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A METHOD ASSOCIATED WITH CHARACTERIZATIONS OF THE EXPONENTIAL DISTRIBUTION

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Summary

The aim of this paper is to give a method for obtaining a characterization of the exponential distribution, which is based on a study of convolution-type equations and on a subsequent selection of positive solutions of these equations. The method is illustrated by two characterizations of exponential distribution through the integrated lack of memory property and relevation type equation. It is proved that each of the equations

$$\int_0^\infty \frac{\bar{F}(x+z)}{\bar{F}(x)} f(x)dx=\bar{F}(z)\ , \qquad z>0$$

and

$$\int_0^z \bar{F}(z-x)f(x)dx=\int_0^z \frac{\bar{F}(z)}{\bar{F}(x)} f(x)dx\ , \qquad z>0\ ,$$

where $\bar{F}(x)=1-F(x)$, $F(x)$ is a distribution function and $f(x)$ is a probability density function, has solutions only of the form

$$\bar{F}(x)=\exp(-\lambda x)\ , \qquad x>0\ (\lambda>0)\ .$$

1. Introduction

Let X be a positive random variable with a nondegenerate distribution function (d.f.) $F(x)$, which has the sense of a d.f. of failures of a technical device. Thus

$$\mathrm{P}\{X<x\}=F(x)$$

is the probability of failure of this device until the moment of time x. Put

$$\bar{F}(x)=1-F(x)\ .$$

The value

$$\mathrm{P}\{X \geqq x+z \mid X \geqq x\} = \frac{\bar{F}(x+z)}{\bar{F}(x)} \qquad (z \geqq 0)$$

is the probability of failure of the device during the time from the moment $x+z$ up to infinity provided that there were no failures until the moment of time x. The well-known property of lack of memory means that

$$\mathrm{P}\{X \geqq x+z \mid X \geqq x\} = \mathrm{P}\{X \geqq z\}$$

or, equivalently,

$$\frac{\bar{F}(x+z)}{\bar{F}(x)} = \bar{F}(z) .$$

If we assume that x is the value of a random variable with the density function $f(x)$ ($f(x)=0$ for $x<0$), then the integrated lack of memory property has the form

$$(1) \qquad \int_0^\infty \frac{\bar{F}(x+z)}{\bar{F}(x)} f(x)dx = \bar{F}(z) , \qquad z \geqq 0 .$$

Equation (1) was studied in the papers by Ahsanullah [1], [2] under a rather restrictive assumption that $F(x)$ possesses a monotone hazard rate. Under this assumption it was proved that if $F(x)$ satisfies equation (1), then it is the d.f. of the exponential law. Grosswald and Kotz [3] studied equation (1) under somewhat different regularity assumptions. However the assumptions of [3] are, nevertheless, rather restrictive and, besides, there is considered only the case when $f(x)$ is the density function of $F(x)$.

Here we give a method of proof of characterization theorems which allows to study equation (1) without the assumption that $f(x)$ is a density function at all. In particular, we may make no assumption on the positiveness of the function $f(x)$. Besides, our regularity assumptions are different from those of [3].

In the papers by Grosswald, Kotz and Johnson [4] and Grosswald and Kotz [3] a related relevation type equation

$$\int_0^z \bar{F}(z-x)f(x)dx = \int_0^z \frac{\bar{F}(z)}{\bar{F}(x)} f(x)dx$$

was considered (for the interpretation of this equation in terms of reliability theory see [4]). They show that the only solutions of this equation (under a priori assumption of analyticity) are d.f.'s of the exponential law. In the present paper we get rid of a priori analyticity assumption, replacing it by weaker regularity assumptions.

2. Results

THEOREM 1. *Let a d.f.* $F(x)$ $(F(x)=0$ *for* $x\leqq 0)$ *satisfy equation* (1). *Assume that for some* $\rho\geqq 0$

$$\bar{F}(x)=e^{-\rho x}\varphi(x)\ ,$$

where the function $\varphi(x)$ *satisfies the following conditions:*

(i) $f(x)/\varphi(x)\leqq Me^{-\alpha x}$ *for some positive constants* M *and* α;

(ii) *the relation* $f(x)/\varphi(x)$ *is differentiable and*

$$\int_0^\infty \left|\frac{d}{dx}(f(x)/\varphi(x))\right|dx<\infty$$

(the positiveness of the function $f(x)$ *is not assumed). Then* $\varphi(x)$ *for* $x>0$ *is a solution of some ordinary linear differential equation with constant coefficients.*

THEOREM 2. *If, in addition to the assumptions of Theorem* 1

$$f(x)\geqq 0 \qquad \text{and} \qquad \int_0^\infty f(x)dx=1\ ,$$

then $F(x)$ *is a d.f. of the exponential law, i.e.* $F(x)=1-e^{-\lambda x}$ *for* $x>0$ $(\lambda>0)$ *and* $F(x)=0$ *for* $x\leqq 0$.

COROLLARY. *Let a differentiable d.f.* $F(x)$ *satisfy equation* (1) *in which* $f(x)=F'(x)$. *Assume that*

$$\bar{F}(x)=e^{-\rho x}\varphi(x) \qquad (\rho\geqq 0)\ ,$$

and $\varphi(x)$ *is such that*

(i) *the second derivative* $\varphi''(x)$ *exists and also*

$$|\varphi'(x)/\varphi(x)|\leqq Me^{-\alpha x}$$

for some positive constants M *and* α,

(ii) $$\int_0^\infty \left|\frac{d}{dx}(\varphi'(x)/\varphi(x))\right|e^{-\rho x}dx<\infty\ .$$

Then $F(x)$ *is a function of the exponential distribution.*

Consider now the relevation type equation.

THEOREM 3. *Let a d.f.* $F(x)$ *be differentiable and such that* $\bar{F}(+0)=1$, $\bar{F}(z)>0$, $\bar{F}'(z)<0$ *for all* $z>0$, $\bar{F}'(+0)<0$ *and suppose that the function* $f(x)$ *is continuous, strictly positive and* $\int_0^\infty f(x)dx=1$. *Then* *the only solutions of the equation*

$$\int_0^z \bar{F}(z-x)f(x)dx=\int_0^z (\bar{F}(z)/\bar{F}(x))f(x)dx\ , \qquad z\geqq 0$$

are of the form

$$F(z)=1-e^{-\lambda z}\ , \qquad z\geqq 0\ (\lambda>0)\ .$$

3. Proofs

The method of proofs of Theorems 1 and 2 is based on the estimation of the number of linear independent solutions of some convolution-type equation on a semi-axis together with subsequent elimination of those solutions which do not have the probability meaning.

Let us make in equation (1) the change of the variable $\bar{F}(x)=e^{-\rho x}\varphi(x)$. We get

$$(2) \qquad \int_0^\infty f(x)\frac{\varphi(x+z)}{\varphi(x)}dx=\varphi(z)\ .$$

Suppose that $\varphi(z)$ is some solution of equation (2). We fix it and denote

$$R(x)=f(x)/\varphi(x)\ .$$

Consider the equation

$$(3) \qquad g(z)=\int_0^\infty R(x)g(x+z)dx\ , \qquad z>0$$

with respect to an unknown function g. It is clear that $g(z)=\varphi(z)$ satisfies this equation. Besides, by virtue of the conditions of Theorem 1 the function $R(x)$ is differentiable and $|R'(x)|$ is integrable. Therefore if we write equation (3) in an equivalent form

$$(4) \qquad g(z)=\int_0^\infty R(y-z)g(y)dy\ , \qquad z>0\ ,$$

differentiate both sides of (4) with respect to z and use integration by parts, we see that $g(z)$ is differentiable and

$$g'(z)=\int_0^\infty R(y-z)g'(y)dy\ , \qquad z>0\ .$$

Hence, if the function $g(z)$ satisfies equation (3), then $g(z)$ is differentiable and its derivative $g'(z)$ also satisfies equation (3). Thus, $g(z)$ is an infinitely differentiable function (for $z>0$) and also all of its derivatives satisfy (3).

Put

$$K(t)=\int_0^\infty \frac{f(z)}{\varphi(z)}e^{-itz}dz\ .$$

By virtue of the conditions of Theorem 1 $K(t)$ is analytical in a band which contains the real axis. Therefore a representation

$$(5)\qquad \frac{1}{1-K(t)}=\frac{\prod_{l=1}^{r}(t-\xi_l)^{\delta_l}}{\prod_{p=1}^{s}(t-\eta_p)^{\sigma_p}}K_1(t)\ ,$$

where ξ_l, η_p are real numbers, δ_l and σ_p are positive integers, $K_1(t)\neq 0$ for $t\in(-\infty,\infty)$ and

$$(6)\qquad \mathfrak{X}=\frac{1}{2\pi}[\arg K_1(t)]_{-\infty}^{\infty}<\infty$$

is valid. Then (see Gahov and Chersky [5], Ch. III, § 9, 10.3) equation (3) has no more than $m=\mathfrak{X}-n$ (if $\mathfrak{X}>n$) linear independent solutions (if $\mathfrak{X}\leqq n$, equation (3) does not possess non-trivial solutions, but in our case it is impossible since (1) and, hence, (3) have the exponential solution). Since $g(z), g'(z),\cdots, g^{(l)}(z),\cdots,$ are solutions of equation (3), the functions $g(z), g'(z),\cdots, g^{(m)}(z)$ must be linearly dependent, i.e. there are real constants $a_0, a_1,\cdots, a_m$, $\sum_{l=0}^{m} a_l^2>0$, such that

$$(7)\qquad a_0g^{(m)}(z)+a_1g^{(m-1)}(z)+\cdots+a_mg(z)=0$$

for all $z>0$. Thus, $g(z)$ is a solution of a linear differential equation of the order $\leqq m$ with constant coefficients. Since $g(z)=\varphi(z)$ is also a solution of equation (3), $\varphi(z)$ satisfies equation (7). Theorem 1 is proved.

PROOF OF THEOREM 2. We have from Theorem 1:

$$(8)\qquad \bar{F}(z)=\sum_{j=1}^{m} c_j p_j(z)e^{-\lambda_j z}\ ,$$

where $\lambda_j=\rho-\rho_j$ and ρ_j's are the roots (in general, complex ones) of a characteristic equation

$$a_0\rho^m+a_1\rho^{m-1}+\cdots+a_m=0\ ,$$

c_j's are constants and $p_j(z)$ are polynomials of z. In addition we can assume that in (8) all $c_j\neq 0$, the numbers λ_j are different and are ordered with respect to the increase of their real parts, i.e. $\mathrm{Re}\,\lambda_1\leqq\mathrm{Re}\,\lambda_2\leqq\cdots\leqq\mathrm{Re}\,\lambda_m$, and also $\mathrm{Re}\,\lambda_1>0$ (otherwise, the real function $\bar{F}(z)$ cannot be non-increasing from 1 to 0).

Let

$$R_1(x)=f(x)/\bar{F}(x)\ .$$

Then (1) implies

$$\int_0^\infty R_1(x)\bar{F}(x+z)dx=\bar{F}(z)\ . \tag{9}$$

Substituting (8) in equation (9) we get

$$\sum_{j=1}^m c_j p_j(z)e^{-\lambda_j z}=\sum_{j=1}^m c_j e^{-\lambda_j z}\int_0^\infty R_1(z)p_j(x+z)e^{-\lambda_j x}dx\ . \tag{10}$$

Since $\int_0^\infty R_1(x)p_j(x+z)e^{-\lambda_j x}dx$ is a polynomial of z, whose degree is not higher than the degree of $p_j(z)$ and the functions $z^l \exp(-\lambda_j z)$ are linearly independent for different l and j it follows from (10) that for any $j=1,\cdots,m$,

$$p_j(z)=\int_0^\infty R_1(x)p_j(x+z)e^{-\lambda_j x}dx \tag{11}$$

(we take advantage of that $c_j\neq 0$, $j=1,\cdots,m$). Let

$$p_j(z)=\sum_{s=0}^{l_j} a_{js}z^s \qquad (a_{jl_j}\neq 0)\ .$$

If we substitute this expression in (11) we get

$$\sum_{s=0}^{l_j} a_{js}z^s=\sum_{s=0}^{l_j}\int_0^\infty R_1(x)a_{js}(x+z)^s e^{-\lambda_j x}dx\ , \qquad j=1,\cdots,m\ . \tag{12}$$

Setting equal the coefficients at z^{l_j} in the left-hand side and the right-hand side of (12) we obtain

$$\int_0^\infty R_1(x)e^{-\lambda_j x}dx=1\ , \qquad j=1,\cdots,m\ . \tag{13}$$

If at least one of the numbers $l_j>0$, then equating in (12) the coefficients at z^{l_j-1}, we find that

$$\int_0^\infty R_1(x)xe^{-\lambda_j x}dx=0\ . \tag{14}$$

Return now to representation (8). Since $\bar{F}(z)$ is a real function, each complex λ_j has a complex-conjugate $\lambda_{j'}=\bar{\lambda}_j$. We can assume that the indexing is such that j' differs from j by one. Show that there is at least one real number among the λ_j's with a minimal real part $\operatorname{Re}\lambda_j=\operatorname{Re}\lambda_1$. Assuming the contrary, we have for some $k\geqq 1$ that $\lambda_1=\bar{\lambda}_2$, $\lambda_3=\bar{\lambda}_4,\cdots,\lambda_{2k-1}=\bar{\lambda}_{2k}$, $\operatorname{Re}\lambda_j=\operatorname{Re}\lambda_1$, $j=1,\cdots,2k$, and $\operatorname{Re}\lambda_j>\operatorname{Re}\lambda_1$ for

$j>2k$. Put $\lambda_j=\xi_1+i\eta_j$, $j=1,\cdots,2k$ $(\eta_{2j-1}=-\eta_{2j})$, $\eta_j\neq 0$, $j=1,\cdots,2k$. Then relation (8) implies

$$\bar{F}(z)=\left\{\sum_{j=1}^{k}[\tilde{p}_j(z)\cos\eta_{2j-1}z+\hat{p}_j(z)\sin\eta_{2j-1}z]\right.\\ \left.+\sum_{j=2k+1}^{m}c_jp_j(z)e^{-(\lambda_j-\xi_1)z}\right\}e^{-\xi_1 z} \tag{15}$$

(here $\tilde{p}_j$ and $\hat{p}_j$ are some polynomials). Since $e^{-(\lambda_j-\xi_1)z}\to 0$ as $z\to\infty$ ($j>2k$), the sum $\sum_{j=2k+1}^{m}c_jp_j(z)e^{-(\lambda_j-\xi_1)z}$ becomes arbitrarily small for sufficiently large z. On the other hand, it is easy to see that the sum of values in square brackets is with alternating signs. Hence, the whole expression in the right-hand side of (15) is with alternating signs, which contradicts the positiveness of $\bar{F}(z)$. So we can assume that λ_1 is a real number and $\lambda_1>0$.

Now we see from relation (13) that

$$\int_0^\infty R_1(x)e^{-\lambda_1 x}dx=1\ . \tag{16}$$

If $\operatorname{Re}\lambda_j>\lambda_1$ for some j, then

$$\left|\int_0^\infty R_1(x)e^{-\lambda_j x}dx\right|\leq\int_0^\infty R_1(x)e^{-(\operatorname{Re}\lambda_j)x}dx<\int_0^\infty R_1(x)e^{-\lambda_1 x}dx=1\ .$$

However, the last relation is in contradiction with (13). Thus,

$$\operatorname{Re}\lambda_j=\lambda_1\ ,\qquad j=1,\cdots,m\ .$$

If there are complex numbers among the λ_j's, i.e. for at least one j, $\lambda_j=\lambda_1+i\eta_j$, $\eta_j\neq 0$, then by virtue of (13)

$$1=\int_0^\infty R_1(x)e^{-\lambda_j x}dx=\int_0^\infty R_1(x)e^{-\lambda_1 x}\cos(\eta_j x)dx+i\int_0^\infty R_1(x)e^{-\lambda_1 x}\sin(\eta_j x)dx\ .$$

The latter is possible only if

$$\int_0^\infty R_1(x)e^{-\lambda_1 x}\cos(\eta_j x)dx=1\ ,\qquad \int_0^\infty R_1(x)e^{-\lambda_1 x}\sin(\eta_j x)dx=0\ .$$

Since $R_1(x)>0$, the first of those equalities under $\eta_j\neq 0$ is in contradiction with the relation

$$\int_0^\infty R_1(x)e^{-\lambda_1 x}dx=1\ .$$

Thus, $\lambda_j=\lambda_1$ for $j=1,\cdots,m$.

If among the polynomials $p_j(z)$ in (8) there is at least one polynomial of a positive degree, then (14) implies

$$\int_0^\infty R_1(x)xe^{-\lambda_1 x}dx=0 .$$

However, the latter is impossible by virtue of positiveness of the integrand function.

Hence, according to what has been said, relation (8) implies

$$\bar{F}(x)=Ce^{-\lambda_1 x} .$$

By virtue of normalization $\bar{F}(0)=1$, we get

$$\bar{F}(x)=e^{-\lambda_1 x} ,$$

i.e. $F(x)$ is a function of the exponential distribution.

Pass now to the proof of Theorem 3. The method of its proof is based on the ideas of the theory of positive operators, i.e. the operators which leave invariant some cone in a Banach space. Namely, the following lemma will be necessary.

LEMMA. *Let $\mathcal{K}$ be a linear operator which operates in the space $C[a, b]$ of continuous functions given on a finite interval $[a, b]$. Assume that*

$$(\mathcal{K}\xi)(x)\geqq 0 , \qquad x \in [a, b]$$

for any nonnegative function $\xi \in C[a, b]$ and, in addition, the condition

$$(\mathcal{K}\xi)(x_0)=0$$

implies $\xi(x)=0$ for $x \in [a, x_0]$. Then to each eigen-value of $\mathcal{K}$ operator corresponds no more than one (to within a constant factor) strictly positive on $[a, b]$ eigen-function.

PROOF. Assume the contrary and let ξ_0 and ξ_1 be two strictly positive eigen-functions of $\mathcal{K}$ operator, corresponding to the eigen-value λ, i.e.

$$\mathcal{K}\xi_0=\lambda\xi_0 \qquad \text{and} \qquad \mathcal{K}\xi_1=\lambda\xi_1 .$$

Choose the number α so that the function $\xi_2=\xi_0-\alpha\xi_1$ is non-negative and vanishes at least in one point t_0 of the interval $[a, b]$ (obviously, it can be done). Then

$$\mathcal{K}\xi_2=\mathcal{K}(\xi_0-\alpha\xi_1)=\lambda(\xi_0-\alpha\xi_1)=\lambda\xi_2 ,$$

i.e. ξ_2 is a non-negative eigen-function of $\mathcal{K}$ operator. Since $\xi_2(t_0)=0$, then $(\mathcal{K}\xi_2)(t_0)=0$ and, by virtue of the property of $\mathcal{K}$ operator, $\xi_2(x)=0$ for $x \in [a, t_0]$. Clearly, there is a number β such that the function $\xi_3=\xi_0-\beta\xi_2$ is non-negative and vanishes at some point $t_1>t_0$ from the interval $[a, b]$. Then

$$\mathcal{K}\xi_3 = \lambda\xi_3$$

and according to the property of $\mathcal{K}$ operator we have $\xi_3(x)=0$ for $x \in [a, t_1]$. It means that $\xi_0(x)=\beta\xi_2(x)$ for $x \in [a, t_1]$. However, $\xi_2(x)=0$ for $x \in [a, t_0] \subset [a, t_1]$, i.e. $\xi_0(x)=0$ for $x \in [a, t_0]$. The latter contradicts the strict positiveness of the function ξ_0. The Lemma is proved.

Begin now the proof of Theorem 3. Fix arbitrarily a number $T > 0$ and consider the equation

$$\int_0^z \bar{F}(z-x) f(x) dx = \int_0^z \frac{f(x)}{\bar{F}(x)} dx \cdot \bar{F}(z) \tag{17}$$

for $z \in [0, T]$. Let $F(x)$ be some solution of equation (17) which satisfies the conditions of Theorem 3. Fix it and consider the equation

$$\int_0^z S(z-x) f(x) dx = \phi(z) S(z) , \tag{18}$$

$z \in [0, T]$, where

$$\phi(z) = \int_0^z \frac{f(x)}{\bar{F}(x)} dx .$$

Consider a linear operator $\mathcal{K}$ given in the space $C[a, b]$ by the relations

$$(\mathcal{K}\xi)(z) = \frac{1}{\phi(z)} \int_0^z \xi(z-x) f(x) dx \qquad \text{for } 0 < z \leqq T ,$$

$$(\mathcal{K}\xi)(0) = \xi(0)\bar{F}(+0) .$$

Clearly, the operator $\mathcal{K}$, introduced like this, satisfies the conditions of Lemma 1. Therefore $\mathcal{K}$ has no more than one (to within a constant factor) strictly positive eigen-function which satisfies the relation

$$\mathcal{K}s = s ,$$

i.e. for a fixed function $F(x)$ equation (18) has no more than one (to within a constant factor) strictly positive and continuous on $[0, T]$ solution. Since $S(z)=\bar{F}(z)$ is the solution of equation (18), then all of its solutions with alternating signs are of the form

$$S(z) = \lambda\bar{F}(z) , \qquad z \in [0, T] ,$$

where λ is a constant.

If we differentiate both sides of (17) with respect to z, it is easy to see that the function $\bar{F}'(z)$ also satisfies equation (18). Since $\bar{F}'(z) < 0$ (by virtue of the assumption of Theorem 2) we have, according to what has been said,

$$\bar{F}'(z) = -\lambda \bar{F}(z) , \qquad z \in [0, T]$$

for some $\lambda > 0$. Integrating this equation and making use of arbitrariness of the number $T > 0$ we obtain

$$\bar{F}(z) = e^{-\lambda z} , \qquad z \geqq 0 .$$

Theorem 3 is proved.

Note that the assertions similar to Theorem 2 can be obtained by means of methods of papers [6], [7].

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