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STORAGE CAPACITY OF A DAM WITH GAMMA TYPE INPUTS

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Summary

Consider mutually independent inputs $X_1, \cdots, X_n$ on n different occasions into a dam or storage facility. The total input is $Y=X_1+\cdots+X_n$. This sum is a basic quantity in many types of stochastic process problems. The distribution of Y and other aspects connected with Y are studied by different authors when the inputs are independently and identically distributed exponential or gamma random variables. In this article explicit exact expressions for the density of Y are given when $X_1, \cdots, X_n$ are independent gamma distributed variables with different parameters. The exact density is written as a finite sum, in terms of zonal polynomials and in terms of confluent hypergeometric functions. Approximations when n is large and asymptotic results are also given.

1. Introduction

Consider the daily, weekly or monthly flow, in excess of certain constant amount, of water into a dam. These excesses may be caused by rains during those periods. Let X_i be the excess flow at the ith occasion. After n such occasions the total flow is $Y=X_1+\cdots+X_n$. The usual assumptions are that $X_1, \cdots, X_n$ are mutually independent and identically distributed as exponential or gamma variates. This problem has been studied by many authors, see for example Prabhu ([4], p. 209). But a more realistic situation is that the expected flows at these n occasions are different and hence $X_1, \cdots, X_n$ are not identically distributed.

Consider the problem of grain storage. Let the yield in the nth year be X_n. Let a portion of the yield every year be stored. Denoting the total storage in the nth year by Z_n let

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$$Z_{n+1}=\alpha(Z_n+X_n)\ ,\qquad n=1,2,\cdots,\qquad Z_0=0,\ 0<\alpha<1\ .$$

Then

$$Z_n=\alpha X_{n-1}+\alpha^2X_{n-2}+\cdots+\alpha^nX_0\ .$$

This problem has been studied by many authors when $X_1, X_2,\cdots$ are independently and identically distributed exponential or gamma variates, see for example Prabhu ([4], p. 179). But a more realistic situation is that

$$Z_n=\alpha_{n-1}X_{n-1}+\alpha_{n-2}X_{n-2}+\cdots+\alpha_0X_0\ ,\qquad 0<\alpha_i<1,\ i=0,1,\cdots$$

where $X_1, X_2,\cdots$ are independent variables but not necessarily identically distributed.

In many types of queueing problems one is interested in the total waiting time $Z=X_1+\cdots+X_n$ where the component waiting times X_1, $X_2,\cdots$ may be mutually independent exponential or Erlangan type random variables. Again the case when the components are identically distributed has been studied in detail, see for example Prabhu ([4], pp. 150–172). Such sums of mutually independent random variables occur as a basic quantity in the discussions of many aspects of stochastic processes, see for example Cox and Miller [2]. In problems such as storage, queues, waiting times, the assumption that X_i is gamma distributed is a realistic one.

In this article we consider a linear combination of the type

$$Y=a_1X_1+\cdots+a_nX_n$$

where $X_1,\cdots,X_n$ are independent real gamma variates with different parameters. For convenience we assume that $a_i>0$, $i=1,\cdots,n$. However the general procedure remains the same even if some of the a_i's are negative but with appropriate modifications of the conditions whenever expansions and inversions are involved. Since the variables can be easily rescaled, without loss of generality, we may consider only the case when

$$Y=X_1+\cdots+X_n \tag{1}$$

where $X_1,\cdots,X_n$ are independent real gamma variates with X_i having the density,

$$f(x)=x^{\alpha_i-1}e^{-x/\beta_i}/[\beta_i^{\alpha_i}\Gamma(\alpha_i)]\ ,\qquad x>0,\ \alpha_i>0,\ \beta_i>0\ , \tag{2}$$

and $f(x)=0$ elsewhere. The aim of this article is to work out the exact density of Y in explicit and computable forms so that exact probabilities of the type $\Pr\{Y\geqq d\}$ can be computed for every d once the parameters α_i, β_i, $i=1,\cdots,n$ are known. In this paper it is shown that the density can be written in terms of a confluent hypergeometric function of many variables when the α_i's are general, in terms of zonal polynomials when the α_i's are equal and as a finite sum of gamma densities when the α_i's are integers. The asymptotic case when $n\rightarrow\infty$ is also discussed.

2. Exact density when the parameters are integers

Consider the Y in (1) when X_i has the density in (2) where $\alpha_1,\cdots,\alpha_n$ are integers. The β_i's are not restricted to be integers. Since the moment generating functions $M_{X_i}(t)$ of X_i, $i=1,\cdots,n$ and $M_Y(t)$ of Y exist we will work with the moment generating functions. It is known that $M_{X_i}(t)=(1-\beta_i t)^{-\alpha_i}$ and due to independence one has

$$M_Y(t)=\prod_{i=1}^{n}(1-\beta_i t)^{-\alpha_i}\,. \tag{3}$$

The density of Y is available by inverting this moment generating function. We will work out this by using a general partial fraction technique. If some of the β_i's are equal then the corresponding factors can be combined. Hence in the following discussion we will assume that all the β_i's are distinct and nonzero. Since the α_i's are integers one can put $M_Y(t)$ as a finite sum by partial fraction technique. That is,

$$\begin{aligned}M_Y(t)&=\prod_{j=1}^{n}(-\beta_j)^{-\alpha_j}\prod_{j=1}^{n}(t-1/\beta_j)^{-\alpha_j}\\&=\left[\prod_{j=1}^{n}(-\beta_j)^{-\alpha_j}\right]\sum_{j=1}^{n}\sum_{r=1}^{\alpha_j}b_{jr}(t-1/\beta_j)^{-r}\end{aligned}$$

where the coefficients b_{jr} are to be determined. But the density corresponding to the moment generating function $(1-\beta_j t)^{-r}$ is $y^{r-1}e^{-y/\beta_j}/[\beta_j^r\Gamma(r)]$ for $y>0$ and zero elsewhere. Hence the density of Y, denoted by $g(y)$, is as follows.

$$g(y)=\left[\prod_{j=1}^{n}(-\beta_j)^{-\alpha_j}\right]\sum_{j=1}^{n}\sum_{r=1}^{\alpha_j}(-1)^r b_{jr}y^{r-1}e^{-y/\beta_j}/(r-1)! \tag{4}$$

for $y>0$ and $g(y)=0$ elsewhere. Hence probabilities of the type $\Pr\{y\geqq d\}$ can be evaluated by using an incomplete gamma table. The coefficients b_{jr} will be evaluated by using the following technique. Since α_j is assumed to be an integer by using the well known results for repeated factors we can write

$$b_{jr}=\lim_{t\to 1/\beta_j}\left\{\frac{1}{(\alpha_j-r)!}\frac{\partial^{\alpha_j-r}}{\partial t^{\alpha_j-r}}\left[(t-1/\beta_j)^{\alpha_j}\prod_{i=1}^{n}(t-1/\beta_i)^{-\alpha_i}\right]\right\}\,.$$

Let

$$\Delta(t)=\prod_{\substack{i=1\\i\neq j}}^{n}(t-1/\beta_i)^{-\alpha_i} \quad \text{and} \quad A(t)=\frac{\partial}{\partial t}\log\Delta(t)=-\sum_{\substack{i=1\\i\neq j}}^{n}[\alpha_i/(t-1/\beta_i)] .$$

Then

$$\frac{\partial}{\partial t}\Delta(t)=A(t)\Delta(t) .$$

Hence

$$\frac{\partial^{\alpha_j-r}}{\partial t^{\alpha_j-r}}\Delta(t)=\frac{\partial^{\alpha_j-r-1}}{\partial t^{\alpha_j-r-1}}[A(t)\Delta(t)]$$
$$=\sum_{j_1=0}^{\alpha_j-r-1}\binom{\alpha_j-r-1}{j_1}A^{(\alpha_j-r-1-j_1)}\Delta^{(j_1)}$$

where for example $A^{(s)}$ and $\Delta^{(s)}$ denote the sth partial derivatives with respect to t of $A(t)$ and $\Delta(t)$ respectively with $A^{(0)}=A(t)$ and $\Delta^{(0)}=\Delta(t)$ and for example $\binom{m}{v}=m!/[v!(m-v)!]$, $0!=1$. Now rewriting $\Delta^{(j_1)}$ and continuing the process and then evaluating all the expressions at $t=1/\beta_j$ one gets the following result.

$$(5)\qquad b_{jr}=\left\{\sum_{j_1=0}^{\alpha_j-r-1}\binom{\alpha_j-r-1}{j_1}A_j^{(\alpha_j-r-1-j_1)}\sum_{j_2=0}^{j_1-1}\binom{j_1-1}{j_2}A_j^{(j_1-1-j_2)}\cdots\right\}\Delta_j/(\alpha_j-r)!$$

where

$$\Delta_j=\prod_{\substack{i=1\\i\neq j}}^{n}(1/\beta_j-1/\beta_i)^{-\alpha_i} \quad \text{and}$$

$$A_j^{(s)}=(-1)^{s+1}s!\sum_{\substack{i=1\\i\neq j}}^{n}\alpha_i(1/\beta_j-1/\beta_i)^{-(s+1)} .$$

For example when $\alpha_j=3$, $b_{j3}=\Delta_j$, $b_{j2}=A_j\Delta_j$ and $b_{j1}=[A_j^{(1)}+(A_j)^2]\Delta_j/2$. This completes the derivation of the density.

3. Exact density when the parameters are equal

When $\beta_1=\beta_2=\cdots=\beta_n$ the problem is a trivial one and the density of Y is obviously a gamma density. Here we consider the case $\alpha_1=\cdots=\alpha_n=\alpha$, $\beta_j>0$, $j=1,\cdots,n$ and not all β_j's equal. In this case the moment generating function of Y can be written as

$$M_Y(t)=\prod_{j=1}^{n}(1-\beta_j t)^{-\alpha}=|I-tB|^{-\alpha}$$

where B is a symmetric positive definite matrix with $\beta_1,\cdots,\beta_n$ being its eigenvalues, I is an identity matrix of order n and $|I-tB|$ denotes the determinant of $I-tB$. For any $\delta>0$ we may write

$$(6)\qquad \begin{aligned} |I-tB|^{-\alpha} &= |I-B/\delta+B(1-\delta t)/\delta|^{-\alpha} \\ &= |B|^{-\alpha}\delta^{n\alpha}(1-\delta t)^{-n\alpha}|I-(I-\delta B^{-1})/(1-\delta t)|^{-\alpha} \\ &= \left[\prod_{j=1}^{n}\beta_j^{-\alpha}\right]\delta^{n\alpha}\sum_{k=0}^{\infty}\sum_{K}\frac{(\alpha)_K}{k!}C_K(I-\delta B^{-1})(1-\delta t)^{-(n\alpha+k)} \end{aligned}$$

where $K=(k_1, k_2, \cdots, k_n)$ denotes a partition of the nonnegative integer k into not more than n parts $k_1\geqq k_2\geqq\cdots\geqq k_n\geqq 0$, $k=k_1+\cdots+k_n$, C_K denotes the zonal polynomial of order k and

$$(\alpha)_K=\prod_{i=1}^{n}(\alpha-(i-1)/2)_{k_i}\ , \qquad K=(k_1, \cdots, k_n)$$

where for example $(x)_m=x(x+1)\cdots(x+m-1)$, $(x)_0=1$. The expansion in (6) is valid when a norm of the matrix $(I-\delta B^{-1})/(1-\delta t)$ is less than unity. The absolute value of the largest eigenvalue being less than unity is a sufficient condition. This condition can always be met by adjusting the value of the arbitrary quantity δ and choosing t. For example $\delta<\beta_i$, $i=1, \cdots, n$ and $t<\min(1/\beta_1, \cdots, 1/\beta_n)$ is a sufficient condition. Also the convergence of the series can be made faster by adjusting δ. For more details about zonal polynomials see Constantine [1]. Thus the density function of Y is the following.

$$(7)\qquad \begin{aligned} g(y) &= \left[\prod_{j=1}^{n}\beta_j^{-\alpha}\right]\delta^{n\alpha}\sum_{k=0}^{\infty}\sum_{K}\frac{(\alpha)_K}{k!}C_K(I-\delta B^{-1})y^{n\alpha+k-1}e^{-y/\delta}/[\delta^{n\alpha+k}\Gamma(n\alpha+k)] \\ &= \left[\prod_{j=1}^{n}\beta_j^{-\alpha}\right]\frac{y^{n\alpha-1}e^{-y/\delta}}{\Gamma(n\alpha)}\sum_{k=0}^{\infty}\sum_{K}(\alpha)_K C_K(I-\delta B^{-1})/[k!(n\alpha)_k] \end{aligned}$$

for $y>0$ and $g(y)=0$ elsewhere.

In order to compute probabilities of the type $\Pr\{y\geqq d\}$ for a given d one needs the zonal polynomials for all values of k. When k is large explicit expressions for the zonal polynomials are not available. They are available for small values of k. By using these available zonal polynomials and by adjusting the value of δ to make the convergence of the series faster one can compute fairly accurate values of the required probabilities by using (7) and the incomplete gamma tables.

4. Exact density in the general case

In the general case we consider general values of α_j's and β_j's, $\alpha_j>0$, $\beta_j>0$, $j=1, \cdots, n$. One can rewrite the factors as follows.

$$\begin{aligned} (1-\beta_2 t)^{-\alpha_2} &= (1-\beta_1 t)^{-\alpha_2}(\beta_1/\beta_2)^{\alpha_2}[1-(1-\beta_1/\beta_2)/(1-\beta_1 t)]^{-\alpha_2} \\ &= (1-\beta_1 t)^{-\alpha_2}(\beta_1/\beta_2)^{\alpha_2}\sum_{r=0}^{\infty}\frac{(\alpha_2)_r}{r!}(1-\beta_1/\beta_2)^r(1-\beta_1 t)^{-r} \end{aligned}$$

for $|(1-\beta_1/\beta_2)/(1-\beta_1 t)|<1$. A sufficient condition for the expansion is that $t<\min(1/\beta_1, 1/\beta_2)$ and $\beta_1<\beta_2$. Rewriting the factors $(1-\beta_i t)^{-\alpha_i}$, $i=2, 3, \cdots, n$ one has

$$M_Y(t)=(1-\beta_1 t)^{-\gamma}\beta_1^{\gamma-\alpha_1}(\beta_2^{\alpha_2}\cdots\beta_n^{\alpha_n})^{-1}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\{(\alpha_2)_{r_2}\cdots(\alpha_n)_{r_n} \cdot(1-\beta_1/\beta_2)^{r_2}\cdots(1-\beta_1/\beta_n)^{r_n}(1-\beta_1 t)^{-r}/[r_2!\cdots r_n!]\}$$

for $|(1-\beta_1/\beta_i)/(1-\beta_1 t)|<1$, $i=2,\cdots,n$, where $\gamma=\alpha_1+\alpha_2+\cdots+\alpha_n$, $r=r_2+\cdots+r_n$. A sufficient condition is that $\beta_1<\beta_i$, $i=2,\cdots,n$ and $t<\min(1/\beta_1,\cdots,1/\beta_n)$. But the density corresponding to the factor $(1-\beta_1 t)^{-(\gamma+r)}$ is $y^{\gamma+r-1}e^{-y/\beta_1}/[\beta_1^{\gamma+r}\Gamma(\gamma+r)]$. Term by term inversion is possible in this case and the density is the following.

$$g(y)=\left[\prod_{j=1}^{n}\beta_j^{\alpha_j}\Gamma(\gamma)\right]^{-1}y^{\gamma-1}e^{-y/\beta_1}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\{(\alpha_2)_{r_2}\cdots(\alpha_n)_{r_n} \cdot[(1/\beta_1-1/\beta_2)y]^{r_2}\cdots[(1/\beta_1-1/\beta_n)y]^{r_n}/[r_2!\cdots r_n!(\gamma)_r]\}\,, \tag{8}$$

for $y>0$ and $g(y)=0$ elsewhere. But the multiple series appearing in (8) is a confluent hypergeometric function of $n-1$ variables, namly ϕ_2. Hence

$$g(y)=\left[\prod_{j=1}^{n}\beta_j^{\alpha_j}\Gamma(\gamma)\right]^{-1}y^{\gamma-1}e^{-y/\beta_1}\phi_2(\alpha_2,\cdots,\alpha_n;\gamma;(1/\beta_1-1/\beta_2)y, \cdots,(1/\beta_1-1/\beta_n)y)\,,\qquad y>0\,. \tag{9}$$

Properties of ϕ_2 are already available and this function is studied in detail in the literature on Special Functions. For a definition of ϕ_2 see Mathai and Saxena ([3], p. 163).

In order to make the convergence of ϕ_2 faster one can use the following procedure.

$$\Pr\{y\geqq d\}=\Pr\{\delta y\geqq \delta d\}\qquad \text{for } \delta>0\,.$$

The density of δY is equivalent to the density in (9) with β_j replaced by $\delta\beta_j$. Now by choosing δ one can make the arguments of ϕ_2 smaller so that the convergence of ϕ_2 is made faster. Again exact probabilities of the type $\Pr\{y\geqq d\}$ can be evaluated from (8) by using incomplete gamma tables.

5. Approximations

Gamma type approximations to the density of Y are available by taking the first few terms of the series representation given in (8). Here we will consider a normal approximation. Taking logarithms of the moment generating function and expanding in powers of t one has the following.

$$\log_e M_Y(t) = -\sum_{j=1}^{n} \alpha_j \log_e (1-\beta_j t) = \sum_{j=1}^{n} \alpha_j [\beta_j t + (\beta_j t)^2/2 + \cdots]$$

for $|\beta_j t| < 1$, $j=1, \cdots, n$. The coefficient of $t^r/r!$ in this expansion is the rth cumulant K_r or the rth semi-invariant of Y. But it is known that K_1 is the mean value and K_2 is the variance of Y. Suppose that n is large such that $\sum_{j=1}^{n} \alpha_j(\beta_j/n^{1/2})^r \to 0$ as $n\to\infty$ for $r \geqq 3$ and $0 < \sum_{j=1}^{n} \alpha_j(\beta_j/n^{1/2})^2 < \infty$. Then the rth cumulant of the random variable $Y/n^{1/2}$ goes to zero as $n \to \infty$ for $r \geqq 3$. Hence we have the following result.

$$Z = \left[\left(Y - \sum_{j=1}^{n} \alpha_j\beta_j\right)\Big/ n^{1/2}\right]\Big/\left[\sum_{j=1}^{n} \alpha_j\beta_j^2/n\right]^{1/2} = \left[Y - \sum_j \alpha_j\beta_j\right]\Big/\left[\sum_j \alpha_j\beta_j^2\right]^{1/2}$$

$$\to N(0, 1) \text{ as } n\to\infty \text{ when } 0 < \sum_{j=1}^{n} \alpha_j(\beta_j/n^{1/2})^2 < \infty \text{ and}$$

$\sum_{j=1}^{n} \alpha_j(\beta_j/n^{1/2})^r \to 0$ as $n\to\infty$ for $r \geqq 3$.

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