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HYPOTHESIS TESTING FOR THE COMMON MEAN OF TWO NORMAL DISTRIBUTIONS IN THE PRESENCE OF AN INDIFFERENCE ZONE

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Summary

Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be independent, random samples from populations which are $N(\theta, \sigma_x^2)$ and $N(\theta, \sigma_y^2)$, respectively, with all parameters unknown. In testing $H_0: \theta=0$ against $H_1: \theta \neq 0$, the t-test based upon either sample is known to be admissible in the two-sample setting. If, however, one tests H_0 against $H_1: |\theta| \geqq \varepsilon > 0$, with ε arbitrary, our main results show: (i) the construction of a test which is better than the particular t-test chosen, (ii) each t-test is admissible under the invariance principle with respect to the group of scale changes, and (iii) there does not exist a test which simultaneously is better than both t-tests.

1. Introduction

In this paper we investigate the problem of hypothesis testing for the common, unknown mean of two normal distributions with unknown variances, when there exists a zone of indifference separating the null and alternative hypotheses.

We shall assume that one has a random sample from each of two normal populations, referred to herein as the X and Y samples (or populations), where the samples are of sizes $m \geqq 2$ and $n \geqq 2$, respectively. We denote the common mean by θ and the variances by σ_x^2 and σ_y^2, where $-\infty < \theta < \infty$, $0 < \sigma_x^2$, and $0 < \sigma_y^2$. A number of recent articles appearing in the literature have been directed at achieving dominance over the classical one-sample inferential procedures for θ in this two-sample framework.

In an estimation setting, Cohen and Sackrowitz [2], [3] offer unbiased minimax estimators of θ which, under mild sample size restric-

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tions, dominate the mean of one sample when loss is taken to be proportional to squared error. Moreover, for equal sample sizes exceeding 9, a procedure simultaneously dominating the sample means $\bar{X}$ and $\bar{Y}$ is given. Brown and Cohen [1] also offer improved two-sample estimators, but additionally exhibit a confidence interval of width equal to that of the classical procedure and with coverage probability greater everywhere in the parameter space.

Within the hypothesis testing framework, Cohen and Sackrowitz [4] have shown that when testing $\theta=0$ against the unrestricted alternative $\theta\neq 0$, the one-sample t-test based upon the X sample (say) is an admissible procedure in the two-sample setting. This is somewhat surprising in light of the aforementioned realities of the estimation problem. They also demonstrate, however, that if one restricts the alternative space by bounding the variance of the X population away from zero, then a similar test based upon both samples may be found whose power exceeds that of the one-sample t-test everywhere in the alternative space. It is our purpose to consider a different restriction in the form of a zone of indifference for the mean and, therefore, we will treat the problem of testing $\theta=0$ against $|\theta|\geqq\varepsilon$, where ε is positive and arbitrary. Our feeling is that this is a somewhat more natural region over which to seek a dominating procedure, since a test developed for the case of bounded variance will not necessarily have high probability of detecting large absolute values of θ when the variance σ_x is small.

In Section 2 we specify a dominating two-sample procedure for the indifference zone problem, while in Section 3 we identify some of the properties of both the proposed similar test as well as of other potentially successful competitors to the t-test, particularly with respect to the notion of invariance under the group of scale changes. We show, specifically, that in an important sense the parametrization of the proposed test is independent of ε, that no invariant test can dominate the one-sample t-test, and also that unless further restriction is imposed on the parameter space it is futile to search for a procedure which simultaneously dominates both one-sample t-tests. The proposed test also suffices as a competitor in the case of bounded variance considered by Cohen and Sackrowitz [4]. Moreover, this section contains some of the basic germination for the form of the test of Section 2, principally a result due to Stein [7] giving a sufficient condition for admissibility of tests of parameters of multiparameter exponential families. Section 4 (Appendix) contains the proof of several auxiliary lemmas.

2. A dominating similar test in the presence of a zone of indifference

We first give some preliminaries necessary for the main result of this section. The approach is initially quite similar to that given by Cohen and Sackrowitz [4]. Lengthy computations arising in the proof of Theorem 2.1 have been deferred to the Appendix.

The distributions of the sample $(X_1, X_2, \cdots, X_m, Y_1, Y_2, \cdots, Y_n)$ constitute a four-parameter exponential family with a minimal sufficient statistic being $\left(\bar{X}, \sum_1^m X_i^2, \bar{Y}, \sum_1^n Y_i^2\right)$ and corresponding natural parameter $(m\theta/\sigma_x^2, -1/2\sigma_x^2, n\theta/\sigma_y^2, -1/2\sigma_y^2)$. Let $T_x=\sum_1^m X_i^2$, $T_y=\sum_1^n Y_i^2$, and $s_y=\sqrt{T_y-n\bar{Y}^2}$. The conditional density of $\bar{X}$ given T_x, which depends upon θ and σ_x only through the noncentrality parameter $\delta_x=m\theta/\sigma_x^2$, is easily seen to be

$$(2.1)\qquad f_{\delta_x}(u\,|\,t_x)=\begin{cases}\dfrac{(1-mu^2/t_x)^{(m-3)/2}\exp(\delta_x u)}{(t_x/m)^{1/2}\int_{-1}^{1}(1-v^2)^{(m-3)/2}\exp(\delta_x(t_x/m)^{1/2}v)dv}, & \text{if } |u|\leqq(t_x/m)^{1/2}\\ 0, & \text{otherwise},\end{cases}$$

from which it follows that the conditional c.d.f. of $\bar{X}$ given T_x is

$$(2.2)\qquad F_{\delta_x}(u\,|\,t_x)=\begin{cases}0, & \text{if } u<-(t_x/m)^{1/2},\\ \dfrac{\int_{-1}^{(m/t_x)^{1/2}u}(1-v^2)^{(m-3)/2}\exp(\delta_x v(t_x/m)^{1/2})dv}{\int_{-1}^{1}(1-v^2)^{(m-3)/2}\exp(\delta_x v(t_x/m)^{1/2})dv}, & \\ 1, & \text{otherwise}.\end{cases}$$

Since T_x/σ_x^2 has a non-central chi-squared distribution with m degrees of freedom and non-centrality parameter δ_x, the marginal density of T_x is given by

$$(2.3)\qquad g_{\theta,\sigma_x}(t_x)=K(\theta,\sigma_x)\exp(-t_x/2\sigma_x^2)t_x^{(m-3)/2}(t_x/m)^{1/2}\times\int_{-1}^{1}(1-v^2)^{(m-3)/2}\exp(\sqrt{m}\,\theta\sqrt{t_x}\,v/\sigma_x^2)dv\,,\qquad \text{for } t_x>0\,,$$

where $K(\theta, \sigma_x)=K_m\exp(-m\theta^2/2\sigma_x^2)/\sigma_x^m$, $K_m=(m/2\pi)^{1/2}/(2^{(m-1)/2}\Gamma((m-1)/2))$, and $\Gamma(\cdot)$ is the Gamma function.

We consider a class $\mathcal{F}$ of test functions of the form

$$(2.4)\qquad \phi_{a,\alpha}(\bar{X}, T_x, \bar{Y}, T_y)=\begin{cases}0, & \text{if } C_\alpha^L(T_x, \bar{Y}, T_y)\leqq\bar{X}\leqq C_\alpha^U(T_x, \bar{Y}, T_y)\\ 1, & \text{otherwise},\end{cases}$$

where $a\in[0, 1]$, $\alpha\in(0, 1/2)$, and C_α^L and C_α^U are functions of the arguments shown. As $\phi_{a,\alpha}$ has an acceptance region composed of convex sections, it follows from a result of Matthes and Truax [5] that $\mathcal{F}$ is complete for testing $\theta=0$ in the two-sample setting. We shall, in fact, restrict ourselves to a subset $\mathcal{F}_1$ of $\mathcal{F}$ consisting of tests of the form (2.4), but where

$$C_a^U(T_x, \bar{Y}, T_y)=F_0^{-1}(1-\alpha/2-aQ_y(\bar{Y}, s_y)Q_x(T_x)\,|\,T_x) \quad \text{and} \quad C_a^L(T_x, \bar{Y}, T_y)=F_0^{-1}(\alpha/2-aQ_y(\bar{Y}, s_y)Q_x(T_x)\,|\,T_x)\,. \tag{2.5}$$

Here $Q_x(\cdot)$, which is non-negative, and $Q_y(\cdot)$ are functions chosen consistent with the conditions that $E_{\theta=0}\{Q_y(\bar{Y}, s_y)\}=0$ and that the arguments of F_0^{-1} in (2.5) are in (0, 1).

We note that $\mathcal{F}_1$ consists of similar tests, as

$$\begin{aligned} E_{0,\sigma_x,\sigma_y}\{\phi_{a,\alpha}\} &= 1-E_{0,\sigma_y}\{E_{0,\sigma_x}\{F_0(C_a^U(T_x, \bar{Y}, T_y)\,|\,T_x) \\ &\quad -F_0(C_a^L(T_x \bar{Y}, T_y)\,|\,T_x)\}\,|\,\bar{Y}, T_y\} \\ &= 1-E_{0,\sigma_y}\{E_{0,\sigma_x}\{1-\alpha\}\,|\,\bar{Y}, T_y\}=\alpha\,. \end{aligned}$$

Define $k_{0,\alpha}=t_{\alpha/2}(m-1)/(m-1+t_{\alpha/2}^2(m-1))^{1/2}$, where $t_{\alpha/2}(m-1)$ represents the $(1-\alpha/2)$ percentile of the t-distribution with $m-1$ degrees of freedom.

Let us now consider a particular member of $\mathcal{F}_1$, denoted $\phi_{a,\alpha}^*$ and determined by choosing

$$Q_x=c_2T_x^{1/2}(b_\varepsilon^{1/2}-T_x^{1/2})I_{[0,b_\varepsilon]}(T_x)\,, \qquad Q_y=\operatorname{sgn}\bar{Y}/(c_1+s_y/b_\varepsilon^{1/2})\,, \tag{2.6}$$

where $c_1>2/\alpha$, $\varepsilon>0$, $k_{0,\alpha}^2\varepsilon^2 m>b_\varepsilon$, c_2 is a constant proportional to ε^{-2} and for which an upper bound is to be subsequently specified, and $I_A(\cdot)$ is the indicator function of the set A. Denoting differentiation with respect to a by priming, we now state the following:

LEMMA 2.1. *Let $\beta_{\theta,\sigma_x,\sigma_y}(a)$ denote the power function of $\phi_{a,\alpha}^*$. Then $\beta'_{\theta,\sigma_x,\sigma_y}(0)>0$, for all $(\theta, \sigma_x, \sigma_y)$ such that $|\theta|\geqq\varepsilon$.*

PROOF. From (2.1)-(2.5) it is straightforward to show that

$$\begin{aligned} \beta'_{\theta,\sigma_x,\sigma_y}(0) &= (K(\theta, \sigma_x)/m^{1/2})\int_{-1}^{1}(1-v^2)^{(m-3)/2}dvE_{\theta,\sigma_y}\{Q_y\} \\ &\quad \times\int_0^\infty(\exp(k_{0,\alpha}\theta m^{1/2}t_x^{1/2}/\sigma_x^2)-\exp(-k_{0,\alpha}\theta m^{1/2}t_x^{1/2}/\sigma_x^2)) \\ &\quad \times\exp(-t_x/2\sigma_x^2)t_x^{(m/2)-1}Q_xdt_x\,, \end{aligned} \tag{2.7}$$

where $K(\theta, \sigma_x)$ and K_m are given in (2.3). The result follows directly.

LEMMA 2.2. *Let $m\geqq3$, $n\geqq4$, and $\alpha\in(0, 1/2)$. Then there exists a positive constant M, not depending upon θ, σ_x, σ_y, or a^*, such that*

$|\beta'_{\theta,\sigma_x,\sigma_y}(0)/\beta''_{\theta,\sigma_x,\sigma_y}(a^*)| \geqq M$, *for all* $a^* \in (0, 1)$ *and* $(\theta, \sigma_x, \sigma_y)$ *such that* $|\theta| \geqq \varepsilon$.

PROOF. See Appendix.

We now state the main result of this section.

THEOREM 2.1. *Let* $X_1, X_2, \cdots, X_m$ *and* $Y_1, Y_2, \cdots, Y_n$ *be independent, random samples from populations which are* $N(\theta, \sigma_x^2)$ *and* $N(\theta, \sigma_y^2)$, *respectively, with all parameters unknown. Let* $\alpha \in (0, 1/2)$, $m \geqq 3$, $n \geqq 4$, *and* $\varepsilon > 0$ *be arbitrary. Then in testing* $H_0: \theta = 0$ *versus* $H_1: |\theta| \geqq \varepsilon$, *the one-sample t-test of size* α *based upon the X sample (say) is dominated by the test* $\phi^*_{a,\alpha}$ *given in Lemma* 2.1, *for a sufficiently small.*

PROOF. Let γ be a constant in (0, 1) and let b_ε denote the quantity $\gamma^2 k^2_{0,\alpha} \varepsilon^2 m$ (a rationale for b_ε will be given in Section 3). Expanding $\beta_{\theta,\sigma_x,\sigma_y}(a)$ in a Taylor series about $a = 0$ with the third term constituting the remainder, we have

$$\beta_{\theta,\sigma_x,\sigma_y}(a) = \beta_{\theta,\sigma_x,\sigma_y}(0) + a\beta'_{\theta,\sigma_x,\sigma_y}(0) + \frac{a^2}{2}\beta''_{\theta,\sigma_x,\sigma_y}(a^*) \ , \tag{2.8}$$

for some $a^* \in (0, a)$. But, as the first term is the t-test's power, it suffices to show that for some a, the second and third terms are positive in sum for all $(\theta, \sigma_x, \sigma_y) \in H_1$ and $a^* \in (0, a)$. It, therefore, suffices to show (i) $\beta'_{\theta,\sigma_x,\sigma_y}(0) > 0$, for all $(\theta, \sigma_x, \sigma_y) \in H_1$, and (ii) $|\beta'_{\theta,\sigma_x,\sigma_y}(0)/\beta''_{\theta,\sigma_x,\sigma_y}(a^*)| \geqq M > 0$, for all $(\theta, \sigma_x, \sigma_y) \in H_1$ and $a^* \in (0, 1)$, where M is a constant not depending on θ, σ_x, σ_y, or a^*. Thus, by Lemmas 2.1 and 2.2 we have the desired result.

Remarks. (i) An upper bound for the constant c_2 of the test $\phi^*_{a,\alpha}$ is specified at the end of the proof of Lemma 4.7, since the constraints on that quantity occur therein.

(ii) The constant a is also specified, by way of an upper bound, at the end of the proof of Lemma 2.2 since this constant depends upon M.

(iii) We have arbitrarily chosen to "beat" the t-test based upon the X sample. It is clear, however, that a test of identical form to $\phi^*_{a,\alpha}$, with the roles of statistics and sample sizes related to the X and Y samples reversed, would dominate the t-test based upon the Y sample.

(iv) If the problem is altered through translation to one of testing $H_0: \theta = \theta_0$ vs. $H_1: |\theta - \theta_0| \geqq \varepsilon$, we encounter no additional difficulties, since letting $X'_i = X_i - \theta_0$ and $Y'_i = Y_i - \theta_0$ permits us to use all of the aforementioned results.

(v) A result directly analogous to Theorem 2.1 applies in the one-sided testing problem of $H_0: \theta = 0$ vs. $H_1: \theta \geqq \varepsilon$, or vs. $H_1: \theta \leqq -\varepsilon$.

3. Invariance, universality of the mixture constant a, and motivational consideration

We shall, in this section, consider some specific properties of the testing problem discussed in Section 2, with a view towards the notion of invariant test procedures. We represent the sample and parameter spaces by $\mathcal{X}$ and Θ, respectively, and begin by noting the following standard definition:

DEFINITION 3.1. A test function ϕ is said to be invariant under a group of transformations, $\mathcal{G}$, if $\phi(g(x))=\phi(x)$, for all $x \in \mathcal{X}$, and $g \in \mathcal{G}$. The following result establishes the futility of searching among the class of invariant tests for a successful competitor to the t-test when $\mathcal{G}$ is taken to be the group of scale changes.

THEOREM 3.1. *In testing* $H_0: \theta=0$ vs. $H_1: |\theta| \geqq \varepsilon > 0$, *there does not exist a test function invariant under the group* $\mathcal{G}$ *of scale changes, which dominates* $\phi_{0,\alpha}$, *the one-sample t-test based upon the X sample.*

PROOF. The orbits of Θ consist of sets indexed by a point, say $(\theta^*, \sigma_x^*, \sigma_y^*)$. Taking $\mathcal{G}=\{g_k: g_k(x_1, \cdots, x_m, y_1, \cdots, y_n)=(kx_1, \cdots, kx_m, ky_1, \cdots, ky_n), k>0\}$, we obtain the transformation induced in Θ by g_k as $\bar{g}_k(\theta, \sigma_x, \sigma_y)=(k\theta, k\sigma_x, k\sigma_y)$. Hence, the orbit generated by $(\theta^*, \sigma_x^*, \sigma_y^*)$ is the line $L(\theta^*/\sigma_x^*, \theta^*/\sigma_y^*)=\{(\theta, \sigma_x, \sigma_y): \theta/\sigma_x=\theta^*/\sigma_x^*, \theta/\sigma_y=\theta^*/\sigma_y^*\}$. Assume that ϕ is invariant under $\mathcal{G}$ and dominates $\phi_{0,\alpha}$ when testing H_0 vs. H_1, and let $\Delta>1$ be arbitrary. Then there exists a point $(\theta', \sigma_x', \sigma_y') \in \Theta$ such that $|\theta'| \geqq \varepsilon$, and for which $\beta_{\theta', \sigma_x', \sigma_y'}(\phi) > \beta_{\theta', \sigma_x', \sigma_y'}(\phi_{0,\alpha})$. Now consider any other point $(\theta, \sigma_x, \sigma_y)$ for which $\theta \neq 0$. Then $\beta_{\theta, \sigma_x, \sigma_y}(\phi)=\beta_{\Delta\varepsilon, \Delta\varepsilon\sigma_x/\theta, \Delta\varepsilon\sigma_y/\theta}(\phi)$, as ϕ is invariant under $\mathcal{G}$ and $\beta(\phi)$ is, therefore, constant on orbits of Θ (note that $(\theta, \sigma_x, \sigma_y)$ and $(\Delta\varepsilon, \Delta\varepsilon\sigma_x/\theta, \Delta\varepsilon\sigma_y/\theta)$ are of common orbit). But the r.h.s. of this last equality is at least $\beta_{\Delta\varepsilon, \Delta\varepsilon\sigma_x/\theta, \Delta\varepsilon\sigma_y/\theta}(\phi_{0,\alpha})$, as Δ has been chosen greater than unity and ϕ dominates $\phi_{0,\alpha}$ for all points for which $|\theta| \geqq \varepsilon$. Moreover, by the well-known invariance of $\phi_{0,\alpha}$ under $\mathcal{G}$, we have $\beta_{\Delta\varepsilon, \Delta\varepsilon\sigma_x/\theta, \Delta\varepsilon\sigma_y/\theta}(\phi_{0,\alpha})=\beta_{\theta, \sigma_x, \sigma_y}(\phi_{0,\alpha})$. Thus, $\beta_{\theta, \sigma_x, \sigma_y}(\phi) \geqq \beta_{\theta, \sigma_x, \sigma_y}(\phi_{0,\alpha})$, for all $\theta \neq 0$, with strict inequality for some $\theta' \neq 0$; i.e., ϕ dominates $\phi_{0,\alpha}$ over the unrestricted alternative. However, $\phi_{0,\alpha}$ is contradictorily known to be admissible if H_1 is unrestricted. This completes the proof of the theorem.

This lack of invariance of a potentially successful competitor, perhaps disconcerting at first glance, is not unreasonable. For ε should logically be in the same units of measurement as the data, and if these units are changed then ε should be changed concordantly. It is with respect to this "scaled" ε that one's test function should reach the same conclusion regarding the rejection of H_0. It is easy to show that our dominating procedure, $\phi^*_{a,\alpha}$, indeed has this property.

We now employ this last result to prove an interesting property relating to the constant a. Ostensibly, the choice of an upper bound for this constant would seem to depend upon the point of indifference ε. The following result demonstrates, on the contrary, that any value of a consistent with the specification of a test of the form indicated by (2.6) for a particular choice of ε is consistent with the choice of a test for any other ε as well.

THEOREM 3.2. *Let $\delta>0$ be arbitrary and $\phi^*_{a_0,\varepsilon_0}$ be a test of the form indicated by* (2.6) *which dominates $\phi_{0,\alpha}$ when testing $H_0: \theta=0$ vs. $H_1: |\theta| \geqq \varepsilon_0$. Then $\phi^*_{a_0,\varepsilon_1}$ also dominates $\phi_{0,\alpha}$ when H_1 is changed to $H_1^*: |\theta| \geqq \varepsilon_1 = \delta\varepsilon_0$.*

PROOF. For any $(\theta, \sigma_x, \sigma_y) \in H_1^*$, we have

$$\begin{aligned}\beta_{\theta,\sigma_x,\sigma_y}(\phi^*_{a_0,\varepsilon_1}) &= E_{\theta,\sigma_x,\sigma_y}\{\phi^*_{a_0,\varepsilon_1}(\bar{X}, T_x, \bar{Y}, T_y)\}\\ &= E_{\theta,\sigma_x,\sigma_y}\{\phi^*_{a_0,\varepsilon_0}(\delta^{-1}\bar{X}, \delta^{-2}T_x, \delta^{-1}\bar{Y}, \delta^{-2}T_y)\}\end{aligned}$$

(by the invariance property alluded to above)

$$= E_{\theta/\delta,\sigma_x/\delta,\sigma_y/\delta}\{\phi^*_{a_0,\varepsilon_0}(\bar{X}, T_x, \bar{Y}, T_y)\}$$

(as the distribution of $(\delta^{-1}\bar{X}, \delta^{-2}T_x, \delta^{-1}\bar{Y}, \delta^{-2}T_y)$ under $(\theta, \sigma_x, \sigma_y)$ is the same as that of $(\bar{X}, T_x, \bar{Y}, T_y)$ under $(\delta^{-1}\theta, \delta^{-1}\sigma_x, \delta^{-1}\sigma_y)$). Now this last expectation exceeds that of $\phi_{0,\alpha}$ if $|\theta/\delta| \geqq \varepsilon_0$. But, for such points, $|\theta| \geqq \delta\varepsilon_0 = \varepsilon_1$. Hence,

$$\begin{aligned}&\beta_{\theta,\sigma_x,\sigma_y}(\phi^*_{a_0,\varepsilon_1}) > \beta_{(\theta/\delta,\sigma_x/\delta,\sigma_y/\delta)}(\phi_{0,\alpha}), && |\theta| \geqq \varepsilon_1,\\ &\Rightarrow \beta_{\theta,\sigma_x,\sigma_y}(\phi^*_{a_0,\varepsilon_1}) > \beta_{\theta,\sigma_x,\sigma_y}(\phi_{0,\alpha}), && |\theta| \geqq \varepsilon_1,\end{aligned}$$

as the power of $\phi_{0,\alpha}$ depends upon θ and σ_x only through $\sqrt{m}\,\theta/\sigma_x$. Thus, the theorem is proved.

We now give a proof of the non-existence of a test based upon both samples which simultaneously dominates the t-tests based upon the X sample and Y sample. We shall let $\phi_{0,\alpha,x}$ and $\phi_{0,\alpha,y}$ represent the respective t-tests, and begin by giving a lemma which is not more than a rephrasing of a result due to Stein [7] with a shift in emphasis. Moreover, this lemma lends considerable insight into the functional form of $\phi_{a,\alpha}$.

LEMMA 3.1. *Suppose $\boldsymbol{Z}$ is distributed as a k-parameter exponential family with absolutely continuous distribution function. Let A be a* *closed, convex subset of the sample space $\mathcal{X}$ and B a subset of the adjoint space $\mathcal{X}'$ (the linear space of all real-valued functions on $\mathcal{X}$). Suppose that for all $\boldsymbol{\xi} \in B$ and real C for which*

$$(*) \qquad \{z: \boldsymbol{\xi}' z > C\} \cap A = \phi ,$$

one can find $\boldsymbol{\omega}^* \in \Omega$ (the natural parameter space) such that there exists arbitrarily large λ for which $\boldsymbol{\omega}^* + \lambda \boldsymbol{\xi} \in \Omega_1 \subset \Omega$. Suppose, moreover that the set $A_* \subset \mathcal{X}$ is such that $A_* - A$ contains a set F of positive k-dimensional Lebesgue measure (m_k) for which $\boldsymbol{\xi} \in B$ implies the existence of a real C_0 such that $(*)$ holds with $m_k\{\{z: \boldsymbol{\xi}' z > C_0\} \cap F\} > 0$. Then if we define the two size α (with respect to some $\Omega_0 \subset \Omega$) tests

$$\phi_{0,\alpha}(z) = \begin{cases} 0 , & \text{if } z \in A \\ 1 , & \text{otherwise} \end{cases}$$

and

$$\phi_*(z) = \begin{cases} 0 , & \text{if } z \in A_* \\ 1 , & \text{otherwise} , \end{cases}$$

ϕ_* does not dominate $\phi_{0,\alpha}$ when testing $H_0: \boldsymbol{\omega} \in \Omega_0$ vs. $H_1: \boldsymbol{\omega} \in \Omega_1$.

PROOF. See Stein [7].

The essence of this key result is captured in the following geometrical interpretation: Let $\Omega_0 \subset \Omega$ and $\Omega_1 \subset \Omega$ correspond to hypotheses H_0 and H_1, respectively. Consider the closed, convex subset of $\mathcal{X}$ corresponding to the acceptance region of the test $\phi_{0,\alpha}$, say $A_{\phi_{0,\alpha}}$. Now any "hopeful" competitor, say ϕ_*, with acceptance region A_{ϕ_*}, must be such that $A_{\phi_*} - A_{\phi_{0,\alpha}}$ contains a set F of positive k-dimensional Lebesgue measure (lest ϕ_* have size exceeding α). Then there exists a hyperplane which separates F from $A_{\phi_{0,\alpha}}$. Moreover, it can be shown that for the exponential family, as one moves in the natural parameter space orthogonal to, and arbitrarily distant from this hyperplane via a sequence $\{\boldsymbol{\omega}_n, n \geqq 1\}$, that $P_{\boldsymbol{\omega}_n}(F)/P_{\boldsymbol{\omega}_n}(A_{\phi_{0,\alpha}}) \to \infty$. In particular, there is a point $\boldsymbol{\omega}_{n_1}$ (say) such that $P_{\boldsymbol{\omega}_{n_1}}(A_{\phi_*}) > P_{\boldsymbol{\omega}_{n_1}}(A_{\phi_{0,\alpha}})$. But, if $\boldsymbol{\omega}_{n_1} \in \Omega_1$, it immediately follows that ϕ_* does not dominate $\phi_{0,\alpha}$.

Turning to the special case engendered by the separated hypothesis problem we have been considering, we recall that we may take (by sufficiency) $\mathcal{X}$ to be the space of points $(\bar{X}, T_x, \bar{Y}, T_y)$ and Ω the space of points $(m\theta/\sigma_x^2, -1/2\sigma_x^2, n\theta/\sigma_y^2, -1/2\sigma_y^2)$. Observe that the value of T_x for which the hyperplane tangent to $A_{\phi_{0,\alpha,x}}$ is orthogonal to the hyperplane described by $\theta = \varepsilon$ is $k_{0,\alpha}^2 m\varepsilon^2$. We have chosen b_ε in $\phi_{\varepsilon,\alpha}^*$ to be less than this quantity, since if it were not and the acceptance region of a competitor "stuck out" from $A_{\phi_{0,\alpha,x}}$ on a set of positive 4-dimensional Lebesgue measure, then there would exist a separating hyperplane as described in Lemma 3.1. (Note that the presence of the indicator function serves to equate $\phi^*_{a,\alpha}$ and $\phi_{0,\alpha,x}$ when T_x is large.) Then it would follow that $\phi_{a,\alpha}$ does not dominate $\phi_{0,\alpha,x}$ if we were able to exhibit a sequence of parameter points $\{\boldsymbol{\omega}_n,\ n\geqq 1\}$ as indicated in the lemma (see Fig. 1).

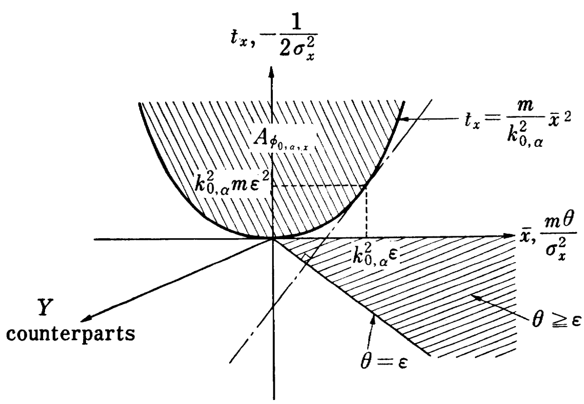

Fig. 1

But, this is easily demonstrable, as any such separating hyperplane in the above setting must be of the form $D_{\boldsymbol{\xi}^*,C}=\{\boldsymbol{\xi}:\ \boldsymbol{\xi}'\boldsymbol{\xi}^*>C\}$, where $\boldsymbol{\xi}^*=(\xi_1^*, \xi_2^*, 0, 0)$, $\xi_2^*<0$ and $|\xi_2^*/\xi_1^*|>1/(2m\varepsilon)$. Now if we choose $\boldsymbol{\omega}^*=(\xi_1^*, \xi_2^*, n\xi_1^*, m\xi_2^*)\in\Omega$, then $\boldsymbol{\omega}_\lambda=\boldsymbol{\omega}^*+\lambda\boldsymbol{\xi}^*\in\Omega_1$ for arbitrarily large $\lambda>0$, where $\Omega_1=\{(\omega_1, \omega_2, \omega_3, \omega_4):\ \omega_1/m\omega_2=\omega_3/n\omega_4,\ \omega_2<0,\ \omega_4<0,\ |\omega_1/\omega_2|\geqq 2m\varepsilon,\ |\omega_3/\omega_4|\geqq 2n\varepsilon\}$. Hence, $\phi^*_{a,\alpha}$ would not dominate $\phi_{0,\alpha,x}$.

We now may prove

THEOREM 3.3. *There does not exist a test of level* $\alpha\in(0,1)$ *which dominates both* $\phi_{0,\alpha,x}$ and $\phi_{0,\alpha,y}$ when testing H_0: $\theta=0$ *vs.* H_1: $|\theta|\geqq\varepsilon>0$.

PROOF. Let ϕ be a test whose non-existence is asserted in the theorem. Then

$$(*)\qquad \sup_{\theta_0} \mathrm{P}\,(\phi=1)\leqq\alpha\ .$$

From Lemma 3.1, it follows that if $T_x>b_{\varepsilon,x}$ and $\phi_{0,\alpha,x}=1$, then $\phi=1$ (except, perhaps, on a set of zero 4-dimensional Lebesgue measure). Here $b_{\varepsilon,x}$ denotes the value of T_x at the point of tangency to $A_{\phi_{0,\alpha,x}}$ as per the discussion following Lemma 3.1. Similarly, if $T_y>b_{\varepsilon,y}$ and $\phi_{0,\alpha,y}=1$, then $\phi=1$ (with the same obvious shift made to the Y sample). But, $P_{0,\sigma_x,\sigma_y}(\phi=1)\geqq P_{0,\sigma_x,\sigma_y}(\{\phi_{0,\alpha,y}=1,\ T_y>b_{\varepsilon,y}\}\cup\{\phi_{0,\alpha,x}=1,\ T_x>b_{\varepsilon,x}\})$, which, by the independence of the X and Y samples, equals

$$P_{0,\sigma_y}(\phi_{0,\alpha,y}=1,\ T_y>b_{\epsilon,y})+P_{0,\sigma_x}(\phi_{0,\alpha,x}=1,\ T_x>b_{\epsilon,x})$$
$$-P_{0,\sigma_y}(\phi_{0,\alpha,y}=1,\ T_y>b_{\epsilon,y})P_{0,\sigma_x}(\phi_{0,\alpha,x}=1,\ T_x>b_{\epsilon,x})\ .$$

Both probabilities involved in this expression $\rightarrow\alpha$ as $\sigma_x\rightarrow\infty$ and $\sigma_y\rightarrow\infty$. (This follows in one case, for example, from observing that $P_{0,\sigma_x}(T_x\leqq b_{\epsilon,x})\rightarrow 0$ as $\sigma_x\rightarrow\infty$. Hence, $P_{0,\sigma_x}(\phi_{0,\alpha,x}=1,\ T_x>b_{\epsilon,x})=P_{0,\sigma_x}(\phi_{0,\alpha,x}=1)-P_{0,\sigma_x}\cdot(\phi_{0,\alpha,x}=1,\ T_x\leqq b_{\epsilon,x})\geqq\alpha-P_{0,\sigma_x}(T_x\leqq b_{\epsilon,x})\rightarrow\alpha$ as $\sigma_x\rightarrow\infty$. But, $P_{0,\sigma_x}(\phi_{0,\alpha,x}=1,\ T_x>b_{\epsilon,x})\leqq\alpha$, giving the result. The other term may be identically dispensed with.)

Hence,

$$\lim_{\sigma_x,\sigma_y\rightarrow\infty} P_{0,\sigma_x,\sigma_y}(\phi=1)\geqq\alpha+\alpha-\alpha^2=2\alpha-\alpha^2>\alpha\ ,$$

for $\alpha\in(0,1)$ contradicting (*) and, thus, proving the theorem.

Remarks. (i) It can be shown that the test proposed in Theorem 2.1 also dominates $\phi_{0,\alpha,x}$ if σ_x^2 is bounded away from zero, the restriction imposed by Cohen and Sackrowitz [4]. Although their test is not invariant in the appropriate sense mentioned earlier, it should be noted that it can be made so.

(ii) The motivation behind the form of the test function $\phi_{a,\alpha}^*$ is quite similar to that found in Cohen and Sackrowitz, the major exception being the introduction of an indicator function over a suitable bounded set of T_x values. Reasons for this "truncation" are essentially those outlined in the discussion following the proof of Lemma 3.1.

(iii) No claim is made that $\phi_{a,\alpha}^*$ is itself admissible. Indeed, the use of $\bar{Y}$ only through its sign (for relative tractability of computations) would intuitively argue for inadmissibility.

4. Appendix

The object of this section is to prove Lemma 2.2 (which requires seven preliminary lemmas) as well as to specify upper bounds for the constants a and c_2 of $\phi_{a,\alpha}^*$. Certain of these preliminary results are given by Cohen and Sackrowitz [4], although it is the existence of, rather than a needed specific form for certain bounds which is given therein.

LEMMA 4.1. *For the test $\phi_{a,\alpha}^*$ of Lemma 2.1, there exist functions $k_{a,\alpha}^-(T_x, \bar{Y}, T_y)$ and $k_{a,\alpha}^+(T_x, \bar{Y}, T_y)$ such that*

$$\begin{aligned} (4.1)\qquad C_a^U(T_x, \bar{Y}, T_y)&=k_{a,\alpha}^-(T_x, \bar{Y}, T_y)(T_x/m)^{1/2}\ , &\text{if } \bar{Y}\leqq 0\\ &=k_{a,\alpha}^+(T_x, \bar{Y}, T_y)(T_x/m)^{1/2}\ , &\text{if } \bar{Y}>0\ , \end{aligned}$$

and

(4.2) $$C^L_a(T_x, \bar{Y}, T_y) = -C^U_a(T_x, -\bar{Y}, T_y) .$$

Moreover, there exist constants k^- *and* k^+ *such that*

(4.3) $$0 < k^+ \leqq k^+_{a,\alpha} \leqq k^-_{a,\alpha} \leqq k^- < 1 .$$

PROOF. Define

$$k^-_{a,\alpha}(T_x, \bar{Y}, T_y) = (m/T_x)^{1/2} F_0^{-1}(1-(\alpha/2) + aQ_x(T_x)(c_1 + s_y/\sqrt{b_\epsilon})^{-1} | T_x) ,$$

and

$$k^+_{a,\alpha}(T_x, \bar{Y}, T_y) = (m/T_x)^{1/2} F_0^{-1}(1-(\alpha/2) - aQ_x(T_x)(c_1 + s_y/\sqrt{b_\epsilon})^{-1} | T_x) .$$

Let $\alpha^* = 1-(\alpha/2) + (\alpha/2) \max\{Q_x(T_x) \colon 0 \leqq T_x \leqq b_\epsilon\}$ and $\alpha^{**} = 1-(\alpha/2) - (\alpha/2) \cdot \max\{Q_x(T_x) \colon 0 \leqq T_x \leqq b_\epsilon\}$, where $Q_x < 1$ by choice of c_2. Then if we define k^- and k^+ via

$$k^- = t_{1-\alpha^*}(m-1)/(m-1+t^2_{1-\alpha^*}(m-1))^{1/2}$$

and

$$k^+ = t_{1-\alpha^{**}}(m-1)/(m-1+t^2_{1-\alpha^{**}}(m-1))^{1/2} ,$$

the result follows directly.

LEMMA 4.2. *There exists a constant* $K_1 > 0$, *independent of* $\theta, \sigma_x, \sigma_y$, *and* a, *such that*

$$k^-_{a,\alpha}(T_x, \bar{Y}, T_y) - k_{0,\alpha} \leqq K_1 Q_x(T_x)(c_1 + s_y/\sqrt{b_\epsilon})^{-1} .$$

PROOF. From (2.2), (4.1) and (4.4), and choosing $a=0$, we obtain

$$\int_{-1}^{k^-_{a,\alpha}} (1-v^2)^{(m-3)/2} dv \Big/ \int_{-1}^{1} (1-v^2)^{(m-3)/2} dv = 1-(\alpha/2) + aQ_x(T_x)(c_1 + s_y/\sqrt{b_\epsilon})^{-1}$$

and

$$\int_{-1}^{k_{0,\alpha}} (1-v^2)^{(m-3)/2} dv \Big/ \int_{-1}^{1} (1-v^2)^{(m-3)/2} dv = 1-\alpha/2 .$$

Subtracting the second ratio from the first and noting that $a<1$ and $k^-_{a,\alpha} \leqq k^- < 1$ from Lemma 4.1, we may obviously choose

$$K_1 = \int_{-1}^{1} (1-v^2)^{(m-3)/2} dv / [1-(k^-)^2]^{(m-3)/2} ,$$

proving the lemma.

LEMMA 4.3. *If* $n \geqq 4$, *there exists a positive constant* K_2, *independent of* θ *and* σ_y, *such that*

$$E_{\sigma_y}\{(c_1 + s_y/\sqrt{b_\epsilon})^{-1}\} / \sigma_y^i E_{\sigma_y}\{(c_1 + s_y/\sqrt{b_\epsilon})^{-2}\} \geqq K_2 , \qquad \textit{for } i=0,1 .$$

Moreover, K_2 *may be taken to be* 1 *if* $i=0$ *and* $\min\{1, b_\varepsilon((n-3)/(n-2))\cdot (c_1+(\mathrm{E}\{\sqrt{V}\}/(b_\varepsilon(n-1))^{1/2})^{-1}\}$ *if* $i=1$, *where* $V\sim\chi^2(n-1)$ *and*

$$\mathrm{E}\{\sqrt{V}\}=\begin{cases}(\pi/2)^{1/2}\prod_{k=1}^{(n-1)/2}(2k-1)\Big/\prod_{k=1}^{(n-3)/2}(2k)\,, & \text{if } n\geqq 5 \text{ and odd}\,,\\ (2/\pi)^{1/2}\prod_{k=1}^{(n-2)/2}(2k)\Big/\prod_{k=1}^{(n-2)/2}(2k-1)\,, & \text{if } n\geqq 4 \text{ and even}\,.\end{cases}$$

PROOF. That there exists a lower bound is shown in Cohen and Sackrowitz [4]. To specify a bound we begin by observing that for $i=0$, the fact that $c_1>(2/\alpha)>2$ assures that $(c_1+s_y/\sqrt{b_\varepsilon})^{-1}>(c_1+s_y/\sqrt{b_\varepsilon})^{-2}$, establishing that we may take $K_2=1$ when $i=0$. For $i=1$, consider two cases:

Case (i). $\sigma_y\leqq 1$. That K_2 may chosen to be 1 is immediate.

Case (ii). $\sigma_y>1$. Multiplying numerator and denominator of the ratio in the lemma by σ_y, then $\sigma_y>1$ implies the ratio is at least

$$b_\varepsilon^{-1}E_{\sigma_y}\{(c_1+s_y/(\sigma_y\sqrt{b_\varepsilon}))^{-1}\}/E_{\sigma_y}\{(s_y^2/\sigma_y^2)^{-1}\}\ .$$

Clearly, the expectation in the denominator is $(n-1)/(n-3)$, as $(n-1)s_y^2/\sigma_y^2\sim\chi^2(n-1)$ and $\mathrm{E}\{V^{-1}\}=(d-2)^{-1}$ if $V\sim\chi^2(d)$ (see, for example, Mood, Graybill, and Boes [6], p. 248). Applying Jensen's inequality to the expectation in the numerator, and observing that $\mathrm{E}\{V^{1/2}\}=\sqrt{2}\,\Gamma(n/2)/\Gamma((n-1)/2)$ if $V\sim\chi^2(n-1)$, as well as that for any positive integer k, $\Gamma(k+(1/2))=(\sqrt{\pi}/2^k)\prod_{j=1}^{k}(2j-1)$ (see Mood, Graybill, and Boes [6], p. 534), we obtain the desired result.

LEMMA 4.4. *If* $\theta/\sigma_y\leqq 1$, *then* $[1-2\Phi(-\theta\sqrt{n}/\sigma_y)]/\theta\geqq K_3/\sigma_y$, *where* $K_3=(2n/\pi)^{1/2}\exp(-n/2)$.

PROOF. See Cohen and Sackrowitz [4].

LEMMA 4.5. *If* $\theta/\sigma_y\geqq 1$, then $[1-2\Phi(-\theta\sqrt{n}/\sigma_y)]\geqq K_4$, *where* $K_4=1-2\Phi(-\sqrt{n})$.

PROOF. See Cohen and Sackrowitz [4].

LEMMA 4.6. *If* $\theta\geqq\varepsilon>0$, *and* $n\geqq 4$, *there exists a positive constant* K_5, *independent of* θ, σ_x, *and* σ_y, *such that*

$$\frac{[1-2\Phi(-\theta\sqrt{n}/\sigma_y)]E_{\sigma_y}\{(c_1+s_y/\sqrt{b_\varepsilon})^{-1}\}}{\Phi(-\theta\sqrt{n}/\sigma_y)E_{\sigma_y}\{(c_1+s_y/\sqrt{b_\varepsilon})^{-2}\}}\geqq K_5\ . \tag{4.4}$$

PROOF. We consider two cases.

Case (i). $\theta/\sigma_y\geqq 1$. Lemmas 4.3 (with $i=0$) and 4.5, and the fact that $\Phi(-\theta\sqrt{n}/\sigma_y)<1/2$ for $\theta>0$ indicates that choosing $K_5=2K_2K_4$ suffices.

Case (ii). $\theta/\sigma_y<1$. Multiplying by θ both numerator and denominator of (4.8), bounding $\Phi(-\theta\sqrt{n}/\sigma_y)$ by 1/2 and θ by ε, and applying Lemma 4.3 (with $i=1$) indicates that we may choose K_5 to be $2\varepsilon K_2K_3$ for this case.

Hence, taking $K_5=\min\{2K_2K_4, 2\varepsilon K_2K_3\}$ completes the proof.

LEMMA 4.7. *Let* $Z\sim N(\sqrt{m}k_{0,\alpha}\theta, \sigma_x^2)$, $0<\gamma<1$, $\sqrt{b_\varepsilon}=\gamma\sqrt{m}k_{0,\alpha}\varepsilon$, $c_2>0$, *and* $Q(z^2)=c_2|z|(\sqrt{b_\varepsilon}-|z|)$. *Then for suitable choice of the constant* c_2, *there exists a constant* $K_6>0$ *independent of* θ, σ_x, *and* σ_y *such that for any positive constant* K *we have*

$$\frac{E_{\theta,\sigma_x}\{Z^{m-1}Q(Z^2)(1-\exp(-2\sqrt{m}k_{0,\alpha}\theta Z/\sigma_x^2))I_{[0,\sqrt{b_\varepsilon}]}(Z)\}}{(\theta/\sigma_x^2)E_{\theta,\sigma_x}\{Z^mQ^2(Z^2)\exp(K\theta ZQ(Z^2)/\sigma_x^2)I_{[0,\sqrt{b_\varepsilon}]}(Z)\}}\geqq K_6\,, \tag{4.5}$$

for all $\theta\geqq\varepsilon>0$.

PROOF. We consider two cases.

Case (i). $2\sqrt{m}k_{0,\alpha}\theta/\sigma_x^2\geqq1$. Denoting the l.h.s. of (4.5) as R and observing the positivity of the integrand in the numerator as well as that $u\leqq e^u$, we get

$$R\geqq\frac{\int_h^{\sqrt{b_\varepsilon}} z^{m-1}Q(z^2)\exp(-(z-\sqrt{m}k_{0,\alpha}\theta)^2/2\sigma_x^2)(1-\exp(-z))dz}{\int_0^{\sqrt{b_\varepsilon}} z^{m-1}Q(z^2)\exp((K+1)\theta zQ(z^2)/\sigma_x^2)\exp(-(z-\sqrt{m}k_{0,\alpha}\theta)^2/2\sigma_x^2)dz}\,, \tag{4.6}$$

for all $h\in(0,\sqrt{b_\varepsilon})$. Constraining c_2 to lie in $(0,4/b_\varepsilon)$ so as to assure that $0\leqq Q(z^2)<1$ for $0\leqq z\leqq\sqrt{b_\varepsilon}$, the definition of $Q(\cdot)$ implies

(4.7)

$$R\geqq\left((h/\sqrt{b_\varepsilon})^{m-3}(1-e^{-h})\int_h^{\sqrt{b_\varepsilon}} z^3(\sqrt{b_\varepsilon}-z)\exp(-(z-\sqrt{m}k_{0,\alpha}\theta)^2/2\sigma_x^2)dz\right)$$
$$\Big/\left((b_\varepsilon)^{3/2}\int_0^{\sqrt{b_\varepsilon}}(\sqrt{b_\varepsilon}-z)\exp(a_1\theta(\sqrt{b_\varepsilon}-z)/\sigma_x^2)\exp(-(z-\sqrt{m}k_{0,\alpha}\theta)^2/2\sigma_x^2)dz\right),$$

where $a_1=(K+1)c_2b_\varepsilon$ may be chosen arbitrarily small by taking c_2 sufficiently close to zero. Multiplying numerator and denominator of (4.7) by $\exp((\sqrt{b_\varepsilon}-\sqrt{m}k_{0,\alpha}\theta)^2/2\sigma_x^2)$, and noting that $\sqrt{b_\varepsilon}+z$ is bounded above and below by $2\sqrt{b_\varepsilon}$ and $\sqrt{b_\varepsilon}+h$, respectively, for $z\in[0,\sqrt{b_\varepsilon}]$ and $z\in[h,\sqrt{b_\varepsilon}]$, respectively we obtain

$$R\geqq\frac{(h/\sqrt{b_\varepsilon})^m(1-\exp(-h))\int_h^{\sqrt{b_\varepsilon}}(\sqrt{b_\varepsilon}-z)\exp(r_1(\sqrt{b_\varepsilon}-z))dz}{\int_0^{\sqrt{b_\varepsilon}}(\sqrt{b_\varepsilon}-z)\exp(r^2(\sqrt{b_\varepsilon}-z))dz}\,, \tag{4.8}$$

where

$$r_1=\frac{[(\sqrt{b_\epsilon}+h)/2]-\sqrt{m}k_{0,\alpha}\theta}{\sigma_x^2}, \qquad r_2=\frac{\sqrt{b_\epsilon}-(1-a_2)\sqrt{m}k_{0,\alpha}\theta}{\sigma_x^2},$$

and $a_2=a_1/\sqrt{m}k_{0,\alpha}$. We note that a_2 may be chosen arbitrarily small by choosing c_2 (and, hence, a_1) sufficiently close to zero. It follows from the definition of $\sqrt{b_\epsilon}$ that both r_1 and r_2 are negative if a_2 is chosen less than $1-\gamma$, $0\leqq h\leqq\sqrt{b_\epsilon}$, and $\theta\geqq\varepsilon$. Following the substitution $u=\sqrt{b_\epsilon}-z$, integration in (4.8) yields

$$\text{(4.9)} \quad R\geqq((h/\sqrt{b_\epsilon})^m(1-e^{-h})(r_2/r_1)^2\{[r_1(\sqrt{b_\epsilon}-h)-1]\exp(r_1(\sqrt{b_\epsilon}-h)+1\}) \\ /((r_2\sqrt{b_\epsilon}-1)\exp(r_2\sqrt{b_\epsilon})+1)\ .$$

By choice of a_2, it easily follows that

$$\text{(4.10)} \qquad (r_2/r_1)^2\geqq(1-\gamma-a_2)^2>0\ ,$$

and the bound is independent of θ, σ_x, and σ_y. It now suffices for case (i) to show that if we define R_1 via

$$\text{(4.11)} \qquad R_1=\frac{[r_1(\sqrt{b_\epsilon}-h)-1]\exp(r_1(\sqrt{b_\epsilon}-h))+1}{(r_2\sqrt{b_\epsilon}-1)\exp(r_2\sqrt{b_\epsilon})+1}\ ,$$

then R_1 is bounded away from zero by a constant for all choices of (r_1, r_2) satisfying $\theta\geqq\varepsilon$. Now by definition of r_1 and r_2 and letting $y=r_1/r_2$, it follows that $dy/d\theta<0$, for all θ, and $\lim_{\theta\to+\infty} y(\theta)=(1-a_2)^{-1}$. Hence,

$$\text{(4.12)} \qquad (1-a_2)r_1<r_2\leqq\frac{[(1-a_2)/\gamma]-1}{[(1/\gamma)-1/2]-h/2b_\epsilon}r_1$$

from which, in conjunction with the positivity of the numerator and denominator of R_1, it follows that

$$\text{(4.13)} \qquad R_1\geqq\frac{(k_2r_1-1)\exp(k_2r_1)+1}{(k_3r_1-1)\exp(k_3r_1)+1}=g(r_1)\ ,$$

where $k_2=\sqrt{b_\epsilon}-h$ and $k_3=(1-a_2)\sqrt{b_\epsilon}$. Clearly, $\lim_{r_1\to-\infty} g(r_1)=1$, $\lim_{r_1\to 0} g(r_1)=(k_2/k_3)^2>0$ and g is both continuous and positive on $(-\infty, \infty)$. Hence, g is bounded away from zero, establishing case (i).

Case (ii). $2\sqrt{m}k_{0,\alpha}\theta/\sigma_x^2<1$. We begin by writing R as

$$\text{(4.14)} \quad R=\frac{2\sqrt{m}k_{0,\alpha}E_{\theta,\sigma_x}\dfrac{\{Z^{m-1}Q(Z^2)(1-\exp(-2\sqrt{m}k_{0,\alpha}\theta Z/\sigma_x^2)I_{[0,\sqrt{b_\epsilon}]}(Z)\}}{2\sqrt{m}k_{0,\alpha}\theta/\sigma_x^2}}{E_{\theta,\sigma_x}\{Z^mQ^2(Z^2)\exp(K\theta ZQ(Z^2)/\sigma_x^2)I_{[0,\sqrt{b_\epsilon}]}(Z)\}}\ .$$

Now letting $l(z)=[1-\exp(-2\sqrt{m}k_{0,\alpha}\theta z/\sigma_x^2)]/(2\sqrt{m}k_{0,\alpha}\theta/\sigma_x^2)$ and following along the lines of the proof of case (i), we easily get

$$R \geqq \frac{2mk_{0,\alpha}}{\sqrt{b_\varepsilon}} \times \frac{\int_h^{\sqrt{b_\varepsilon}} z^{m-1}Q(z^2)\exp(-(z-\sqrt{m}\,k_{0,\alpha}\theta)^2/2\sigma_x^2)l(z)dz}{\int_0^{\sqrt{b_\varepsilon}} z^{m-1}Q(z^2)\exp(K\theta zQ(z^2)/\sigma_x^2)\exp(-(z-\sqrt{m}\,k_{0,\alpha}\theta)^2/2\sigma_x^2)dz}, \tag{4.15}$$

as $Q(z^2)<1$ on $(0, \sqrt{b_\varepsilon})$ by choice of c_2. We observe that the ratio of the two integrals in (4.15) is identical to the ratio in (4.6), except that K and $l(z)$ are replaced therein by $K+1$ and $(1-\exp(-z))$, respectively. The former change surely does not affect the boundedness of R, and it therefore suffices for case (ii) to show that $l(z)$ is bounded away from zero over the region $[h, \sqrt{b_\varepsilon}]$. Letting $d=2\sqrt{m}\,k_{0,\alpha}\theta/\sigma_x^2$, we consider $l_z(d)$ as a function of d for fixed z, with $d \in (0, 1)$ and $z \in [h, \sqrt{b_\varepsilon}]$. It is easy to see that $l_z(1)=1-e^{-z}$, $l_z(d)$ is continuous in d over $(0, 1)$, $\lim_{d\downarrow 0} l_z(d)=z$, $\lim_{d\to\infty} l_z(d)=0$ (extending the domain of $l_z(\cdot)$), $\lim_{d\to-\infty} l_z(d)=+\infty$ (again, extending the domain), and $dl_z(d)/dd<0$ for all d. It follows that $l_z(d)\geqq 1-\exp(-h)$, for all $d \in (0, 1)$ and $z \in [h, \sqrt{b_\varepsilon}]$, proving case (ii).

Note. The specification of a value for the positive constant c_2 may be achieved by observing that it is subject to the constraints that it be both less than $4/b_\varepsilon$ (to assure $Q_x<1$) and less than $[\sqrt{m}\,k_{0,\alpha}(1-\gamma)]/[(\sqrt{m}\,K_1\alpha/2)+1]b_\varepsilon$ (to assure $a_2<1-\gamma$). This implies that

$$c_2<(1-\gamma)/[\sqrt{b_\varepsilon}\,\gamma\varepsilon((\sqrt{m}\,K_1\alpha/2)+1)] \tag{4.16}$$

where K_1 is specified in Lemmas 4.1 and 4.2 via

$$K_1=\int_{-1}^{1}(1-v^2)^{(m-3)/2}dv/[1-(k^-)^2]\ , \tag{4.17}$$

$$k^-=t_{1-\alpha^*}(m-1)/\sqrt{m-1+t_{1-\alpha^*}^2(m-1)}\ , \tag{4.18}$$

and

$$\alpha^*=1-(\alpha/2)+\alpha c_2 b_\varepsilon/8\ . \tag{4.19}$$

It is, moreover, easy to show that k^- is increasing in c_2. Hence, for any $\tau \in (0, 1)$, if we define

$$D(c_2)=\frac{(1-\gamma)}{\sqrt{b_\varepsilon}\,\gamma\varepsilon[(\sqrt{m}\,K_1\alpha/2)+1]}\ , \tag{4.20}$$

then we may choose

$$0<c_2<\min\{4\tau/b_\varepsilon, D(4\tau/b_\varepsilon)\}\ . \tag{4.21}$$

We finally are in a position to prove Lemma 2.2.

LEMMA 2.2. *Let $m\geqq 3$, $n\geqq 4$, and $\alpha\in(0, 1/2)$. Then there exists a positive constant M, not depending θ, σ_x, σ_y, or a^*, such that $|\beta'_{\theta,\sigma_x,\sigma_y}(0)/\beta''_{\theta,\sigma_x,\sigma_y}(a^*)|\geqq M$, for all $a^*\in(0, 1)$ and $(\theta, \sigma_x, \sigma_y)$ for which $|\theta|\geqq\varepsilon$.*

PROOF. We have from (2.2) and (2.4) that

$$\beta_{\theta,\sigma_x,\sigma_y}(a)=1-E_{\theta,\sigma_x,\sigma_y}\{F_{\delta_x}(C^U_a(T_x, \bar{Y}, T_y)|T_x)-F_{\delta_x}(C^L_a(T_x, \bar{Y}, T_y)|T_x)\} .$$

Thus,

$$\beta'_{\theta,\sigma_x,\sigma_y}(a)=-E_{\theta,\sigma_x,\sigma_y}\{f_{\delta_x}(C^U_a|T_x)C^{U'}_a-f_{\delta_x}(C^L_a|T_x)C^{L'}_a\} ,$$

where $f_{\delta_x}(\cdot)$ is given in (2.1). Then

$$\beta''_{\theta,\sigma_x,\sigma_y}(a)=-E_{\theta,\sigma_x,\sigma_y}\left\{\left[f'_{\delta_x}(C^U_a|T_x)\left(\frac{\partial C^U_a}{\partial a}\right)^2+f_{\delta_x}(C^U_a|T_x)\frac{\partial^2 C^U_a}{\partial a^2}\right]\right.$$
$$\left.-\left[f'_{\delta_x}(C^L_a|T_x)\left(\frac{\partial C^L_a}{\partial a}\right)^2+f_{\delta_x}(C^L_a|T_x)\frac{\partial^2 C^L_a}{\partial a^2}\right]\right\} .$$

From (2.1) and the form of $\phi^*_{a,\alpha}$ we have

$$\frac{\partial^2 C^U_a}{\partial a^2}=\frac{(Q_yQ_x)^2}{f_0^2(C^U_a|T_x)}\cdot\frac{m(m-3)C^U_a}{(T_x-m(C^U_a)^2)} ,$$

with an identical relationship obtaining with C^L_a replacing C^U_a, from which it follows that

$$\beta''_{\theta,\sigma_x,\sigma_y}(a)=-\delta_x E_{\theta,\sigma_x,\sigma_y}\left\{\left[\frac{f_{\delta_x}(C^U_a|T_x)}{f_0^2(C^U_a|T_x)}-\frac{f_{\delta_x}(C^L_a|T_x)}{f_0^2(C^L_a|T_x)}\right](Q_yQ_x)^2\right\} .$$

Thus,

$$\begin{aligned}(4.22)\qquad \beta''_{\theta,\sigma_x,\sigma_y}(a)=&(-\theta/\sigma_x^2)K(\theta, \sigma_x)\left[\int_{-1}^{1}(1-v^2)^{(m-3)/2}dv\right]^2\\ &\times\int_0^\infty t_x^{m-2}\exp(-t_x/2\sigma_x^2)Q_x^2(t_x)\\ &\times E_{\theta,\sigma_y}\{[(t_x-m(C^U_a)^2)^{-(m-3)/2}\exp(m\theta C^U_a/\sigma_x^2)\\ &-(t_x-m(C^L_a)^2)^{-(m-3)/2}\exp(m\theta C^L_a/\sigma_x^2)]Q_y^2\}dt_x .\end{aligned}$$

Letting $Q(t_x)=c_2\sqrt{t_x}(\sqrt{b_\varepsilon}-\sqrt{t_x})$, $z=\sqrt{t_x}$, and taking note of (2.7) and (4.22), we have

$$\begin{aligned}(4.23)\quad &|\beta'_{\theta,\sigma_x,\sigma_y}(0)/\beta''_{\theta,\sigma_x,\sigma_y}(a^*)|\\ &=\Big(2\sqrt{2\pi}\sigma_x\exp(m\theta^2k^2_{0,\alpha}/2\sigma_x^2)\\ &\quad\times\left(\sqrt{m}\int_{-1}^{1}(1-v^2)^{(m-3)/2}dv\right)^{-1}E_{\theta,\sigma_x}\{Z^{m-1}Q(Z^2)\\ &\quad\times(1-\exp(-2\sqrt{m}k_{0,\alpha}\theta Z/\sigma_x^2))I_{[0,\sqrt{b_\varepsilon}]}(Z)\end{aligned}$$

$$\times[1-2\Phi(-\theta\sqrt{n}/\sigma_y)]E_{\sigma_y}(c_1+s_y/\sqrt{b_\varepsilon})^{-1}\}\Big)$$
$$\div\Big(\theta/\sigma_x^2\Big|\int_0^{b_\varepsilon} t_x^{m-2}\exp(-t_x/2\sigma_x^2)Q^2(t_x)$$
$$\times E_{\theta,\sigma_y}\{[(t_x-m(C_{a^*}^U)^2)^{-(m-3)/2}\exp(m\theta C_{a^*}^U/\sigma_x^2)$$
$$-(t_x-m(C_{a^*}^L)^2)^{-(m-3)/2}\exp(m\theta C_{a^*}^L/\sigma_x^2)](c_1+s_y/\sqrt{b_\varepsilon})^{-2}\}dt_x\Big|\Big)\ ,$$

where $Z\sim N(\sqrt{m}k_{0,\alpha}\theta,\sigma_x^2)$. From Lemma 4.1, we may define constants $m_1=1-(k^-)^2$ and $m_2=1-(k^+)^2$, both in the interval (0, 1), such that

$$m_1T_x\leqq T_x-m(C_\alpha^U)^2\leqq m_2T_x\ ,\qquad m_1T_x\leqq T_x-m(C_\alpha^L)^2\leqq m_2T_x\ . \tag{4.24}$$

By (4.24) Lemma 4.1, letting $z=\sqrt{t_x}$, noting the independence of $\bar{Y}$ and s_y, and conditioning on the sign of $\bar{Y}$, we have

$$|\beta'_{\theta,\sigma_x,\sigma_y}(0)/\beta''_{\theta,\sigma_x,\sigma_y}(a^*)|\geqq A\Big/\sum_{i=1}^4 B_i\ , \tag{4.25}$$

where

$$A=m_1^{(m-3)/2}[1-2\Phi(-\theta\sqrt{n}/\sigma_y)]\Big(\sqrt{m}\int_{-1}^1(1-v^2)^{(m-3)/2}dv\Big)^{-1}$$
$$\times E_{\sigma_y}\{(c_1+s_y/\sqrt{b_\varepsilon})^{-1}\}E_{\theta,\sigma_x}\{Z^{m-1}(Q(Z^2)$$
$$\times(1-\exp(-2\sqrt{m}k_{0,\alpha}\theta Z/\sigma_x^2))I_{[0,\sqrt{b_\varepsilon}]}(Z)\}\ ,$$
$$B_i=(\theta/\sigma_x^2)u_i(\theta,\sigma_y)E_{\theta,\sigma_x}\{Z^mQ^2(Z^2)\exp(-k_{0,\alpha}\sqrt{m}\theta Z/\sigma_x^2)$$
$$\times E_{\theta,\sigma_y}\{\exp(\sqrt{m}\theta k_iZ/\sigma_x^2)(c_1+s_y/\sqrt{b_\varepsilon})^{-2}|Z\}I_{[0,\sqrt{b_\varepsilon}]}(Z)\}\ ,$$
$$u_1(\theta,\sigma_y)=u_4(\theta,\sigma_y)=\Phi(-\theta\sqrt{n}/\sigma_y)\ ,$$
$$u_2(\theta,\sigma_y)=u_3(\theta,\sigma_y)=1-\Phi(-\theta\sqrt{n}/\sigma_y)\ ,$$
$$k_1=k_{a^*,\alpha}^-\ ,\quad k_2=-k_{a^*,\alpha}^-\ ,\quad k_3=k_{a^*,\alpha}^+\ ,\qquad\text{and}\qquad k_4=-k_{a^*,\alpha}^+\ ,$$

and $Z-N(\sqrt{m}k_{0,\alpha}\theta,\sigma_x^2)$.

Thus, it suffices to show $A/B_i\geqq M_i>0$, $i=1,2,3,4$, for $|\theta|\geqq\varepsilon$ and where M_i is independent of θ, σ_x, σ_y, and a^*. Moreover, we need only treat the case of positive θ, since (2.7), (4.22), Lemma 4.1, the definition of $Q_y(\bar{Y},T_y)$, the eveness of $K(\theta,\sigma_x)$ as a function of θ, and the fact that the distribution of $\bar{Y}$ under $-\theta$ is identical to the distribution of $-\bar{Y}$ under θ assure that $\beta'_{\theta,\sigma_x,\sigma_y}(0)=\beta'_{-\theta,\sigma_x,\sigma_y}(0)$ and $\beta''_{\theta,\sigma_x,\sigma_y}(a^*)=\beta''_{-\theta,\sigma_x,\sigma_y}(a^*)$.

Term 1. A/B_1. From Lemma 4.2, the fact that $c_1>2/\alpha$, the independence of the X and Y samples and Lemmas 4.6 and 4.7, we have that

$$A/B_1\geqq K_5K_6K_7=M_1\qquad\text{(say)}\ , \tag{4.26}$$

where $K_7=m_1^{(m-3)/2}m^{-1/2}\left(\int_{-1}^{1}(1-v^2)^{(m-3)/2}dv\right)^{-1}$.

Term 2. A/B_2. From (4.25) it is clear that A/B_2 differs from A/B_1 only in that $1-\Phi(-\theta\sqrt{n}/\sigma_y)$ replaces $\Phi(-\theta\sqrt{n}/\sigma_y)$ and $-k^-_{a^*,\alpha}$ replaces $k^-_{a^*,\alpha}$ in the interior expectation. Observing that Lemma 4.6 holds with K_5 halved under the former change, while the latter alteration serves only to increase A/B_1, we obtain $M_2=M_1/2$.

Term 3. A/B_3. Noting that $k^+_{a^*,\alpha}\leqq k^-_{a^*,\alpha}$, it is apparent that we may choose $M_3=M_1/2$.

Term 4. A/B_4. Here, we may choose $M_4=M_1$. Thus,

$$|\beta'_{\theta,\sigma_x,\sigma_y}(0)/\beta''_{\theta,\sigma_x,\sigma_y}(a^*)|\geqq[(1/M_1)+(2/M_1)+(2/M_1)+(1/M_1)]^{-1} = M_1/6=M\,, \tag{4.27}$$

completing the proof.

Note. We may give an interval within which we can choose the constant a of Theorem 2.1, by observing that from (2.8) and (4.27) we may specify that $a\in(0, M_1/3)$. Reference to the proofs of Lemmas 4.2 and 4.7 indicates, after some computation, that a may be chosen in $(0, K_5K_7K_8/3)$, where $K_8=\min\{K_8', K_8''\}$,

$$K_8'=(h/\sqrt{b_\epsilon})^m(1-\exp(-h))[1-\gamma-((\sqrt{m}K_1a/2)+1)c_2b_\epsilon)/\sqrt{m}k_{0,\alpha}]\,,$$

and

$$K_8''=(2mk_{0,\alpha}/\sqrt{b_\epsilon})(h/\sqrt{b_\epsilon})^m(1-\exp(-h))(1-\gamma-(K_1ac_2b_\epsilon)/2k_{0,\alpha})^2\,, \quad \text{for any } h\in(0, a_2\sqrt{b_\epsilon})\,.$$

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