+03~ 0.3468D+03  0.2368D+00
 19   0.3468D+03~ 0.3660D+03  0.1074D+00
 20   0.3660D+03~ 0.3652D+03  0.4448D-01
```

c) $c=30$

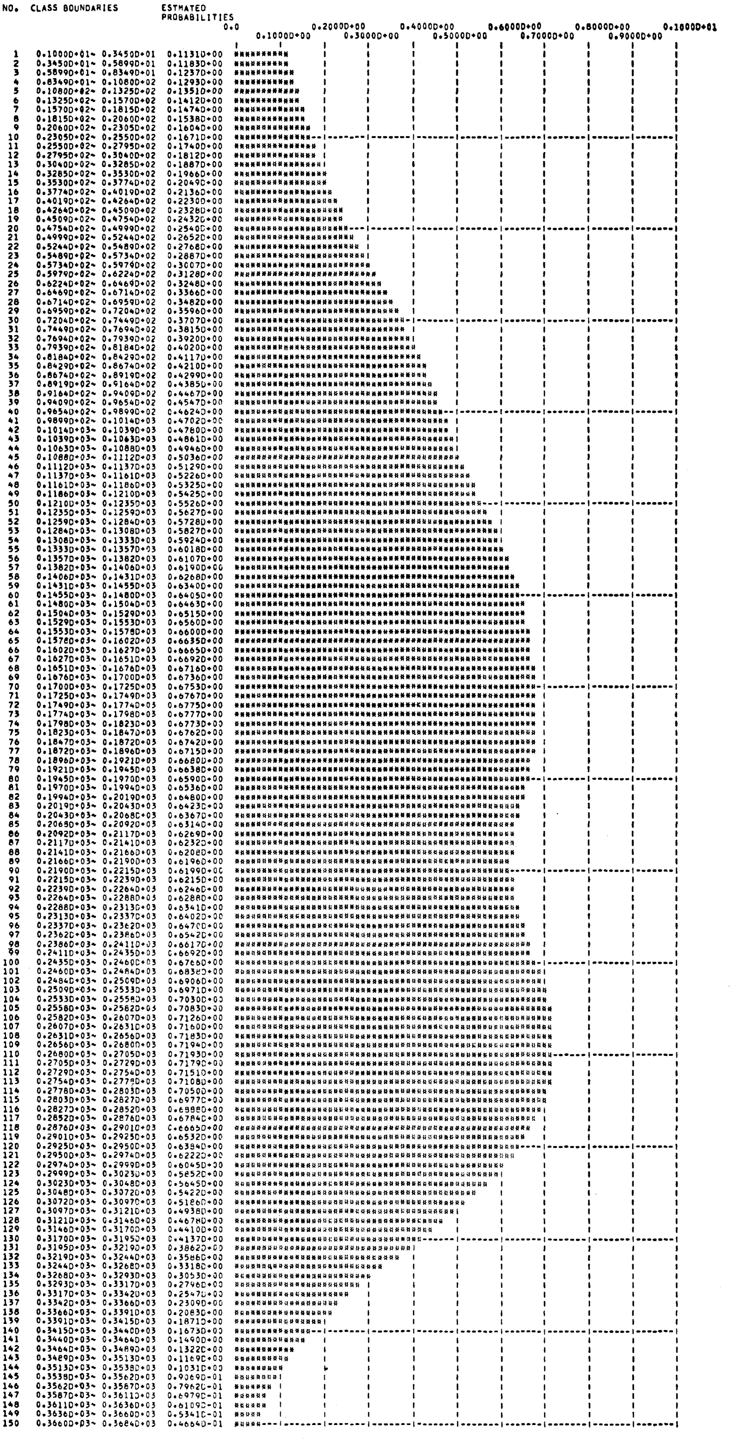
```
NO.  CLASS BOUNDARIES           ESTMATED
                                PROBABILITIES
                                         0.0          0.2000D+00       0.4000D+00       0.6000D+00       0.8000D+00       0.1000D+01
                                              0.1000D+00       0.3000D+00       0.5000D+00       0.7000D+00       0.9000D+00
  1   0.1000D+01~ 0.1359D+02  0.1257D+00
  2   0.1359D+02~ 0.2617D+02  0.1581D+00
  3   0.2617D+02~ 0.3876D+02  0.1907D+00
  4   0.3876D+02~ 0.5134D+02  0.2358D+00
  5   0.5134D+02~ 0.6393D+02  0.2970D+00
  6   0.6393D+02~ 0.7652D+02  0.3620D+00
  7   0.7652D+02~ 0.8910D+02  0.4185D+00
  8   0.8910D+02~ 0.1017D+03  0.4585D+00
  9   0.1017D+03~ 0.1143D+03  0.4843D+00
 10   0.1143D+03~ 0.1269D+03  0.5385D+00
 11   0.1269D+03~ 0.1394D+03  0.5963D+00
 12   0.1394D+03~ 0.1520D+03  0.6415D+00
 13   0.1520D+03~ 0.1646D+03  0.6645D+00
 14   0.1646D+03~ 0.1772D+03  0.6795D+00
 15   0.1772D+03~ 0.1898D+03  0.6864D+00
 16   0.1898D+03~ 0.2024D+03  0.6593D+00
 17   0.2024D+03~ 0.2150D+03  0.6068D+00
 18   0.2150D+03~ 0.2276D+03  0.5766D+00
 19   0.2276D+03~ 0.2401D+03  0.6153D+00
 20   0.2401D+03~ 0.2527D+03  0.6811D+00
 21   0.2527D+03~ 0.2653D+03  0.7342D+00
 22   0.2653D+03~ 0.2779D+03  0.7451D+00
 23   0.2779D+03~ 0.2905D+03  0.7126D+00
 24   0.2905D+03~ 0.3031D+03  0.6381D+00
 25   0.3031D+03~ 0.3157D+03  0.5192D+00
 26   0.3157D+03~ 0.3282D+03  0.3648D+00
 27   0.3282D+03~ 0.3408D+03  0.2299D+00
 28   0.3408D+03~ 0.3534D+03  0.1259D+00
 29   0.3534D+03~ 0.3660D+03  0.6199D-01
 30   0.3660D+03~ 0.3786D+03  0.2938D-01
```

d) $c=50$

```
NO.  CLASS BOUNDARIES           ESTMATED
                                PROBABILITIES
                                         0.0          0.2000D+00       0.4000D+00       0.6000D+00       0.8000D+00       0.1000D+01
                                              0.1000D+00       0.3000D+00       0.5000D+00       0.7000D+00       0.9000D+00
  1   0.1000D+01~ 0.8449D+01  0.1242D+00
  2   0.8449D+01~ 0.1590D+02  0.1409D+00
  3   0.1590D+02~ 0.2335D+02  0.1593D+00
  4   0.2335D+02~ 0.3080D+02  0.1792D+00
  5   0.3080D+02~ 0.3824D+02  0.2014D+00
  6   0.3824D+02~ 0.4569D+02  0.2271D+00
  7   0.4569D+02~ 0.5314D+02  0.2567D+00
  8   0.5314D+02~ 0.6059D+02  0.2691D+00
  9   0.6059D+02~ 0.6804D+02  0.3226D+00
 10   0.6604D+02~ 0.7549D+02  0.3562D+00
 11   0.7549D+02~ 0.8294D+02  0.3889D+00
 12   0.8294D+02~ 0.9039D+02  0.4195D+00
 13   0.9039D+02~ 0.9784D+02  0.4460D+00
 14   0.9784D+02~ 0.1053D+03  0.4750D+00
 15   0.1053D+03~ 0.1127D+03  0.5023D+00
 16   0.1127D+03~ 0.1202D+03  0.5307D+00
 17   0.1202D+03~ 0.1276D+03  0.5586D+00
 18   0.1276D+03~ 0.1351D+03  0.5856D+00
 19   0.1351D+03~ 0.1425D+03  0.6101D+00
 20   0.1425D+03~ 0.1500D+03  0.6307D+00
 21   0.1500D+03~ 0.1574D+03  0.6463D+00
 22   0.1574D+03~ 0.1649D+03  0.6572D+00
 23   0.1649D+03~ 0.1723D+03  0.6652D+00
 24   0.1723D+03~ 0.1798D+03  0.6708D+00
 25   0.1798D+03~ 0.1872D+03  0.6722D+00
 26   0.1872D+03~ 0.1947D+03  0.6669D+00
 27   0.1947D+03~ 0.2021D+03  0.6615D+00
 28   0.2021D+03~ 0.2096D+03  0.6529D+00
 29   0.2096D+03~ 0.2170D+03  0.6464D+00
 30   0.2170D+03~ 0.2245D+03  0.6459D+00
 31   0.2245D+03~ 0.2319D+03  0.6527D+00
 32   0.2319D+03~ 0.2394D+03  0.6643D+00
 33   0.2394D+03~ 0.2468D+03  0.6781D+00
 34   0.2468D+03~ 0.2543D+03  0.6902D+00
 35   0.2543D+03~ 0.2617D+03  0.6991D+00
 36   0.2617D+03~ 0.2692D+03  0.7012D+00
 37   0.2692D+03~ 0.2766D+03  0.6946D+00
 38   0.2766D+03~ 0.2841D+03  0.6772D+00
 39   0.2841D+03~ 0.2915D+03  0.6480D+00
 40   0.2915D+03~ 0.2990D+03  0.6074D+00
 41   0.2990D+03~ 0.3064D+03  0.5547D+00
 42   0.3064D+03~ 0.3139D+03  0.4907D+00
 43   0.3139D+03~ 0.3213D+03  0.4184D+00
 44   0.3213D+03~ 0.3288D+03  0.3441D+00
 45   0.3288D+03~ 0.3362D+03  0.2733D+00
 46   0.3362D+03~ 0.3437D+03  0.2094D+00
 47   0.3437D+03~ 0.3511D+03  0.1555D+00
 48   0.3511D+03~ 0.3586D+03  0.1127D+00
 49   0.3586D+03~ 0.3660D+03  0.8044D-01
 50   0.3660D+03~ 0.3734D+03  0.5679D-01
```

Fig. 11.a-d

e) $c=150$



```
NO.  CLASS BOUNDARIES          ESTMATED
                               PROBABILITIES
                                       0.0          0.2000D+00     0.4000D+00     0.6000D+00     0.8000D+00     0.1000D+01
                                              0.1000D+00     0.3000D+00     0.5000D+00     0.7000D+00     0.9000D+00
  1  0.1000D+01~ 0.3450D+01  0.1131D+00
  2  0.3450D+01~ 0.5899D+01  0.1183D+00
  3  0.5899D+01~ 0.8349D+01  0.1237D+00
  4  0.8349D+01~ 0.1080D+02  0.1293D+00
  5  0.1080D+02~ 0.1325D+02  0.1351D+00
  6  0.1325D+02~ 0.1570D+02  0.1412D+00
  7  0.1570D+02~ 0.1815D+02  0.1474D+00
  8  0.1815D+02~ 0.2060D+02  0.1538D+00
  9  0.2060D+02~ 0.2305D+02  0.1604D+00
 10  0.2305D+02~ 0.2550D+02  0.1671D+00
 11  0.2550D+02~ 0.2795D+02  0.1740D+00
 12  0.2795D+02~ 0.3040D+02  0.1812D+00
 13  0.3040D+02~ 0.3285D+02  0.1887D+00
 14  0.3285D+02~ 0.3530D+02  0.1966D+00
 15  0.3530D+02~ 0.3774D+02  0.2049D+00
 16  0.3774D+02~ 0.4019D+02  0.2136D+00
 17  0.4019D+02~ 0.4264D+02  0.2230D+00
 18  0.4264D+02~ 0.4509D+02  0.2328D+00
 19  0.4509D+02~ 0.4754D+02  0.2432D+00
 20  0.4754D+02~ 0.4999D+02  0.2540D+00
 21  0.4999D+02~ 0.5244D+02  0.2652D+00
 22  0.5244D+02~ 0.5489D+02  0.2768D+00
 23  0.5489D+02~ 0.5734D+02  0.2887D+00
 24  0.5734D+02~ 0.5979D+02  0.3007D+00
 25  0.5979D+02~ 0.6224D+02  0.3128D+00
 26  0.6224D+02~ 0.6469D+02  0.3248D+00
 27  0.6469D+02~ 0.6714D+02  0.3366D+00
 28  0.6714D+02~ 0.6959D+02  0.3482D+00
 29  0.6959D+02~ 0.7204D+02  0.3596D+00
 30  0.7204D+02~ 0.7449D+02  0.3707D+00
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 33  0.7939D+02~ 0.8184D+02  0.4020D+00
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 36  0.8674D+02~ 0.8919D+02  0.4299D+00
 37  0.8919D+02~ 0.9164D+02  0.4385D+00
 38  0.9164D+02~ 0.9409D+02  0.4467D+00
 39  0.9409D+02~ 0.9654D+02  0.4547D+00
 40  0.9654D+02~ 0.9899D+02  0.4624D+00
 41  0.9899D+02~ 0.1014D+03  0.4702D+00
 42  0.1014D+03~ 0.1039D+03  0.4780D+00
 43  0.1039D+03~ 0.1063D+03  0.4861D+00
 44  0.1063D+03~ 0.1088D+03  0.4946D+00
 45  0.1088D+03~ 0.1112D+03  0.5036D+00
 46  0.1112D+03~ 0.1137D+03  0.5129D+00
 47  0.1137D+03~ 0.1161D+03  0.5226D+00
 48  0.1161D+03~ 0.1186D+03  0.5325D+00
 49  0.1186D+03~ 0.1210D+03  0.5425D+00
 50  0.1210D+03~ 0.1235D+03  0.5526D+00
 51  0.1235D+03~ 0.1259D+03  0.5627D+00
 52  0.1259D+03~ 0.1284D+03  0.5728D+00
 53  0.1284D+03~ 0.1308D+03  0.5827D+00
 54  0.1308D+03~ 0.1333D+03  0.5924D+00
 55  0.1333D+03~ 0.1357D+03  0.6018D+00
 56  0.1357D+03~ 0.1382D+03  0.6107D+00
 57  0.1382D+03~ 0.1406D+03  0.6190D+00
 58  0.1406D+03~ 0.1431D+03  0.6268D+00
 59  0.1431D+03~ 0.1455D+03  0.6340D+00
 60  0.1455D+03~ 0.1480D+03  0.6405D+00
 61  0.1480D+03~ 0.1504D+03  0.6463D+00
 62  0.1504D+03~ 0.1529D+03  0.6515D+00
 63  0.1529D+03~ 0.1553D+03  0.6560D+00
 64  0.1553D+03~ 0.1578D+03  0.6600D+00
 65  0.1578D+03~ 0.1602D+03  0.6635D+00
 66  0.1602D+03~ 0.1627D+03  0.6665D+00
 67  0.1627D+03~ 0.1651D+03  0.6692D+00
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 77  0.1872D+03~ 0.1896D+03  0.6715D+00
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 79  0.1921D+03~ 0.1945D+03  0.6638D+00
 80  0.1945D+03~ 0.1970D+03  0.6590D+00
 81  0.1970D+03~ 0.1994D+03  0.6536D+00
 82  0.1994D+03~ 0.2019D+03  0.6480D+00
 83  0.2019D+03~ 0.2043D+03  0.6423D+00
 84  0.2043D+03~ 0.2068D+03  0.6367D+00
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 87  0.2117D+03~ 0.2141D+03  0.6232D+00
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 89  0.2166D+03~ 0.2190D+03  0.6196D+00
 90  0.2190D+03~ 0.2215D+03  0.6199D+00
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 97  0.2362D+03~ 0.2386D+03  0.6542D+00
 98  0.2386D+03~ 0.2411D+03  0.6617D+00
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100  0.2435D+03~ 0.2460D+03  0.6766D+00
101  0.2460D+03~ 0.2484D+03  0.6838D+00
102  0.2484D+03~ 0.2509D+03  0.6906D+00
103  0.2509D+03~ 0.2533D+03  0.6971D+00
104  0.2533D+03~ 0.2558D+03  0.7030D+00
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144  0.3513D+03~ 0.3538D+03  0.1031D+00
145  0.3538D+03~ 0.3562D+03  0.9069D-01
146  0.3562D+03~ 0.3587D+03  0.7962D-01
147  0.3587D+03~ 0.3611D+03  0.6979D-01
148  0.3611D+03~ 0.3636D+03  0.6109D-01
149  0.3636D+03~ 0.3660D+03  0.5341D-01
150  0.3660D+03~ 0.3684D+03  0.4664D-01
```

Fig. 11.e

The curve estimated from the Kanna River data is more complicated in structure. We shall examine the stability of the present procedure by applying it to this set of data. Figures 11.a–e are the final estimates where $c=10$, 20, 30, 50 and 150, respectively. These figures demonstrate the following facts: If the value of c is not less than 20, the procedure picks up the two peaks; If it is 30 and over, one more hump emerges; If $c\geqq 50$, the estimated probabilities are fairly stable regardless of the value of c. These observations show that the estimation of a response curve by the procedure is usually good enough for practical purposes if only the number of initial categories is sufficiently large, say, $c\geqq 50$.

Our procedure has a close relation to the linear logistic models (Cox [3], for example). It is easily shown as follows:

Assume that p_j is linearly parametrized by

$$p_j=\log\frac{\exp(\theta_0+\theta_1 j)}{1+\exp(\theta_0+\theta_1 j)} \qquad (j=1, 2, \cdots, c).$$

Let $\tilde{\theta}_0$ and $\tilde{\theta}_1$ be maximum likelihood estimates of parameters θ_0 and θ_1, respectively, of this model. Then it can be seen that our Bayesian estimate $\hat{q}_j$ converges to $\tilde{\theta}_0+\tilde{\theta}_1 j$ for each j as the hyper-parameter v^2 tends to zero. Such is the case for the result given as Figure 5.b. The extension of this relation to Kth order logistic models defined by

$$(25) \qquad p_j=\log\frac{\exp\left(\sum_{k=0}^{K}\theta_k j^k\right)}{1+\exp\left(\sum_{k=0}^{K}\theta_k j^k\right)} \qquad (K\geqq 0)$$

is straightforward. When the second order difference of $\{q_j\}$ within the definition of the prior density (14) is replaced by the $(K+1)$th order difference, our estimate $\hat{q}_j$ converges to

$$\tilde{q}_j\equiv\sum_{k=0}^{K}\tilde{\theta}_k j^k .$$

for each j as v^2 tends to zero. Of course, $\{\tilde{\theta}_k\}$ are maximum likelihood estimates of the parameters of the model (25).

Concluding remarks

There are two types of random variables, categorical and continuous, and then there are four types of relations between two sets of variables as follows:

Response variables	Explanatory variables	
	Continuous	Categorical
Continuous	Case 1	Case 2
Categorical	Case 3	Case 4

Regression analysis is the most familiar statistical technique for 'Case 1'; analysis of variance is for 'Case 2'; discriminant analysis is for 'Case 3'; and contingency table analysis is for 'Case 4'. Sakamoto and Akaike [13] and Sakamoto [14] proposed a practically useful procedure for 'Case 3' and 'Case 4', especially for the purpose of variable selection. The procedure in the present paper refines the previous procedure but covers only a part of 'Case 3' and 'Case 4', that is, a single explanatory variable case. The observations in the preceding sections, however, clearly show that the present procedure as it is has wide range of applications. If we further refine the procedure so that it takes care of the multinomial-multivariate cases it will find a wider range of applications. This will be the subject for further study. Also the present procedure can be extended to the estimation of the probability density function [5].

Appendix. Derivatives of $\log L(q)$

Let $f(q)$ be defined by

$$f(q)=\log L(q)=\sum\,[n(1,\,j)q_j-n(j)\log\{1+\exp(q_j)\}]\,,$$

then the first derivative is given by

$$\frac{\partial f}{\partial q_k}=n(1,\,k)-n(k)\frac{\exp(q_k)}{1+\exp(q_k)}=n(1,\,k)-n(k)p_k\,.$$

Since $\partial p_k/\partial q_k$ is calculated to be

$$\frac{\partial p_k}{\partial q_k}=\frac{\partial}{\partial q_k}\left(\frac{\exp(q_k)}{1+\exp(q_k)}\right)=\frac{\exp(q_k)}{1+\exp(q_k)}-\left(\frac{\exp(q_k)}{1+\exp(q_k)}\right)^2=p_k(1-p_k)\,,$$

it follows

$$\frac{\partial^2 f}{\partial q_k^2}=-n(k)p_k(1-p_k)\,.$$

It can easily be seen that higher order derivatives also are given as polinomials of p_k's and bounded for the entire space of the parameter vector $\boldsymbol{q}$.

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REFERENCES

[1] Akaike, H. (1980). Likelihood and Bayes procedure, *Bayesian Statistics* (eds. J. M. Bernardo, M. H. De Groot, D. U. Lindley and A. F. M. Smith), University Press, Valencia, Spain.

[2] Akaike, H. and Ishiguro, M. (1980). Trend estimation with missing observations, *Ann. Inst. Statist. Math.*, **32**, B, 481-488.

[3] Cox, D. R. (1970). *Analysis of Binary Data*, Methuen & Co. Ltd., London.

[4] Ishiguro, M. and Akaike, H. (1981). A Bayesian approach to the trading-day adjustment of monthly data, *Time Series Analysis* (eds. O. D. Anderson and M. R. Perryman), North-Holland, Amsterdam.

[5] Ishiguro, M. and Sakamoto, Y. (1982). A Bayesian approach to probability density estimation, *Research Memorandum* No. 239, The Institute of Statistical Mathematics, Tokyo.

[6] Kashiwagi, N. (1982). A Bayesian estimation procedure for fertilities in field experiments, *Research Memorandum* No. 220, The Institute of Statistical Mathematics, Tokyo.

[7] Lewis, P. A. W. (1964). A branching Poisson process model for the analysis of computer failure patterns, *J. R. Statist. Soc.*, B, **26**, 398-456.

[8] Mays, C. W. and Lloyed, R. D. (1971). Bone sarcoma risk from ^{90}Sr, Biomedical Implications of Radiostrontium Exposure (eds. M. Goldman and L. K. Bustad), *AEC Sympo.*, **25**, 354.

[9] Mays, C. W. and Lloyed, R. D. (1972). Bone sarcoma incidence vs. alpha particle dose, *Radiobiology of Plutonium* (eds. B. J. Stover and W. S. S. Jee), J. W. Press, Salt Lake City.

[10] Nakamura, T. (1982). A Bayesian cohort model for standard cohort table analysis, *Proc. Inst. Statist. Math.*, **29**, 77-79 (in Japanese).

[11] Noda, K. and Murakami, M. (1979). Estimation of two-phase segmented lines on the basis of binary data, *J. Japan Statist. Soc.*, **12**, 63-75.

[12] Ozaki, T. (1981). Marked point processes and non-linear systems modeling of daily rainfall and riverflow time series in stochastic hydrology, *Technical Report* No. 148, Department of Mathematics, University of Manchester, Institute of Science and Technology, Manchester.

[13] Sakamoto, Y. and Akaike, H. (1978). Analysis of cross-classified data by AIC, *Ann. Inst. Statist. Math.*, **30**, B, 185-197.

[14] Sakamoto, Y. (1983). Efficient use of Akaike's Information Criterion for model selection in high dimensional contingency table analysis, *Metron* (to appear).