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THE $GI/G/1$ QUEUE WITH LAST-COME-FIRST-SERVED

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Introduction

This paper is concerned with the following single server queueing system. Customers arrive at the system at epochs $t_0, t_1, t_2, \cdots$ ($t_0=0$; $t_0<t_1<t_2<\cdots$) where the interarrival times $T_n=t_n-t_{n-1}$ are independent identically distributed random variables (i.i.d. r.v.'s) having distribution function (d.f.) $A(x)$ with a finite mean α. The service requirements $S_0, S_1, S_2, \cdots$ of the successive arriving customers are i.i.d. r.v.'s having d.f. $B(x)$ with a finite mean β. The r.v.'s $T_1, T_2, \cdots$ and $S_0, S_1, \cdots$ are stochastically independent. We assume $\beta/\alpha<1$. In this paper it is shown that when the queue discipline is last-come-first-served with preemption the stationary distribution of the number of customers in the system immediately before (after) an arrival (departure) epoch is geometric. Also it is shown that the remaining service requirements of customers immediately before (after) an arrival (departure) epoch are i.i.d. r.v.'s, and that the distribution of the time between a departure and the next arrival epochs is independent of the state of the system. Our approach method is based on Kelly [2].

Preliminaries

We cite below some preliminary results which are necessary for the study of the queueing system described in preceding section.

Consider a random walk defined by $U_{n+1}=U_n+Y_n$ ($n=0, 1, 2, \cdots$), where $Y_n=S_n-T_{n+1}$ ($n=0, 1, 2, \cdots$). The points at which the random walk first jumps above its latest maximum value will be called 'ascending ladder indices'. Since $\beta/\alpha<1$, it follows that $U_n\to-\infty$ as $n\to\infty$ and consequently the total number K of ascending ladder indices is a finite r.v. with probability one. It follows (see Kleinrock [3]) that

$$\Pr(K=n)=(1-\sigma)\sigma^n \qquad (n=0, 1, 2, \cdots)\,, \tag{1}$$

where $1-\sigma\equiv\Pr(U_n\leqq U_0;\ n=1, 2, \cdots)$. Moreover given that $K=k$ and that the successive ladder indices are the points $n_1, n_2, \cdots, n_k$, we put

$$I_i = U_{n_i} - U_{n_{i-1}} \qquad (i=1, 2, \cdots, k;\ n_0=0)\,.$$

It is well-known that $I_1, I_2, \cdots, I_k$ are i.i.d. r.v.'s having d.f.

$$(2) \qquad F(x) \equiv \Pr(I_i \leqq x) \qquad (x \geqq 0)\,.$$

Define

$$(3) \qquad F_n(x) = \int_0^x F_{n-1}(x-u)\cdot dF(u) \qquad (x \geqq 0;\ n=1, 2, \cdots)\,,$$

$$(4) \qquad F_0(x) = 1 \qquad (x \geqq 0)\,,$$

and

$$(5) \qquad C(x) = \Pr(Y_n \leqq x) = \int_0^\infty B(x+y)\cdot dA(y) \qquad (-\infty < x < \infty)\,.$$

The following equations, then, were derived by Fakinos [1]:

$$(6) \quad \Pr(U_n \leqq x;\ n=1, 2, \cdots \mid U_0=0) = \sum_{n=0}^\infty (1-\sigma)\sigma^n F_n(x) \qquad \text{for any } x \geqq 0\,,$$

$$(7) \quad \sigma\bar{F}(x) = \bar{C}(x) + \sum_{n=0}^\infty \sigma^{n+1} \int_{-\infty}^0 \int_0^{-u} \bar{F}(-u+x-z)\cdot dF_n(z)\cdot dC(u)^{*)}\,,$$

where $\bar{F}(x) = 1-F(x)$ and $\bar{C}(x) = 1-C(x)$. From the definition of $F_0(x)$ it follows that

$$(8) \qquad \int_{-\infty}^0 F_0(-v-y)\cdot dC(v) = \int_{-\infty}^{-y} dC(v) = \Pr(Y_n \leqq -y)$$
$$= \int_0^\infty \bar{A}(y+v)\cdot dB(v) \qquad (y \geqq 0)\,,$$

where $\bar{A}(x) = 1-A(x)$. On the basis of the random walk, we define $\bar{D}(x)$ for $x \geqq 0$ as follows:

$$(9) \quad \bar{D}(x) = \Pr(U_n \leqq -x;\ n=1, 2, \cdots \mid U_0=0,\ K=0)$$
$$= \frac{\Pr(U_n \leqq -x;\ n=1, 2, \cdots \mid U_0=0)}{\Pr(K=0)}$$
$$= \frac{\int_{-\infty}^{-x} \Pr(U_n \leqq -x;\ n=2, 3, \cdots \mid Y_0=u,\ U_0=0)\cdot dC(u)}{1-\sigma}$$

(by (1))

*) A possible misunderstanding was involved in the derivation of this result of Fakinos [1], i.e. Equation (11) of his paper, although the result is true. The corrected result of Equation (11) of Fakinos's paper becomes as follows:

$$\Pr(K=1;\ I_1 > x \mid Y_0 = u\ U_0=0) = \sum_{n=1}^\infty (1-\sigma)\sigma^n \int_0^{-u} F(-u+x-y)\cdot dF_{n-1}(y) \qquad (u \leqq 0)\,.$$

$$=\frac{\int_{-\infty}^{-x} \Pr(U_n \leqq -x-u;\ n=1,2,\cdots | U_0=0)\cdot dC(u)}{1-\sigma}$$

$$=\sum_{n=0}^{\infty} \sigma^n \int_{-\infty}^{0} F_n(-x-u)\cdot dC(u) \qquad \text{(by (6))},$$

where $F_n(y)=0$ $(n=0, 1, 2, \cdots)$ for $y<0$.

Results

Let $\mathcal{A}$ $(\mathcal{D})$ be the set of arrival (departure) epochs after time t_0 $(=0)$. In this paper the system will be observed exclusively at epochs $e \in \mathcal{E} = \mathcal{A} \cup \mathcal{D}$. At epoch $e \in \mathcal{E}$ let $\rho_e = +1$ or -1 according as $e \in \mathcal{A}$ or $e \in \mathcal{D}$. If $e \in \mathcal{A}$ $(\mathcal{D})$ let ν^e be the number of customers in the queue immediately before (after) epoch e. If a customer arrives at epoch $a \in \mathcal{A}$, he takes up position ν_a+1 in the queue. The server devotes his attention to the last arrival and so if a customer leaves the queue at epoch $d \in \mathcal{D}$ he leaves from position ν_d+1. Let χ_{je} be the remaining service requirement at epoch e of the customer occupying position j in the queue, for $j=1, 2, \cdots, \nu_e$. If a customer service is interrupted because of the arrival of another customer, his remaining service requirement remains constant until the server can attend to him again. Let η_e be the time between an epoch e and the next arrival epoch. Where its ommision can cause no confusion the subscript e will be dropped. It is clear that the process $(\rho, \nu, \chi_1, \chi_2, \cdots, \chi_\nu, \eta)$ observed at successive epochs $e \in \mathcal{E}$ is a Markov process.

THEOREM. *Stationary distributions for* $(\rho=+1, \nu, \chi_1, \chi_2, \cdots, \chi_\nu, \eta)$ *and* $(\rho=-1, \nu, \chi_1, \chi_2, \cdots, \chi_\nu, \eta)$ *observed at successive epochs* $e \in \mathcal{E}$ *are*

(10)
$$\Pr(\rho=+1,\ \nu=n,\ \chi_1>x_1, \chi_2>x_2, \cdots, \chi_n>x_n,\ \eta>y) = \frac{1}{2} P_n \bar{A}(y) \prod_{j=1}^{n} \bar{F}(x_j),$$

and

(11)
$$\Pr(\rho=-1,\ \nu=n,\ \chi_1>x_1, \chi_2>x_2, \cdots, \chi_n>x_n,\ \eta>y) = \frac{1}{2} P_n \bar{D}(y) \prod_{j=1}^{n} \bar{F}(x_j),$$

where $P_n=(1-\sigma)\sigma^n$.

PROOF. We show that if at the first epoch in $\mathcal{E}$ $(\rho, \nu, \chi_1, \chi_2, \cdots, \chi_\nu, \eta)$ has distribution (10) or (11), then at every subsequent epoch in $\mathcal{E}$ it will have the same distributions, i.e. the following equations hold:

$$(12)\qquad \frac{1}{2}P_0\bar{A}(y)=\frac{1}{2}P_0\Pr(\tilde{w}_1>y)\,,$$

$$(13)\qquad \frac{1}{2}P_n\bar{A}(y)\prod_{j=1}^{n}\bar{F}(x_j)=\frac{1}{2}P_{n-1}\prod_{j=1}^{n-1}\bar{F}(x_j)\Pr(\tilde{z}>\tilde{w}_1+x_n,\ \tilde{w}_2>y)$$
$$+\frac{1}{2}P_n\prod_{j=1}^{n-1}\bar{F}(x_j)\Pr(\tilde{x}>\tilde{y}+x_n,\ \tilde{w}_1>y)$$

for $n=1,2,\cdots$, and

$$(14)\qquad \frac{1}{2}P_n\bar{D}(y)\prod_{j=1}^{n}\bar{F}(x_j)=\frac{1}{2}P_n\prod_{j=1}^{n}\bar{F}(x_j)\Pr(\tilde{w}_1>\tilde{z}+y)$$
$$+\frac{1}{2}P_{n+1}\prod_{j=1}^{n}\bar{F}(x_j)\Pr(\tilde{y}>\tilde{x}+y)$$

for $n=0,1,2,\cdots$, where $\tilde{w}_i$ $(i=1,2)$, $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ are independent r.v.'s with d.f.'s $A(w)$, $F(x)$, $D(y)$, $B(z)$ respectively. The validity of Equation (12) is easily checked noting that the event $\{\rho=+1,\ \nu=0,\ \eta>y\}$ will occur at an epoch e iff at the preceding epoch the event $\{\rho=-1,\ \nu=0\}$ occured and the time between the next arrival and e is $>y$. Equation (13) is obtained from the consideration that the event $\{\rho=+1,\ \nu=n\}$ can occur at an epoch only if at the preceding epoch either of the events $\{\rho=+1,\ \nu=n-1\}$ or $\{\rho=-1,\ \nu=n\}$ occured. The validity is checked as follows:

[the right-hand side of (13)]

$$=\frac{1}{2}P_{n-1}\prod_{j=1}^{n-1}\bar{F}(x_j)\bar{A}(y)\int_0^\infty\bar{B}(x_n+v)\cdot dA(v)$$
$$+\frac{1}{2}P_n\prod_{j=1}^{n-1}\bar{F}(x_j)\bar{A}(y)\int_0^\infty\bar{F}(x_n+v)\cdot dD(v)$$
$$=\frac{1}{2}P_{n-1}\bar{A}(y)\prod_{j=1}^{n-1}\bar{F}(x_j)\Big[\bar{C}(x_n)+\sigma\int_0^\infty\bar{F}(x_n+v)$$
$$\times\Big\{\sum_{m=0}^{\infty}\sigma^m\int_{-\infty}^{0}dC(u)\Big\}dF_m(-u-v)\Big]$$
$$=\frac{1}{2}P_{n-1}\bar{A}(y)\prod_{j=1}^{n-1}\bar{F}(x_j)\Big[\bar{C}(x_n)+\sum_{m=0}^{\infty}\sigma^{m+1}\int_{-\infty}^{0}dC(u)$$
$$\times\int_0^{-u}\bar{F}(x_n+v)\cdot dF_m(-u-v)\Big]$$
$$=\frac{1}{2}P_{n-1}\bar{A}(y)\prod_{j=1}^{n-1}\bar{F}(x_j)\Big[\bar{C}(x_n)+\sum_{m=0}^{\infty}\sigma^{m+1}\int_{-\infty}^{0}dC(u)$$
$$\times\int_0^{-u}\bar{F}(-u-v+x_n)\cdot dF_m(v)\Big]$$
$$=\frac{1}{2}P_{n-1}\bar{A}(y)\prod_{j=1}^{n-1}\bar{F}(x_j)\sigma\bar{F}(x_n)\qquad\text{(by (7))}$$

$$=\frac{1}{2}P_n\bar{A}(y)\prod_{j=1}^{n}\bar{F}(x_j)\ ,$$

where $\bar{B}(x)=1-B(x)$. Equation (14) reflects the fact that the event $\{\rho=-1,\ \nu=n\}$ must be preceded by one of the events $\{\rho=+1,\ \nu=n\}$ or $\{\rho=-1,\ \nu=n+1\}$. The validity is checked as follows:

[the right-hand side of (14)]

$$=\frac{1}{2}P_n\prod_{j=1}^{n}\bar{F}(x_j)\int_0^\infty \bar{A}(y+v)\cdot dB(v)$$

$$+\frac{1}{2}P_{n+1}\prod_{j=1}^{n}\bar{F}(x_j)\int_0^\infty \bar{D}(y+v)\cdot dF(v)$$

$$=\frac{1}{2}P_n\prod_{j=1}^{n}\bar{F}(x_j)\Big[\int_{-\infty}^{0} F_0(-u-y)\cdot dC(u)$$

$$+\sigma\int_0^\infty dF(v)\Big\{\sum_{m=0}^{\infty}\sigma^m\int_{-\infty}^{0} F_m(-u-v-y)\cdot dC(u)\Big\}\Big] \qquad \text{(by (8))}$$

$$=\frac{1}{2}P_n\prod_{j=1}^{n}\bar{F}(x_j)\Big[\int_{-\infty}^{0} F_0(-u-y)\cdot dC(u)$$

$$+\sum_{m=0}^{\infty}\sigma^{m+1}\int_{-\infty}^{0} dC(u)\Big\{\int_0^\infty F_m(-u-v-y)\cdot dF(v)\Big\}\Big]$$

$$=\frac{1}{2}P_n\prod_{j=1}^{n}\bar{F}(x_j)\Big[\int_{-\infty}^{0} F_0(-u-y)\cdot dC(u)$$

$$+\sum_{m=0}^{\infty}\sigma^{m+1}\int_{-\infty}^{0} F_{m+1}(-u-y)\cdot dC(u)\Big]$$

$$=\frac{1}{2}P_n\bar{D}(y)\prod_{j=1}^{n}\bar{F}(x_j)\ .$$

The process $(\rho,\ \nu,\ \chi_1, \chi_2, \cdots, \chi_\nu,\ \eta)$ observed at successive epochs $e\in\mathcal{E}$ is a periodic process since $\Pr\{\rho=+1(-1),\ \nu=n$ at the mth epoch of $e\in\mathcal{E}\,|\,\rho=+1(-1),\ \nu=n$ at the 0th epoch of $e\in\mathcal{E}\}=0$ for any n and any $m\equiv1$ (mod. 2). Hence, although expression (10) or (11) is a stationary distribution it is not in general a limiting distribution. The process $(\rho,\ \nu,\ \chi_1, \chi_2, \cdots, \chi_\nu,\ \eta)$ observed at successive epochs $a\in\mathcal{A}$ or $d\in\mathcal{D}$, however, is an aperiodic process (Kelly [2]). Indeed, we can obtain the following corollaries from the above theorem (for the proofs, see Kelly [2]):

COROLLARY 1. *The unique stationary distribution for* $(\nu,\ \chi_1, \chi_2, \cdots, \chi_\nu)$ *observed at successive epochs* $a\in\mathcal{A}$ *is*

$$\Pr(\nu=n,\ \chi_1>x_1, \chi_2>x_2, \cdots, \chi_n>x_n)=P_n\prod_{j=1}^{n}\bar{F}(x_j)\ . \tag{15}$$

(*This result has been obtained by Fakinos* [1].)

COROLLARY 2. *The unique stationary distribution for* $(\nu, \chi_1, \chi_2, \cdots, \chi_\nu, \eta)$ *observed at successive epochs* $d \in \mathcal{D}$ *is*

$$\Pr(\nu=n,\ \chi_1>x_1, \chi_2>x_2, \cdots, \lambda_n>x_n,\ \eta>y)=P_n\bar{D}(y)\prod_{j=1}^{n}\bar{F}(x_j)\,. \tag{16}$$

The distribution of the time between a departure and the next arrival epochs is independent of the state of the system.

Remarks. An idle time distribution in the $GI/G/1$ queue with first-come-first-served is identical with that in the queue considered in this paper. Hence $\bar{D}(x)$ defined by (9) is an expression of the idle time d.f. in the $GI/G/1$ queue with first-come-first-served.

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