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ON A RESULT OF ROY AND GNANADESIKAN CONCERNING MULTIVARIATE VARIANCE COMPONENTS

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Summary

Roy and Gnanadesikan [5] showed that inference for a general multivariate variance components model may be carried out using the standard multivariate F distribution under certain conditions. It is shown in this note that the theory of zonal polynomials, and their extension by the author to invariant polynomials in two matrix arguments, provide a concise approach to the derivation of these conditions. Relevant distributions are also derived for the general case.

1. Introduction

The multivariate Model II with a k-way classification has been formulated by Roy and Gnanadesikan [5] as

$$(1) \qquad X=A\Xi+\varepsilon=[A_1, A_2, \cdots, A_k][\xi_1', \cdots, \xi_k']'+\varepsilon ,$$

where X is an $N\times p$ observable matrix, A is the $N\times M$ design matrix of rank $r\leqq M\leqq N$, the A_i are $N\times m_i$ $(i=1, 2, \cdots, k, \sum_{i=1}^{k} m_i=M)$, Ξ is $M\times p$, and where

(i) ξ_i is an $m_i\times p$ matrix whose rows are a random sample from the p-variate nonsingular normal population $N(\boldsymbol{\mu}_i, \Sigma_i)$, $i=1, 2, \cdots, k$;

(ii) ε is an $N\times p$ matrix whose rows are a random sample from the p-variate nonsingular normal population $N(\mathbf{0}, \Sigma_0)$; $p\leqq N-r$. The ξ_i and ε are mutually independent.

To present a precise treatment of problems of estimation and hypothesis testing associated with the variance components ξ_i, Roy and Gnanadesikan imposed the restriction

$$(2) \qquad \Sigma_i=\sigma_i^2\Sigma_0 ,$$

where the σ_i^2 are positive scalars $(i=1, 2, \cdots, k)$. They then introduced the statistics appropriate for testing the hypotheses H_{0i} of equality of the rows of ξ_i when (1) is interpreted in the Model I sense, i.e.

$$H_{0i}: C_i\Xi = 0$$

where C_i is $q_i \times M$ ($q_i = m_i - 1$), and is partitioned like A in the form

$$C_i = [0, 0, \cdots, \tilde{C}_i, 0, \cdots, 0] ,$$

$$(3) \qquad \tilde{C}_i = \begin{bmatrix} 1 & 0 & 0 \cdots 0 & -1 \\ 0 & 1 & 0 \cdots 0 & -1 \\ \vdots & \vdots & \vdots \quad\; \vdots & \vdots \\ 0 & 0 & 0 \cdots 1 & -1 \end{bmatrix}$$

$\tilde{C}_i$ being $q_i \times m_i$. Let A_I be an $N \times r$ matrix consisting of a selection of r linearly independent columns of A, and let C_{iI} be the $q_i \times r$ matrix containing the corresponding columns of C_i. Then the sum of squares and products matrix for the hypothesis H_{0i} (assumed testable) is $Y_i'Y_i$, where

$$(4) \qquad Y_i = B_i^{-1/2} C_{iI}(A_I'A_I)^{-1}A_I'X \qquad B_i = C_{iI}(A_I'A_I)^{-1}C_{iI}'$$

are $q_i \times p$ and $q_i \times q_i$ matrices respectively ($i = 1, 2, \cdots, k$). As usual, the error matrix is

$$S = X'(I_N - A_I(A_I'A_I)^{-1}A_I')X ,$$

where I_N denotes the $N \times N$ unit matrix. Under the Model II (1), S has the distribution $W_p(n, \Sigma_0)$, $n = N - r$, and is independent of the Y_i's.

It was shown by Roy and Gnanadesikan that, provided the condition (2) holds, and provided also that B_i is of the form

$$(5) \qquad B_i = \nu_i^{-1}(I_{q_i} + J) ,$$

where ν_i is a scalar and J is the $q_i \times q_i$ matrix of ones, $\lambda_i^{-1}Y_i'Y_i$ has the p-variate central Wishart distribution $W_p(q_i, \Sigma_0)$ with q_i degrees of freedom and covariance matrix Σ_0, where

$$\lambda_i = \nu_i \sigma_i^2 + 1$$

($i = 1, 2, \cdots, k$). If $q_i < p$ the distribution is pseudo-Wishart. Further, the $Y_i'Y_i$ are mutually independent if

$$(6) \qquad C_{iI}(A_I'A_I)^{-1}C_{jI}' = 0 , \qquad i \neq j = 1, 2, \cdots, k .$$

From (5),

$$\nu_i = 2(m_i - 1)/\mathrm{trace}\,(B_i) .$$

An alternative expression given by Roy and Gnanadesikan ([5], p. 333) follows from equation (12) below. The conditions (5) and (6) are satisfied in particular by the multivariate analogues of the usual univariate complete block designs. Their fulfilment enables inferences to be made on the σ_i^2 using the standard distribution theory for the latent roots of the multivariate F matrix.

In Section 2 we present expansions for the joint distribution of the roots of $Y_i S^{-1} Y_i'$ when $q_i \leqq p$ in the general case, and indicate the corresponding result for $q_i \geqq p$. The conditions (2), (5) and (6) of Roy and Gnanadesikan are shown to follow quite straightforwardly. Chakravorti [1] has derived the distribution of $(Y_1'Y_1 + Y_2'Y_2)S^{-1}$ and its trace under the latter conditions.

Approximate confidence bounds for measures of dispersion associated with random effects in univariate and multivariate mixed models were derived by Roy and Cobb [4], both for the general normal case, and for possibly nonnormal situations.

2. Some distribution theory, and derivation of the conditions

The subscript i will be omitted throughout this section for convenience. Assuming that H_0 is testable, it follows from (1) and (4) that

$$Y = V + G\xi ,$$

where

$$V = B^{-1/2} C_I (A_I' A_I)^{-1} A_I' \varepsilon , \qquad G = B^{-1/2} \tilde{C}$$

are $q \times p$ and $q \times m$ matrices respectively, and the rows of V are independent $N(0, \Sigma_0)$. Since the ξ's are independent, condition (6) for independence of the $Y'Y$'s readily follows.

If $q \leqq p$ then, conditional upon ξ, the latent roots $f_1 \geqq f_2 \geqq \cdots \geqq f_q \geqq 0$ of the $q \times q$ matrix $F = YS^{-1}Y'$ have the multivariate noncentral F distribution, derivable from James [3] equation (72),

$$\operatorname{etr}\left(-\frac{1}{2}\Omega\right) {}_1F_1^{(q)}\left(\frac{1}{2}(q+n); \frac{1}{2}p; \frac{1}{2}\Omega, (I_q + F^{-1})^{-1}\right) \psi(F) ,$$

where

$$(7) \qquad \psi(F) = \left[\Gamma_q\left(\frac{1}{2}(q+n)\right)\pi^{q^2/2} \Big/ \Gamma_q\left(\frac{1}{2}p\right)\Gamma_q\left(\frac{1}{2}(q+n-p)\right)\Gamma_q\left(\frac{1}{2}q\right)\right] \cdot |F|^{(p-q-1)/2} |I_q + F|^{-(q+n)/2} \prod_{u<v} (f_u - f_v)$$

is the corresponding null distribution. Here ${}_1F_1^{(q)}$ denotes a hypergeometric function of two matrix arguments, Γ_q is the multivariate gamma function, and the noncentrality matrix in the present situation is

$$\Omega = G\eta\eta'G' , \qquad \eta = \xi \Sigma_0^{-1/2} .$$

From (3), the rows of the $m\times p$ matrix η may be regarded as a random sample from $N(\mathbf{0}, \Psi)$, where

$$\Psi=\Sigma_0^{-1/2}\Sigma\Sigma_0^{-1/2} .$$

Hence, in order to derive the unconditional distribution of F, it is necessary to evaluate terms of the form

$$(8)\qquad (2\pi)^{-mp/2}|\Psi|^{-m/2}\int_\eta \mathrm{etr}\left(-\frac{1}{2}\Psi^{-1}\eta'\eta-\frac{1}{2}\eta'G'G\eta\right)C_\kappa\left(\frac{1}{2}\eta'G'G\eta\right)d\eta ,$$

where $\mathrm{etr}(\cdot)=\exp(\mathrm{trace}(\cdot))$, and $C_\kappa(\cdot)$ is the zonal polynomial corresponding to the ordered partition $\kappa=[k_1, k_2,\cdots]$ of k into not more than q parts (James, [3]). For a scalar α, define

$$\Delta=\alpha^{-1}I_p-\Psi^{-1} ,\qquad \Theta=\alpha^{-1}I_m+G'G .$$

Then (8) may be written

$$(9)\qquad (2\pi)^{-mp/2}|\Psi|^{-m/2}\int_\eta \mathrm{etr}\left(-\frac{1}{2}\Theta\eta\eta'\right)\mathrm{etr}\left(\frac{1}{2}\Delta\eta'\eta\right)C_\kappa\left(\frac{1}{2}G'G\eta\eta'\right)d\eta .$$

We now transform to $\zeta=\eta H$, where H is an arbitrary orthogonal $p\times p$ matrix. Integration with respect to the invariant Haar measure (dH) over the orthogonal group $O(p)$ leaves the value of (9) unchanged, and by James [3] equations (13) and (23) we obtain

$$(2\pi)^{-mp/2}|\Psi|^{-m/2}\int_\zeta \mathrm{etr}\left(-\frac{1}{2}\Theta\zeta\zeta'\right){}_0F_0^{(p)}\left(\Delta, \frac{1}{2}\zeta\zeta'\right)C_\kappa\left(\frac{1}{2}G'G\zeta\zeta'\right)d\zeta .$$

Defining W to have the Wishart distribution $W_m(p, \Theta^{-1})$, the evaluation of (8) thus reduces to calculating terms

$$E_W\left\{C_\kappa\left(\frac{1}{2}G'GW\right)C_\lambda\left(\frac{1}{2}W\right)\right\}=\sum_{\phi\in\kappa\cdot\lambda}\left(\frac{1}{2}p\right)_\phi\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(\Theta^{-1}G'G, \Theta^{-1})$$

by Davis [2] equation (2.6). Here λ and ϕ denote partitions of l ($l=0, 1, 2,\cdots$) and $k+l$, respectively; $(p/2)_\phi$ is a multivariate hypergeometric coefficient; and $\phi\in\kappa\cdot\lambda$ means that the irreducible representation of the real linear group of nonsingular $m\times m$ matrices indexed by 2ϕ occurs in the decomposition of the Kronecker product of the representations indexed by 2κ and 2λ. $C_\phi^{\kappa,\lambda}$ is an invariant polynomial with two matrix arguments, and $\theta_\phi^{\kappa,\lambda}=C_\phi^{\kappa,\lambda}(I_m, I_m)/C_\phi(I_m)$. The joint distribution of the roots of F in the general case may now be written

$$(10)\qquad |\Psi|^{-m/2}|\Theta|^{-p/2}\psi(F)\sum_{k=0}^\infty\sum_\kappa\frac{((q+n)/2)_\kappa C_\kappa((I_q+F^{-1})^{-1})}{k!(p/2)_\kappa C_\kappa(I_q)}$$
$$\cdot\sum_{l=0}^\infty\sum_\lambda\frac{C_\lambda(\Delta)}{l!C_\lambda(I_p)}\sum_{\phi\in\kappa\cdot\lambda}\left(\frac{1}{2}p\right)_\phi\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(\Theta^{-1}G'G, \Theta^{-1}) .$$

If a value $\alpha=\sigma^2$ can be chosen such that $\Delta=0$, whence $\Psi=\sigma^2 I_p$ and (2) is satisfied, then only terms with $l=0$ are retained in (10). Write

$$\Phi=I_q+\sigma^2 GG' .$$

Then the distribution reduces in this case to

$$(11)\qquad |\Phi|^{-p/2}\psi(F)\,{}_1F_0^{(q)}\left(\frac{1}{2}(q+n);\,(I_q+F^{-1})^{-1},\,\sigma^2\Phi^{-1}GG'\right)$$
$$=K|\Phi|^{-p/2}|F|^{(p-q-1)/2}\prod_{u<v}(f_u-f_v)$$
$$\cdot\int_{O(q)}|I_q+F\mathcal{H}\Phi^{-1}\mathcal{H}'|^{-(q+n)/2}(d\mathcal{H})\ ,$$

where K is the multiplicative constant in (7). Equation (11) thus provides the distribution of the roots when (2) holds, but not (5). Clearly, $\lambda^{-1}F$ will have the standard q-variate F distribution (7) provided that $\Phi=\lambda I_q$; that is, if

$$(12)\qquad GG'=B^{-1/2}\tilde{C}\tilde{C}'B^{-1/2}=\nu I_q\ ,$$

where $\lambda=\nu\sigma^2+1$. Condition (5) now follows using (3).

For $q\geqq p$, the starting point is the noncentral latent roots distribution of Constantine (James [3], equation (73)). The evaluation of (8) remains unchanged, and a form corresponding to (11) is obtained with the $p\times p$ matrix $F=Y'YS^{-1}$ bordered by zeros in the integral to form a $q\times q$ matrix.

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REFERENCES

[1] Chakravorti, S. (1968). On the analysis of variance test in multivariate variance components model, *Calcutta Statist. Ass. Bull.*, **17**, 57–78.

[2] Davis, A. W. (1979). Invariant polynomials with two matrix arguments extending the zonal polynomials: applications to multivariate distribution theory, *Ann. Inst. Statist. Math.*, **31**, A, 465–485.

[3] James, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples, *Ann. Math. Statist.*, **35**, 475–501.

[4] Roy, S. N. and Cobb, W. (1960). Mixed model variance analysis with normal error and possibly non-normal other random effects, I. The univariate case, II. The multivariate case, *Ann. Math. Statist.*, **31**, 939–957, 958–968.

[5] Roy, S. N. and Gnanadesikan, R. (1959). Some contributions to ANOVA in one or more dimensions: II, *Ann. Math. Statist.*, **30**, 318–339.