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A CHARACTERIZATION OF GAMMA, MEIXNER HYPERGEOMETRIC AND NEGATIVE BINOMIAL DISTRIBUTIONS BASED ON CANONICAL MEASURES

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Summary

In this paper, we show that gamma, Meixner hypergeometric and negative binomial distributions can be characterized by their canonical measures.

1. Introduction

By the Meixner hypergeometric distribution we mean the distribution whose characteristic function $f(t)$ has the form

$$f(t)=\{\cosh ct-i\theta \sinh ct\}^{-\rho}, \qquad -\infty<t<\infty \tag{1.1}$$

where ρ, c and θ are real parameters such that $\rho>0$. For $\theta=0$, we have

$$f(t)=\{\cosh ct\}^{-\rho} \tag{1.2}$$

which is known as the generalized hyperbolic secant distribution (Harkness and Harkness [4]).

Meixner hypergeometric distributions were first defined and studied by Meixner [10] and [11]. Since then, many authors have written about these distributions. In particular, Laha and Lukacs [6] has shown that (1.1) is an infinitely divisible characteristic function. It follows from (1.1) that the mean and the variance are, respectively,

$$\mu=\rho c\theta , \qquad \sigma^2=\rho c^2(1+\theta^2) .$$

As noted in Feller [3], p. 503 and Lai and Vere-Jones [8], Meixner hypergeometric distributions exhibit some curious properties. A brief survey of this family of distributions was given in Lai [7].

The characteristic function of a gamma distribution is given by

$$f(t)=\{1-it/\lambda\}^{-\rho} , \qquad -\infty<t<\infty,\ \lambda, \rho>0 \tag{1.3}$$

whereas the characteristic function of a negative binomial is given by

$$f(t)=\{p(1-qe^{it})^{-1}\}^{\rho}\,, \qquad -\infty<t<\infty \tag{1.4}$$

where

$$\rho>0\,,\quad 0<p<1 \quad \text{and} \quad q=1-p\,.$$

Unlike the Meixner hypergeometric distributions, both gamma and negative binomial are very well known and they can be characterized in many ways (see Kagan etc. [5]).

All of these three families of distributions belong to Meixner class of distributions (Lancaster [9]) which can also be characterized in many ways, for examples, by the generating function of their orthogonal polynomials (Meixer [10]), by the quadratic regression on the sample mean (Laha and Lukacs [6]), by the Bhattacharya matrix (Shanbhag [12] and [13]) and by conditional moments (Bolger and Harkness [2]).

2. Canonical measure and a characterization

Suppose $f(t)$ is an infinitely divisible characteristic function such that the distribution corresponding to f has a finite second movement. Let $\psi(t)=\log f(t)$, then $\psi(\cdot)$ can be represented as

$$\psi(t)=\int_{-\infty}^{\infty}\frac{e^{itx}-1-it\sin x}{x^2}\,dM(x)+ibt \tag{2.1}$$

where ibt is the centering constant (Feller [3], p. 563), and $M(\cdot)$ is a finite measure which is called the canonical measure (see Feller [3], pp. 558–563). We note that an infinitely divisible distribution is uniquely determined by its canonical measure. It is easy to obtain, by differentiating (2.1) with respect to t twice, that

$$-\psi''(t)=\int_{-\infty}^{\infty}e^{itx}dM(x) \tag{2.2}$$

and to see that $\psi''(t)/\psi''(0)$ is a characteristic function of a probability distribution (see Feller [3], p. 559), i.e.

$$M(dx)=\sigma^2G(dx) \tag{2.3}$$

where $G(\cdot)$ is a probability distribution function and σ^2 is the variance that corresponds to $f(t)$.

THEOREM. *Let $f(t)$ be an infinitely divisible characteristic function such that the second moment corresponding to it is finite. Let μ and σ^2 be the mean and the variance of the distribution corresponding to $f(t)$. Then, for some $\rho>0$, the canonical measure $M(\cdot)$ corresponding to f*

satisfies

$$\int_{-\infty}^{\infty} e^{itx}dM(x)=\sigma^2\{f(t)\}^{2/\rho}, \qquad -\infty<t<\infty \tag{2.4}$$

if and only if

(i) $$\mu^2-\rho\sigma^2=0 \quad \textit{and}$$

$$f(t)=\{1-it\mu/\rho\}^{-\rho}, \qquad -\infty<t<\infty$$

(*in which case* f *is gamma or conjugate gamma according as* $\mu>0$ *or* $\mu<0$), *or*

(ii) $$\mu^2-\rho\sigma^2<0 \quad \textit{and}$$

$$f(t)=\{\cosh(\sqrt{\beta}\,t/\rho)-i(\mu/\sqrt{\beta})\sinh(\sqrt{\beta}\,t/\rho)\}^{-\rho}, \quad -\infty<t<\infty,$$

where

$$\beta=\rho\sigma^2-\mu^2$$

or

(iii) $$\mu^2-\rho\sigma^2>0 \quad \textit{and}$$

$$f(t)=e^{it\lambda}\left\{\frac{(2\lambda/(\mu+\lambda))}{1-((\mu-\lambda)/(\mu+\lambda))e^{2it\lambda/\rho}}\right\}^{\rho}, \qquad -\infty<t<\infty$$

with λ *such that* $\lambda\mu>0$ *and* $\lambda^2=\mu^2-\rho\sigma^2$ (*in which case* f *is, except for scale and location changes, negative binomial or conjugate negative binomial according as* $\mu>0$ *or* $\mu<0$).

PROOF. Let $\psi(t)=\log f(t)$. It follows from (2.4) that

$$-\psi''(t)=\sigma^2\{f(t)\}^{2/\rho} \tag{2.5}$$

which is obviously equivalent to

$$f''-\frac{\{f'\}^2}{f}+\sigma^2\{f\}^{2/\rho+1}=0, \qquad \rho>0 \tag{2.6}$$

which is a second order differential equation with initial conditions $f(0)=1$, $f'(0)=i\mu$. The solution of this initial value problem is a function of t, μ, and σ^2. Equation (2.6) implies that $\psi(t)$ is differentiable any number of times with respect to t. By differentiating (2.5) with respect to t, we obtain

$$\psi'''(t)=\frac{2}{\rho}\psi'(t)\psi''(t), \qquad -\infty<t<\infty \tag{2.7}$$

which is obviously equivalent to

$$\psi''(t)=\frac{1}{\rho}(\psi'(t))^2+\left(\frac{\mu^2}{\rho}-\sigma^2\right), \qquad -\infty<t<\infty . \tag{2.8}$$

The solution $\psi(t)$ of (2.8) is such that $f(t)=\exp\{\psi(t)\}$, $-\infty<t<\infty$ is given by

$$f(t)=\begin{cases}(1-it\mu/\rho)^{-\rho} & \text{if } \mu^2-\rho\sigma^2=0\\ \{\cosh(\sqrt{\beta}\,t/\rho)-i(\mu/\sqrt{\beta})\sinh(\sqrt{\beta}\,t/\rho)\}^{-\rho} & \text{if } \mu^2-\rho\sigma^2<0\\ e^{it\lambda}\left\{\dfrac{(2\lambda/(\mu+\lambda))}{1-((\mu-\lambda)/(\mu+\lambda))e^{2it\lambda/\rho}}\right\}^{\rho} & \text{if } \mu^2-\rho\sigma^2>0\end{cases} \tag{2.9}$$

where β and λ are as required by the theorem. The method of solution implies uniqueness if the solution exists; however the uniqueness theorem of a normal system of ordinary differential equations (Birkoff and Rota [1], pp. 108–109) also implies uniqueness of the solution in each of the three cases in (2.9). Hence the 'only if' part of the theorem follows. The 'if' part of the theorem is obvious.

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