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PROPER BAYES MINIMAX ESTIMATORS FOR A MULTIVARIATE NORMAL MEAN WITH UNKNOWN COMMON VARIANCE UNDER A CONVEX LOSS FUNCTION

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Summary

Let $X \sim N_p(\mu, \sigma^2 I_p)$ and let $s/\sigma^2 \sim \chi_n^2$, independent of X, where μ and σ^2 are unknown. This paper considers the estimation of μ (by δ) relative to a convex loss function given by $(\delta-\mu)'[(1-\alpha)I_p/\sigma^2+\alpha Q](\delta-\mu)/[(1-\alpha)p/\sigma^2+\alpha\,\mathrm{tr}\,(Q)]$, where Q is a known $p\times p$ diagonal matrix and $0\leqq \alpha\leqq 1$. Two classes of minimax estimators are obtained for μ when $p\geqq 3$; the first is a new result and the second is a generalization of a result of Strawderman (1973, *Ann. Statist.*, 1, 1189–1194). A proper Bayes estimator is also obtained which is shown to satisfy the conditions of the second class of minimax estimators. The paper concludes by discussing the estimation of μ relative to another convex loss function.

1. Introduction

Let $X=(X_1,\cdots,X_p)'$ be a single observation vector from a p-variate ($p\geqq 3$) normal distribution $N_p(\mu, \sigma^2 I_p)$ with mean μ and covariance matrix $\sigma^2 I_p$, where $\sigma^2>0$ is unknown. Assume that $s/\sigma^2\sim\chi_n^2$ is available and is independent of X. This paper will consider the problem of estimating μ by $\delta=\delta(X, s)$. The estimation problem has received considerable attention since Stein [5] showed that the maximum likelihood estimator, X, is inadmissible, even though it is minimax, relative to the loss function

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(1.1) $$L_1(\boldsymbol{\delta};\boldsymbol{\mu},\sigma^2)=(\boldsymbol{\delta}-\boldsymbol{\mu})'(\boldsymbol{\delta}-\boldsymbol{\mu})/(p\sigma^2)\ .$$

To improve upon the maximum likelihood estimator, various authors have looked into a Stein-type estimator for an answer. Successful results have been reported. Among others, we cite Lin and Tsai [4] for a class of generalized Bayes minimax estimators and Strawderman [7] for a family of proper Bayes minimax estimators. The loss function given by (1.1) is a squared error loss weighted by the reciprocal of the common variance. An advantage of using (1.1) is that it reduces the risk function to a simpler form by a scale-invariance technique as is done by Lin and Tsai [4]. More recently, considerable attention is given to the following arbitrary quadratic loss function

(1.2) $$L_2(\boldsymbol{\delta};\boldsymbol{\mu},\sigma^2)=(\boldsymbol{\delta}-\boldsymbol{\mu})'Q(\boldsymbol{\delta}-\boldsymbol{\mu})/[\sigma^2\operatorname{tr}(Q)]\ ,$$

where Q is a known $p\times p$ diagonal matrix with diagonal elements $q_i>0$, $i=1,\cdots,p$. See, e.g., Berger *et al.* [1] and the references contained therein for the case of unknown covariace matrix Σ.

In this paper, we will consider the estimation of $\boldsymbol{\mu}$ relative to a new quadratic loss function $L(\boldsymbol{\delta};\boldsymbol{\mu},\sigma^2)$, which is a convex combination of $L_1(\boldsymbol{\delta};\boldsymbol{\mu},\sigma^2)$ and $L_2(\boldsymbol{\delta};\boldsymbol{\mu},\sigma^2)$, i.e., for $0\leqq\alpha\leqq1$,

(1.3) $$L(\boldsymbol{\delta};\boldsymbol{\mu},\sigma^2)=(\boldsymbol{\delta}-\boldsymbol{\mu})'[(1-\alpha)I_p/\sigma^2+\alpha Q](\boldsymbol{\delta}-\boldsymbol{\mu})/C$$

where $C=(1-\alpha)p+\alpha\sigma^2\operatorname{tr}(Q)$. More specifically, in Section 2, a class of minimax estimators of the form

(1.4) $$\boldsymbol{\delta}(\boldsymbol{X},w)=[I_p-r(\boldsymbol{X},w)\|\boldsymbol{X}\|_w^{-2}w^{-1}Q^{-1}]\boldsymbol{X}$$

will be obtained, where $\|\boldsymbol{X}\|_w^2=\boldsymbol{X}'Q^{-1}\boldsymbol{X}/w^2$ with $w=s/(n-2)$ and where $r(\boldsymbol{X},w)$ satisfies certain mild conditions. In Section 3, another class of minimax estimators will be obtained. A typical estimator in this class is given componentwise by

(1.5) $$d_i(F,s)=\left[1-\frac{r_i(F,s)}{Fq_i}\right]X_i\ ,\qquad i=1,\cdots,p\ ,$$

where $F=\boldsymbol{X}'Q^{-1}\boldsymbol{X}/s$ and $r_i(F,s)$ satisfies certain conditions. It is noted that the two classes of minimax estimators are not mutually exclusive nor is one a subclass of the other. The primary difference lies in the shrinkage factor. More specifically, the estimator (1.5) is more general in that the $r_i(F,s)$ function is allowed to depend on the ith component; on the other hand, it is slightly less general in some sense since the same function depends on $\boldsymbol{X}$ and s through $F=\boldsymbol{X}'Q^{-1}\boldsymbol{X}/s$ and s. In Section 4, we will produce a family of proper Bayes estimators which will be shown to satisfy the minimaxity conditions of Section 3. Finally, in Section 5, another loss function, which is also a convex combination of $L_1(\boldsymbol{\delta}; \boldsymbol{\mu}, \sigma^2)$ and $L_2(\boldsymbol{\delta}; \boldsymbol{\mu}, \sigma^2)$, will be discussed.

2. A class of minimax estimators

For an estimator $\boldsymbol{\delta}(\boldsymbol{X}, w)$ let $R(\boldsymbol{\delta}; \boldsymbol{\mu}, \sigma^2) = \mathrm{E}_{w,\boldsymbol{X}} L(\boldsymbol{\delta}; \boldsymbol{\mu}, \sigma^2)$ denote the risk function. It is easily verified that the maximum likelihood estimator, $\boldsymbol{X}$, is minimax relative to the loss function given by (1.3) with constant risk equal to 1. Thus an estimator $\boldsymbol{\delta}(\boldsymbol{X}, w)$ will be minimax if and only if $R(\boldsymbol{\delta}; \boldsymbol{\mu}, \sigma^2) \leqq 1$ for all $\boldsymbol{\mu}$ and σ^2. In this section, the estimator $\boldsymbol{\delta}(\boldsymbol{X}, w)$ given by (1.4) will be shown to be minimax by proving that $R(\boldsymbol{X}; \boldsymbol{\mu}, \sigma^2) - R(\boldsymbol{\delta}; \boldsymbol{\mu}, \sigma^2) \geqq 0$ for all $\boldsymbol{\mu}$ and σ^2. The following lemmas are useful in evaluating the difference in risks; they are stated here without proof. Assume all expectations exist and are finite.

LEMMA 2.1 (Stein [6]). *Let $Y \sim N(0, 1)$ and let g be an absolutely continuous function, $g\colon R \to R$ such that $g(y) \exp(-y^2/2) \to 0$ as $y \to \pm\infty$. Then*

$$\mathrm{E}_Y [g'(Y)] = \mathrm{E}_Y [Y g(Y)] .$$

LEMMA 2.2 (Efron and Morris [2]). *Let $U \sim \chi^2_n$ and let g be an absolutely continuous function, $g\colon R^+ \to R^+$ such that $g(u) u^{n/2} \exp(-u/2) \to 0$ as $u \to 0^+$ or as $u \to \infty$. Then*

$$\mathrm{E}_U [U g(U)] = n\, \mathrm{E}_U [g(U)] + 2\, \mathrm{E}_U [U g'(U)] .$$

COROLLARY 2.1. *Let U and g be as defined in Lemma 2.2. Let $Z = cU/(n-2)$, $c > 0$, and $h(Z) = g[(n-2)Z/c]$. Then*

$$\mathrm{E}_Z [(n-2) Z h(Z)/c] = n\, \mathrm{E}_Z [h(Z)] + 2\, \mathrm{E}_Z [Z h'(Z)] .$$

LEMMA 2.3 (Lehmann [3]). *Let S be any random variable, and let $p_1(S)$ and $p_2(S)$ map the real line into itself. If $p_1(S)$ and $p_2(S)$ are either both nonincreasing in S or both nondecreasing in S, then*

$$\mathrm{E}_S [p_1(S) p_2(S)] \geqq \mathrm{E}_S [p_1(S)]\, \mathrm{E}_S [p_2(S)] .$$

The above lemmas have been frequently used to establish the minimaxity of an estimator for a multivariate normal mean; they are included here for ease of reference.

The following theorem will prove the minimaxity of $\boldsymbol{\delta}(\boldsymbol{X}, w)$ given by (1.4). In the theorem we will use $\mathrm{tr}(A)$ and $\mathrm{ch}_{\max}(A)$ (or $\mathrm{ch}_{\min}(A)$) to denote the trace and the maximum (or minimum) characteristic root of a square matrix A.

THEOREM 2.1. *Let $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \sigma^2 I_p)$ and $s/\sigma^2 \sim \chi^2_n$, independent of $\boldsymbol{X}$. Then, relative to the convex loss (1.3), the estimator*

$$\boldsymbol{\delta}(\boldsymbol{X}, w)=[I_p-r(\boldsymbol{X}, w)\|\boldsymbol{X}\|_w^{-2}w^{-1}Q^{-1}]\boldsymbol{X} \tag{2.1}$$

is minimax for $\boldsymbol{\mu}$, *where* $\|\boldsymbol{X}\|_w^2=\boldsymbol{X}'Q^{-1}\boldsymbol{X}/w^2$ *and* $w=s/(n-2)$, *provided that the following conditions hold*:

(i) $0\leqq r(\boldsymbol{X}, w)\leqq 2[(n-2)/(n+2)][\mathrm{tr}\,(Q^{-1})/\mathrm{ch}_{\max}(Q^{-1})-2]$ *with* $\mathrm{tr}\,(Q^{-1})\geqq 2\,\mathrm{ch}_{\max}(Q^{-1})$,

(ii) $r(\boldsymbol{X}, w)$ *is nondecreasing in* $|X_i|$ *for* $i=1,\cdots,p$, *and*

(iii) $r(\boldsymbol{X}, w)$ *is nonincreasing in* w.

PROOF. Let $\Delta=R(\boldsymbol{X};\boldsymbol{\mu},\sigma^2)-R(\boldsymbol{\delta};\boldsymbol{\mu},\sigma^2)$. Then

$$C\Delta=2[(1-\alpha)\Delta_1+\alpha\Delta_2] \tag{2.2}$$

where

$$\Delta_1=(2\sigma^2)^{-1}\,\mathrm{E}_{w,X}[(\boldsymbol{X}-\boldsymbol{\mu})'(\boldsymbol{X}-\boldsymbol{\mu})-(\boldsymbol{\delta}-\boldsymbol{\mu})'(\boldsymbol{\delta}-\boldsymbol{\mu})]$$

and

$$\Delta_2=(1/2)\,\mathrm{E}_{w,X}[(\boldsymbol{X}-\boldsymbol{\mu})'Q(\boldsymbol{X}-\boldsymbol{\mu})-(\boldsymbol{\delta}-\boldsymbol{\mu})'Q(\boldsymbol{\delta}-\boldsymbol{\mu})]\,.$$

In the following, appropriate lower bounds for Δ_1 and Δ_2 will be obtained. These bounds, together with Assumption (i), will imply that $C\Delta\geqq 0$ for all $\boldsymbol{\mu}$ and σ^2, establishing the minimaxity of the estimator given by (2.1). Write $r=r(\boldsymbol{X}, w)$. Then

$$\begin{aligned}\sigma^2\Delta_1&=\mathrm{E}_{w,X}\left[\frac{r}{\|\boldsymbol{X}\|_w^2 w}(\boldsymbol{X}-\boldsymbol{\mu})'Q^{-1}\boldsymbol{X}-\frac{r^2}{2\|\boldsymbol{X}\|_w^4 w^2}\boldsymbol{X}'Q^{-2}\boldsymbol{X}\right]\\&\geqq \mathrm{E}_{w,X}\left[\frac{r}{\|\boldsymbol{X}\|_w^2 w}(\boldsymbol{X}-\boldsymbol{\mu})'Q^{-1}\boldsymbol{X}-\frac{r^2}{2\|\boldsymbol{X}\|_w^2}\mathrm{ch}_{\max}(Q^{-1})\right]\\&=\Delta_{11}+\Delta_{12}\,,\qquad \text{say}\,,\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}\Delta_{11}&=\mathrm{E}_{w,X}\left[\frac{r}{\|\boldsymbol{X}\|_w^2 w}(\boldsymbol{X}-\boldsymbol{\mu})'Q^{-1}\boldsymbol{X}\right]\\&=\mathrm{E}_{w,X}\left[\frac{\sigma}{w}\sum_{i=1}^p\frac{1}{q_i}(r\|\boldsymbol{X}\|_w^{-2}X_i)\left(\frac{X_i-\mu_i}{\sigma}\right)\right]\\&=\mathrm{E}_{w,X}\left[\frac{\sigma^2}{w}\sum_{i=1}^p\frac{1}{q_i}\frac{\partial}{\partial X_i}(r\|\boldsymbol{X}\|_w^{-2}X_i)\right]\\&=\mathrm{E}_{w,X}\left[\frac{\sigma^2}{w}\sum_{i=1}^p\frac{1}{q_i}\left(\frac{r}{\|\boldsymbol{X}\|_w^2}-\frac{2rX_i^2}{\|\boldsymbol{X}\|_w^4 w^2 q_i}+\frac{X_i}{\|\boldsymbol{X}\|_w^2}\frac{\partial r}{\partial X_i}\right)\right]\\&\geqq \mathrm{E}_{w,X}\left\{\frac{r\sigma^2}{\|\boldsymbol{X}\|_w^2 w}[\mathrm{tr}\,(Q^{-1})-2\,\mathrm{ch}_{\max}(Q^{-1})]\right\}\,.\end{aligned} \tag{2.4}$$

The third equality in (2.4) is obtained by an application of Lemma 2.1 where, for each $i=1,\cdots,p$, we have set $Y_i=(X_i-\mu_i)/\sigma$ and $g(Y_i)=$

$r\|\sigma \boldsymbol{Y}+\boldsymbol{\mu}\|_w^{-2}(\sigma Y_i+\mu_i)$ with $\boldsymbol{Y}=(Y_1,\cdots,Y_p)'\sim N_p(\mathbf{0}, I_p)$; and the inequality follows from Assumption (ii) since $X_i(\partial r/\partial X_i)\geqq 0$ for $i=1,\cdots,p$. To evaluate the expectation in the last expression of (2.4), we use Corollary 2.1 by letting $w=s/(n-2)$ and $h(w)=r\sigma^2/(\|\boldsymbol{X}\|_w^2 w)$. Then, taking the expectation first with respect to w and then with respect to $\boldsymbol{X}$, we have

$$\begin{aligned} n\,\mathrm{E}_X&\left\{\mathrm{E}_w\left[\frac{r\sigma^2}{\|\boldsymbol{X}\|_w^2 w}\right]\right\}\\ &=\mathrm{E}_X\left\{\mathrm{E}_w\left[\frac{(n-2)wh(w)}{\sigma^2}\right]-2\,\mathrm{E}_w\,[wh'(w)]\right\}\\ &=\mathrm{E}_X\left\{\mathrm{E}_w\left[\frac{(n-2)r}{\|\boldsymbol{X}\|_w^2}\right]-2\,\mathrm{E}_w\left[\frac{r\sigma^2}{\|\boldsymbol{X}\|_w^2 w}+\frac{\sigma^2}{\|\boldsymbol{X}\|_w^2}\frac{\partial r}{\partial w}\right]\right\}\\ &\geqq\mathrm{E}_X\left\{\mathrm{E}_w\left[\frac{(n-2)r}{\|\boldsymbol{X}\|_w^2}\right]-2\,\mathrm{E}_w\left[\frac{r\sigma^2}{\|\boldsymbol{X}\|_w^2 w}\right]\right\}\end{aligned}$$

since $(\partial r/\partial w)\leqq 0$ by Assumption (iii). After collecting similar terms, the above inequality reduces to

$$\mathrm{E}_{w,X}\left[\frac{r\sigma^2}{\|\boldsymbol{X}\|_w^2 w}\right]\geqq\left(\frac{n-2}{n+2}\right)\mathrm{E}_{w,X}\left[\frac{r}{\|\boldsymbol{X}\|_w^2}\right]. \tag{2.5}$$

Now, substituting (2.5) into (2.4) and (2.3), we have

(2.6)
$$\begin{aligned}\sigma^2\Delta_1&\geqq\mathrm{E}_{w,X}\left\{\left(\frac{n-2}{n+2}\right)\frac{r}{\|\boldsymbol{X}\|_w^2}[\mathrm{tr}\,(Q^{-1})-2\,\mathrm{ch}_{\max}(Q^{-1})]-\frac{r^2}{2\|\boldsymbol{X}\|_w^2}\mathrm{ch}_{\max}(Q^{-1})\right\}\\ &=\mathrm{E}_{w,X}\left\{(r\|\boldsymbol{X}\|_w^{-2})\left\{\left(\frac{n-2}{n+2}\right)[\mathrm{tr}\,(Q^{-1})-2\,\mathrm{ch}_{\max}(Q^{-1})]-(r/2)\,\mathrm{ch}_{\max}(Q^{-1})\right\}\right\}.\end{aligned}$$

Similarly, for a lower bound of Δ_2, we have

$$\begin{aligned}\Delta_2&=\mathrm{E}_{w,X}\left[\frac{r}{\|\boldsymbol{X}\|_w^2 w}(\boldsymbol{X}-\boldsymbol{\mu})'\boldsymbol{X}-\frac{r^2}{2\|\boldsymbol{X}\|_w^2}\right]\\ &\geqq\mathrm{E}_{w,X}\left\{(r\|\boldsymbol{X}\|_w^{-2})\left[\left(\frac{n-2}{n+2}\right)(p-2)-(r/2)\right]\right\}.\end{aligned} \tag{2.7}$$

The lower bounds in (2.6) and (2.7) will now provide an appropriate lower bound for $C\Delta$. More specifically, it follows from (2.2), (2.6), and (2.7) that

$$\begin{aligned}C\Delta/2\geqq\mathrm{E}_{w,X}&\left\{(r\|\boldsymbol{X}\|_w^{-2})\right.\\ &\times\left\{\left(\frac{1-\alpha}{\sigma^2}\right)\left[\left(\frac{n-2}{n+2}\right)(\mathrm{tr}\,(Q^{-1})-2\,\mathrm{ch}_{\max}(Q^{-1}))-(r/2)\,\mathrm{ch}_{\max}(Q^{-1})\right]\right.\end{aligned}$$

$$+\alpha\left[\left(\frac{n-2}{n+2}\right)(p-2)-(r/2)\right]\Big\}\Big\}$$

$$=\mathrm{E}_{w,X}\Big\{(r\|X\|_w^{-2})$$

$$\times\Big\{\left(\frac{n-2}{n+2}\right)\left[\left(\frac{1-\alpha}{\sigma^2}\right)(\mathrm{tr}\,(Q^{-1})-2\,\mathrm{ch}_{\max}\,(Q^{-1}))+\alpha(p-2)\right]$$

$$-(r/2)\left[\left(\frac{1-\alpha}{\sigma^2}\right)\mathrm{ch}_{\max}\,(Q^{-1})+\alpha\right]\Big\}\Big\}$$

which is nonnegative if r is bounded inclusively by 0 and

$$\frac{2[(n-2)/(n+2)]\{(1-\alpha)[\mathrm{tr}\,(Q^{-1})-2\,\mathrm{ch}_{\max}(Q^{-1})]/\sigma^2+\alpha(p-2)\}}{(1-\alpha)\,\mathrm{ch}_{\max}\,(Q^{-1})/\sigma^2+\alpha}\,. \tag{2.8}$$

It is noted that the upper bound given by (2.8) depends on the unknown variance σ^2. To avoid this dependence on σ^2, Assumption (i) is thus imposed. Under this assumption, it is clear from (2.6) that $\Delta_1\geqq 0$ for all $\boldsymbol{\mu}$ and σ^2. Furthermore, since $\mathrm{tr}\,(Q^{-1})\leqq p\,\mathrm{ch}_{\max}\,(Q^{-1})$ for any $p\times p$ positive definite matrix Q, it is also clear from (2.7) that Assumption (i) implies $\Delta_2\geqq 0$ for all $\boldsymbol{\mu}$ and σ^2. Therefore $C\Delta\geqq 0$ for all $\boldsymbol{\mu}$ and σ^2 completing the proof of the theorem.

3. Another class of minimax estimators

In the previous section, the estimator (2.1) is shown to be minimax for $\boldsymbol{\mu}$. It is noted that the function $r(\boldsymbol{X}, w)$ in the shrinkage factor of $\boldsymbol{\delta}(\boldsymbol{X}, w)$ is the same for each of the p components. In the following theorem, we will obtain another minimax estimator for $\boldsymbol{\mu}$, relative to the loss function (1.3), where the function $r_i(F, s)$ will be allowed to depend on the ith component.

THEOREM 3.1. *The estimator* $\boldsymbol{d}(F, s)$ *given componentwise by*

$$d_i(F, s)=\left[1-\frac{r_i(F, s)}{Fq_i}\right]X_i\,, \qquad i=1,\cdots,p\,, \tag{3.1}$$

is minimax for $\boldsymbol{\mu}$, *under the loss function* (1.3), *where* $F=\boldsymbol{X}'Q^{-1}\boldsymbol{X}/s$, *provided that the following conditions hold*

(i) $r_i(F, s)\geqq 0$ *for all* F *and* s, $i=1,\cdots,p$,

(ii) $r_i(F, s)$ *is nondecreasing in* F *and nonincreasing in* s, $i=1,\cdots,p$,

(iiia) $2\sum_{i=1}^{p}[r_i(F, s)/q_i]\geqq\max_{1\leqq i\leqq p}[4r_i(F, s)/q_i+(n+2)r_i^2(F, s)/q_i]$ *and*

(iiib) $2\sum_{i=1}^{p}[r_i(F, s)]\geqq\max_{1\leqq i\leqq p}[4r_i(F, s)+(n+2)r_i^2(F, s)]\,.$

PROOF. As in Theorem 2.1, let $\Delta=R(\boldsymbol{X};\boldsymbol{\mu},\sigma^2)-R(\boldsymbol{d};\boldsymbol{\mu},\sigma^2)$. Then

$$C\Delta=(1-\alpha)p\Delta_1^*+\alpha\sigma^2\operatorname{tr}(Q)\Delta_2^*\,,\qquad \text{say}\,, \tag{3.2}$$

where

$$\Delta_i^*=\mathrm{E}_{s,X}[L_i(\boldsymbol{X};\boldsymbol{\mu},\sigma^2)-L_i(\boldsymbol{d};\boldsymbol{\mu},\sigma^2)]\,,\qquad i=1,2\,. \tag{3.3}$$

To show that the estimator (3.1) is minimax, it suffices to show that $\Delta_i^*\geqq 0$, $i=1,2$, for all $\boldsymbol{\mu}$ and σ^2. It is noted that Strawderman ([8], Theorem 1), under the loss function

$$L_s(\boldsymbol{\delta};\boldsymbol{\mu},\sigma^2)=(\boldsymbol{\delta}-\boldsymbol{\mu})'D(\boldsymbol{\delta}-\boldsymbol{\mu})/\sigma^2 \tag{3.4}$$

where $D=\operatorname{diag}(d_1,\cdots,d_p)$ is a known $p\times p$ diagonal matrix with $d_i>0$, $i=1,\cdots,p$, obtains a minimax estimator given (componentwise) by

$$\eta_i(\boldsymbol{X},s)=\left\{1-\frac{c_i s r_i\left[\sum_{j=1}^{p}(e_jX_j^2)/s,\,s\right]}{\sum_{j=1}^{p}(e_jX_j^2)+gs+h}\right\}X_i \tag{3.5}$$

where

(a) $\{c_i\}$ and $\{e_i\}$ are sets of positive numbers,

(b) g and h are nonnegative,

(c) $r_i(F,s)$ is nonnegative, and nondecreasing in F and nonincreasing in s, and

(d) $2\sum_{i=1}^{p}c_id_ir_i(F,s)\geqq\max_{1\leqq i\leqq p}\left[4c_id_ir_i(F,s)+\frac{c_i^2d_i(n+2)}{e_i}r_i^2(F,s)\right]$ for all F and s.

It is clear that the estimator $d_i(F,s)$ given by (3.1) is a special case of $\eta_i(\boldsymbol{X},s)$ with

$$c_i=e_i=1/q_i\qquad\text{and}\qquad g=h=0\,.$$

Furthermore, the loss functions $L_i(\boldsymbol{\delta};\boldsymbol{\mu},\sigma^2)$, $i=1,2$, are of the form (3.4) with the weight matrix D properly identified, i.e.,

$$L_1(\boldsymbol{\delta};\boldsymbol{\mu},\sigma^2)=L_s(\boldsymbol{\delta};\boldsymbol{\mu},\sigma^2)\qquad\text{with}\qquad D=(1/p)I_p$$

and

$$L_2(\boldsymbol{\delta};\boldsymbol{\mu},\sigma^2)=L_s(\boldsymbol{\delta};\boldsymbol{\mu},\sigma^2)\qquad\text{with}\qquad D=Q/\operatorname{tr}(Q)\,.$$

Therefore, it is easily verified that $\Delta_1^*\geqq 0$ if Conditions (i), (ii), and (iiia) hold, and that $\Delta_2^*\geqq 0$ if Conditions (i), (ii), and (iiib) hold. This completes the proof.

Remarks. (1) If $r_i(F,s)=r(F,s)/(n-2)$ for all $i=1,\cdots,p$, and if the function $r(\boldsymbol{X},w)$ in (2.1) depends on $\boldsymbol{X}$ and w through $F=\boldsymbol{X}'Q^{-1}\boldsymbol{X}/w^2$

and w, then the minimax estimators given by (2.1) and (3.1) are identical. Moreover, Conditions (i), (iiia), and (iiib) of Theorem 3.1 reduce to Assumption (i) of Theorem 2.1, and Condition (ii) of Theorem 3.1 is essentially equivalent to Assumptions (ii) and (iii) of Theorem 2.1.

(2) Conditions (iiia) and (iiib) seem rather complicated. It would be nice if they could be simplified. But, since

$$\sum_{i=1}^{p} [r_i(F, s)/q_i] \geqq \left[\sum_{i=1}^{p} r_i(F, s)\right] \operatorname{ch}_{\min}(Q^{-1})$$

and

$$\max_{1\leqq i\leqq p} \{[4r_i(F, s)+(n+2)r_i^2(F, s)]/q_i\} \leqq \max_{1\leqq i\leqq p} [4r_i(F, s)+(n+2)r_i^2(F, s)] \operatorname{ch}_{\max}(Q^{-1}) ,$$

a sufficient condition for (iiia) to hold would be

$$2\sum_{i=1}^{p} [r_i(F, s)] \geqq \max_{1\leqq i\leqq p} [4r_i(F, s)+(n+2)r_i^2(F, s)] \times \operatorname{ch}_{\max}(Q^{-1})/\operatorname{ch}_{\min}(Q^{-1}) . \tag{3.6}$$

It is noted that (3.6) also implies Condition (iiib). Thus, Theorem 3.1 remains valid with Conditions (iiia) and (iiib) replaced by (3.6).

4. Proper Bayes minimax estimators

In this section, we will employ the same family of prior distributions as that of Strawderman [7] to obtain a class of proper Bayes minimax estimators for $\boldsymbol{\mu}$ under the convex loss function (1.3). Since the method of proof is the same as that of Strawderman [7], we will also adopt most of his notations. Let $\eta^2=1/\sigma^2$. Assume that

$$\boldsymbol{\mu}\,|\,(\lambda, \eta^2) \sim N_p(\mathbf{0}, V) \tag{4.1}$$

where $V=\operatorname{diag}(v_1,\cdots, v_p)$ with $v_i=(q_i-\lambda)/(\lambda\eta^2)$, $i=1,\cdots, p$. The joint density of λ and η^2 is given by

$$f(\lambda, \eta^2) \propto \lambda^{-a}(\eta^2)^{-K} \tag{4.2}$$

where $0<\lambda<\operatorname{ch}_{\min}(Q)\equiv q$ (say), $a<1$, $0<\gamma\leqq\eta^2<\infty$, and $K>1/2$. Then the proper Bayes minimax estimators of $\boldsymbol{\mu}$ will be obtained through the following lemmas. (Note that Strawderman [7] assumes $0<\gamma\leqq\eta<\infty$, but for ease of integration we take $0<\gamma\leqq\eta^2<\infty$; otherwise there is no major difference.)

LEMMA 4.1. *The conditional distribution of* $\boldsymbol{\mu}$, *given* $\boldsymbol{X}$, s, λ, *and* η^2, *is*

$$N_p[(I_p-\lambda Q^{-1})\boldsymbol{X},\ (I_p-\lambda Q^{-1})\eta^{-2}]\ .$$

PROOF. Immediate.

LEMMA 4.2. *The proper Bayes estimator of* $\boldsymbol{\mu}$ *with respect to the above priors under the loss function* (1.3) *is given componentwise by*

$$\hat{\mu}_i=\left[1-\frac{\mathrm{E}\left\{\lambda\eta^2\left[\dfrac{(1-\alpha)\eta^2+\alpha q_i}{(1-\alpha)p\eta^2+\alpha\,\mathrm{tr}\,(Q)}\right]\Big|\,\boldsymbol{X},s\right\}}{q_i\,\mathrm{E}\left\{\eta^2\left[\dfrac{(1-\alpha)\eta^2+\alpha q_i}{(1-\alpha)p\eta^2+\alpha\,\mathrm{tr}\,(Q)}\right]\Big|\,\boldsymbol{X},s\right\}}\right]X_i\ , \tag{4.3}$$

for $i=1,\cdots,p$.

PROOF. Under the loss function (1.3), it is clear that the Bayes estimator of $\boldsymbol{\mu}$ is given componentwise by

$$\hat{\mu}_i=\frac{\mathrm{E}\,[\mu_i\eta^2 J(\alpha,\eta^2,Q)\,|\,\boldsymbol{X},s]}{\mathrm{E}\,[\eta^2 J(\alpha,\eta^2,Q)\,|\,\boldsymbol{X},s]}\ ,\qquad i=1,\cdots,p\ , \tag{4.4}$$

where for convenience we have set

$$J(\alpha,\eta^2,Q)=\frac{(1-\alpha)\eta^2+\alpha q_i}{(1-\alpha)p\eta^2+\alpha\,\mathrm{tr}\,(Q)}\ . \tag{4.5}$$

Using Lemma 4.1, the numerator in the right-hand side of (4.4) can be further evaluated as

$$\begin{aligned}&\mathrm{E}\,\{\mathrm{E}\,[\mu_i\eta^2 J(\alpha,\eta^2,Q)\,|\,\boldsymbol{X},s,\lambda,\eta^2]\,|\,\boldsymbol{X},s\}\\ &\quad=\mathrm{E}\,\{(1-\lambda/q_i)\eta^2 J(\alpha,\eta^2,Q)\,|\,\boldsymbol{X},s\}X_i\ ,\end{aligned}$$

establishing the lemma.

LEMMA 4.3. *Define, for* $i=1,\cdots,p$,

$$r_i(F,s)=\frac{\mathrm{E}\,[u\eta^2 J(\alpha,\eta^2,Q)\,|\,\boldsymbol{X},s]}{\mathrm{E}\,[\eta^2 J(\alpha,\eta^2,Q)\,|\,\boldsymbol{X},s]} \tag{4.6}$$

where $u=\lambda F$ *and* $J(\alpha,\eta^2,Q)$ *is given by* (4.5). *Then*

$$r_i(F,s)\geqq 0 \tag{4.7}$$

and

$$\hat{\mu}_i=\left[1-\frac{r_i(F,s)}{Fq_i}\right]X_i\ . \tag{4.8}$$

PROOF. Immediate.

LEMMA 4.4. *The function* $r_i(F,s)$ *given by* (4.6) *is nondecreasing in* F *for fixed* s.

PROOF. Since the joint probability density function (pdf) of X, s, λ, and η^2 is given by

$$f(X, s, \lambda, \eta^2) \propto \lambda^{p/2-a}(\eta^2)^{(n+p)/2-K} s^{n/2-1} \exp\left[-\frac{1}{2}\eta^2 s(1+\lambda X'Q^{-1}X/s)\right],$$

it follows that the conditional pdf of $u=\lambda F$ and η^2, given X and s, is

$$f(u, \eta^2 | X, s) = \frac{u^{p/2-a}(\eta^2)^{(n+p)/2-K} \exp\left[-\frac{1}{2}\eta^2 s(1+u)\right]}{\int_0^{qF}\int_r^\infty u^{p/2-a}(\eta^2)^{(n+p)/2-K} \exp\left[-\frac{1}{2}\eta^2 s(1+u)\right] d\eta^2 du}.$$

Therefore, the function $r_i(F, s)$ given by (4.6) may be rewritten as

$$r_i(F, s) = G_1/G_0 \tag{4.9}$$

where, for each fixed $i=1,\cdots,p$, and $j=0, 1$, we have set

$$G_j \equiv G_j(F, s) = \int_0^{qF}\int_r^\infty u^{B+j}\eta^{2A} J(\alpha, \eta^2, Q) \exp\left[-\frac{1}{2}\eta^2 s(1+u)\right] d\eta^2 du \tag{4.10}$$

with

$$A=(n+p)/2-K+1 \quad \text{and} \quad B=p/2-a, \tag{4.11}$$

and $J(\alpha, \eta^2, Q)$ is given by (4.5). Now, a straightforward calculation shows

$$\frac{\partial r_i(F, s)}{\partial F} = (q/G_0^2)\int_r^\infty \eta^{2A} J(\alpha, \eta^2, Q) \exp\left[-\frac{1}{2}\eta^2 s(1+qF)\right] d\eta^2$$
$$\times \int_0^{qF}\int_r^\infty u^B \eta^{2A} J(\alpha, \eta^2, Q)(qF)^B (qF-u) \exp\left[-\frac{1}{2}\eta^2 s(1+u)\right] d\eta^2 du$$

which is nonnegative since $qF \geqq u$.

LEMMA 4.5. *The function $r_i(F, s)$ given by* (4.6) *is nonincreasing in s for fixed F.*

PROOF. It is clear that, for each $i=1,\cdots,p$,

$$\frac{\partial r_i(F, s)}{\partial s} = -\frac{1}{2}\operatorname{Cov}[U, H^2(1+U)] = -\frac{1}{2}\operatorname{Cov}\{U, \operatorname{E}[H^2(1+U)|U]\}$$

where the joint pdf of U and H^2 is given by

$$g_{U,H^2}(u, \eta^2) = u^B \eta^{2A} J(\alpha, \eta^2, Q) \exp\left[-\frac{1}{2}\eta^2 s(1+u)\right] \Big/ G_0,$$

with $0 \leqq u \leqq qF$ and $0 < r \leqq \eta^2 < \infty$. To show that $\operatorname{Cov}\{U, \operatorname{E}[H^2(1+U)|U]\}$

$\geqq 0$, it suffices to show, in view of Lemma 2.3, that $g(u)=\mathrm{E}[H^2(1+U)|U=u]$ is nondecreasing in u or $(dg(u)/du)\geqq 0$ for all $u\geqq 0$. But, this follows immediately from the proof of Lemma 2 of Strawderman [7] with obvious changes, completing the proof of the lemma.

We can now summarize the results in the following theorem.

THEOREM 4.1. *The estimator* $\hat{\mu}$ *given componentwise by* (4.8), *with the function* $r_i(F, s)$ *specified by* (4.6), *is a proper Bayes minimax estimator of* μ *relative to the loss function* (1.3) *provided that Conditions* (iiia) *and* (iiib) *of Theorem* 3.1 *hold.*

In practice, the computation of the proper Bayes estimator $\hat{\mu}_i$ or of the function $r_i(F, s)$ is not an easy task without the use of electronic computers. However, if the prior joint density of λ and η^2 given by (4.2) is suitably chosen, the computation may be much simplified. For example, take

$$a=0 \quad \text{and} \quad K=(n-4)/2 \qquad \text{if } p \text{ is even}$$

and

$$a=1/2 \quad \text{and} \quad K=(n-3)/2 \qquad \text{if p is odd}$$

in (4.2) where $n>5$. Then (4.5) becomes

$$J(\alpha, \eta^2, Q)=(1/p)\{1+\alpha(q_i-\bar{q})/[(1-\alpha)\eta^2+\alpha\bar{q}]\}$$

where $\bar{q}=\mathrm{tr}\,(Q)/p$, and the integral in (4.10) reduces to

$$\begin{aligned} pG_j &= \int_0^{qF}\int_r^\infty u^{B+j}\eta^{2A}\left[1+\frac{\alpha(q_i-\bar{q})}{(1-\alpha)\eta^2+\alpha\bar{q}}\right]\exp\left[-\frac{1}{2}\eta^2 s(1+u)\right]d\eta^2 du \\ &= I_1+\alpha(q_i-\bar{q})I_2\,, \qquad \text{say}\,, \end{aligned}$$

where

$$I_1=\int_0^{qF}\int_r^\infty u^{B+j}\eta^{2A}\exp\left[-\frac{1}{2}\eta^2 s(1+u)\right]d\eta^2 du$$

and

$$I_2=\int_0^{qF}\int_r^\infty \{u^{B+j}\eta^{2A}/[(1-\alpha)\eta^2+\alpha\bar{q}]\}\exp\left[-\frac{1}{2}\eta^2 s(1+u)\right]d\eta^2 du$$

with $A=B+3$ and $B=p/2-a$ being positive integers. The integral I_1 may be evaluated as follows: Let

$$v=\eta^2 su \qquad \text{and} \qquad z=\eta^2 s\,. \tag{4.12}$$

Then, after some simplification, we have

$$(4.13)\quad I_1=\int_r^\infty \eta^{2A}\left[\int_0^{qF} u^{B+j}\exp\left(-\frac{1}{2}\eta^2 su\right)du\right]\exp\left(-\frac{1}{2}\eta^2 s\right)d\eta^2$$
$$=s^{-(B+3)}\int_{rs}^\infty z^{1-j}\exp\left(-\frac{1}{2}z\right)\left[\int_0^{qFz} v^{B+j}\exp\left(-\frac{1}{2}v\right)dv\right]dz\ .$$

The bracket in the right-hand side of (4.13) is an incomplete gamma function. Since $B+j$ is a positive integer, the incomplete gamma function may be expressed as a finite sum, upon repeated application of an integration by parts, with a typical term (except constant factor) equal to $(qFz)^k \exp((-1/2)qFz)$ for some positive integer k. Thus I_1 may be written as a finite sum of simple integrals with a typical term taking the form

$$s^{-(B+3)}(qF)^k\int_{rs}^\infty z^{k+1-j}\exp\left[-\frac{1}{2}(1+qF)z\right]dz$$

which, again, is an incomplete gamma function with $(k+1-j)$ being a positive integer. Therefore, the integral I_1 can finally be expressed explicitly as a sum of a finite number of simple terms. As for the integral I_2, we may use the same change of variables (4.12) to obtain

$$I_2=s^{-(B+3)}\int_{rs}^\infty \frac{z^{1-j}\exp((-1/2)z)}{(1-\alpha)z/s+\alpha\bar{q}}\left[\int_0^{qFz} v^{B+j}\exp\left(-\frac{1}{2}v\right)dv\right]dz\ .$$

In view of the above discussion, it is clear that we need only evaluate a typical term of the form

$$I_3=\int_{rs}^\infty \frac{z^{k+1-j}\exp((-1/2)z)}{(1-\alpha)z/s+\alpha\bar{q}}dz$$

where k is a positive integer. If $\alpha=1$, then the evaluation of I_2 is the same as that of I_1. Now assume $\alpha<1$. Then

$$I_3=\frac{s}{1-\alpha}\int_{rs}^\infty \frac{z^{k+1-j}\exp((-1/2)z)}{z+\alpha\bar{q}s/(1-\alpha)}dz\ .$$

Making the change of variable $w=z+\alpha\bar{q}s/(1-\alpha)$, we have

$$(4.14)\quad I_3=\frac{s}{1-\alpha}\exp\left(\frac{\alpha\bar{q}s}{2(1-\alpha)}\right)\int_t^\infty \frac{[w-\alpha\bar{q}s/(1-\alpha)]^{k+1-j}}{w}\exp\left(-\frac{1}{2}w\right)dw$$

where $t=[\gamma+\alpha\bar{q}/(1-\alpha)]s$. After writing the factor $[w-\alpha\bar{q}s/(1-\alpha)]^{k+1-j}$ in terms of the powers of w, it is clear that the integral in the right-hand side of (4.14) is equal to the sum of $(k-j)$ incomplete gamma functionsa nd $\exp((-1/2)t)+\int_t^\infty w^{-1}\exp((-1/2)w)dw$, except their coefficients which are functions of F, s, α, and Q. Consequently, the integral I_2

can also be expressed explicitly as a sum of a finite number of simple terms.

5. Another convex loss function

Relative to the convex loss function (1.3) we have obtained classes of minimax estimators and proper Bayes minimax estimators for $\boldsymbol{\mu}$ in the previous sections. The convex loss function may be rewritten as

$$L(\boldsymbol{\delta}; \boldsymbol{\mu}, \sigma^2)=[1-\beta(\sigma^2)]L_1(\boldsymbol{\delta}; \boldsymbol{\mu}, \sigma^2)+\beta(\sigma^2)L_2(\boldsymbol{\delta}; \boldsymbol{\mu}, \sigma^2) \tag{5.1}$$

where $L_1(\boldsymbol{\delta}; \boldsymbol{\mu}, \sigma^2)$ and $L_2(\boldsymbol{\delta}; \boldsymbol{\mu}, \sigma^2)$ are given by (1.1) and (1.2), and

$$\beta(\sigma^2)=\alpha\sigma^2 \operatorname{tr}(Q)/[(1-\alpha)p+\alpha\sigma^2 \operatorname{tr}(Q)] . \tag{5.2}$$

It is noted that the convex coefficient $\beta(\sigma^2)$ depends on the unknown σ^2. In this section, we will consider another convex loss function which takes the form of (5.1) except now the convex coefficient does not depend on any unknown parameters, i.e.,

$$\begin{aligned} L^*(\boldsymbol{\delta}; \boldsymbol{\mu}, \sigma^2)&=(1-\beta)L_1(\boldsymbol{\delta}; \boldsymbol{\mu}, \sigma^2)+\beta L_2(\boldsymbol{\delta}; \boldsymbol{\mu}, \sigma^2) \\ &=(\boldsymbol{\delta}-\boldsymbol{\mu})'D(\boldsymbol{\delta}-\boldsymbol{\mu})/\sigma^2 \end{aligned} \tag{5.3}$$

where β is a known constant ranging between 0 and 1, and $D=\operatorname{diag}(d_1, \cdots, d_p)$ with

$$d_i=(1-\beta)/p+\beta q_i/\operatorname{tr}(Q) , \qquad i=1, \cdots, p . \tag{5.4}$$

As noted in Section 3, Strawderman [8] obtains a wide class of minimax estimators for $\boldsymbol{\mu}$, under the new loss function (5.3), where the shrinkage factor is allowed to depend on the ith component, $i=1, \cdots, p$. But, for the purpose of verifying the minimaxity of a proper Bayes estimator to be obtained in this section, we will need the following specialized result.

THEOREM 5.1. *Let* $\boldsymbol{X}\sim N_p(\boldsymbol{\mu}, \sigma^2 I_p)$ *and* $s/\sigma^2\sim\chi_n^2$, *independent of* $\boldsymbol{X}$. *Then, relative to the loss function* (5.3), *the estimator given componentwise by*

$$\hat{\mu}_i^*=\left(1-\frac{r(F^*, s)}{F^* d_i}\right)X_i , \qquad i=1, \cdots, p , \tag{5.5}$$

is minimax for $\boldsymbol{\mu}$, *where* $F^*=\boldsymbol{X}'D^{-1}\boldsymbol{X}/s$, *provided that*

(i) $0\leq r(F^*, s)\leq 2(p-2)/(n+2)$

and

(ii) $r(F^*, s)$ *is nondecreasing in* F^* *and nonincreasing in* s.

PROOF. The proof follows immediately from Theorem 1 of Strawderman [8] by setting $c_i=e_i=1/d_i$, $g=h=0$, and $r_i(F^*, s)=r(F^*, s)$ for all $i=1,\cdots,p$.

Now consider the family of prior distributions of $(\boldsymbol{\mu}, \lambda, \eta^2)$ having the density

$$(5.6)\quad f(\boldsymbol{\mu}, \lambda, \eta^2)\propto\lambda^{-a}(\eta^2)^{-K}(\lambda\eta^2)^{p/2}|D-\lambda I_p|^{-1/2}\exp\left[-\frac{1}{2}\lambda\eta^2\boldsymbol{\mu}'(D-\lambda I_p)^{-1}\boldsymbol{\mu}\right]$$

where

$$(5.7)\quad a<1\,,\quad K>1/2\,,\quad 0<\lambda\leqq d\equiv\mathrm{ch}_{\min}(D)\quad\text{and}\quad 0<\gamma\leqq\eta^2<\infty\,.$$

Using the procedure of Section 4, the following theorem can be established.

THEOREM 5.2. *Let $\boldsymbol{X}\sim N_p(\boldsymbol{\mu}, \sigma^2 I_p)$ and $s/\sigma^2\sim\chi^2_n$, independent of $\boldsymbol{X}$. Define $F^*=\boldsymbol{X}'D^{-1}\boldsymbol{X}/s$ where D is given by* (5.4) *and $d=(1-\beta)/p+\beta\,\mathrm{ch}_{\min}(Q)/\mathrm{tr}(Q)$, $0\leqq\beta\leqq1$. Then, the estimator $\hat{\boldsymbol{\mu}}^*=(\hat{\mu}_1^*,\cdots,\hat{\mu}_p^*)'$ given componentwise by* (5.5) *where $r(F^*, s)=G_1(F^*, s)/G_0(F^*, s)$ with*

$$(5.8)\quad G_j(F^*, s)=\int_0^{dF^*}\int_\gamma^\infty u^{p/2-a+j}(\eta^2)^{(n+p)/2-K}\exp\left[-\frac{1}{2}\eta^2 s(1+u)\right]d\eta^2 du\,,\qquad j=0, 1\,,$$

is proper Bayes with respect to the prior distribution (5.6) *and under the loss function* (5.3). *The proper Bayes estimator is also minimax if $p>4+4(3+\varepsilon)/(n-4-2\varepsilon)$, $\varepsilon>0$, and $n\geqq5$.*

PROOF. The first part of the theorem may be established following the same steps as those of Lemmas 4.1 through 4.5. It is noted that the proper Bayes estimator $\hat{\boldsymbol{\mu}}^*$ resembles a proper Bayes minimax estimator $\boldsymbol{\delta}(\boldsymbol{X})$, given by (3.1) of Strawderman [7]. A careful comparison between Strawderman's estimator and ours, it is not difficult to see that, for the minimaxity of $\hat{\boldsymbol{\mu}}^*$, we need only verify Condition (i) of Theorem 5.1. But, as in (3.4) of Strawderman [7],

$$0\leqq r(F^*, s)\leqq\lim_{\substack{F^*\to\infty\\ s\to0}}[G_1(F^*, s)/G_0(F^*, s)]\,.$$

Change the variable η^2 to $v=\eta^2 s$ in (5.8), we get

$$G_j(F^*, s)=s^{-(n+p)/2+K-1}\int_0^{dF^*}\int_{\gamma s}^\infty u^{p/2-a+j}v^{(n+p)/2-K}\exp\left[-\frac{1}{2}v(1+u)\right]dvdu\,.$$

Hence the upper bound is

$$\lim_{\substack{F^*\to\infty \\ s\to 0}} \frac{G_1(F^*, s)}{G_0(F^*, s)} = \frac{\int_0^\infty \int_0^\infty u^{p/2-a+1} v^{(n+p)/2-K} \exp\left[-\frac{1}{2}v(1+u)\right] dv du}{\int_0^\infty \int_0^\infty u^{p/2-a} v^{(n+p)/2-K} \exp\left[-\frac{1}{2}v(1+u)\right] dv du}$$

$$= \frac{\beta(n/2-K+a-1,\ p/2-a+2)}{\beta(n/2-K+a,\ p/2-a+1)}$$

$$= \frac{p-2a+2}{n-2K+2a-2}$$

where we have used $\beta(a, b)$ to denote the Beta function. Thus Condition (i) of Theorem 5.1 will be satisfied if

$$\frac{p-2a+2}{n-2K+2a-2} \leqq \frac{2(p-2)}{n+2}\ . \tag{5.10}$$

Choose $K=(1+\varepsilon)/2$ for some $\varepsilon>0$, and define

$$a(n, p, \varepsilon) = \frac{(n+2)(p+2)-2(p-2)(n-3-\varepsilon)}{2[2(p-2)+n+2]}\ . \tag{5.11}$$

Then (5.10) is equivalent to

$$a \geqq a(n, p, \varepsilon)\ . \tag{5.12}$$

If $a(n, p, \varepsilon)<1$ for those p and n for which there exists an $\varepsilon>0$, then we will be able to find values of a and K such that (5.10) holds and hence $\hat{\mu}^*$ will be minimax. But $a(n, p, \varepsilon)<1$ is equivalent to

$$p>4+4(3+\varepsilon)/(n-4-2\varepsilon) \quad \text{and} \quad n \geqq 5\ . \tag{5.13}$$

For $p=3$ or 4, inequality (5.13) cannot hold for any n. For $p=5$, $n \geqq 17$, for $p=6$, $n \geqq 11$, and so on. This completes the proof of the theorem.

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