Ann. Inst. Statist. Math.
34 (1982), Part A, 151-160

PARTITIONS, SUFFICIENCY AND UNDOMINATED FAMILIES OF PROBABILITY MEASURES

G. Trenkler

(Received Oct. 20, 1980; revised June 11, 1981)

Summary

This article is concerned with a class of statistical structures which has been introduced by Basu and Ghosh and where the underlying family of probability measures is not dominated. Using the concept of partition-inducible subfields it is shown that the intersection of arbitrarily many subfields is sufficient again. This gives rise to the notion of the coarsest sufficient subfield containing a given family of sets. This generated subfield may be calculated as a function of the minimal sufficient subfield which always exists in these structures. Finally some attention is given to invariance and sufficiency.

1. Introduction

Let X be a set, $\mathfrak{A}$ a σ-field of subsets of X and $\mathfrak{P}$ a family of probability measures on $\mathfrak{A}$. The triplet $(X, \mathfrak{A}, \mathfrak{P})$ will be called a statistical structure. As is well known the existence of minimal sufficient subfields of $\mathfrak{A}$ is assured, if the family $\mathfrak{P}$ is dominated [1]. We consider here an interesting class of statistical structures where $\mathfrak{P}$ is not dominated. This is the class of the so called Basu-Ghosh-structures, which has been introduced by Basu and Ghosh [3] and has further been studied by Morimoto [7]. In [3] the existence of minimal sufficient subfields of $\mathfrak{A}$ was established, if the underlying structure is Basu-Ghosh, a fact which is widely used in the results to be presented.

A Basu-Ghosh-structure satisfies the following assumptions:

(i) X is not countable.

(ii) $\mathfrak{A}$ is the power-set of X.

(iii) $\mathfrak{P}$ consists of discrete probability measures.

(iv) $P(A)=0$ for all $P \in \mathfrak{P}$ implies $A=\phi$.

The aim of the present paper is to consider these structures and to derive some new results, which should be of some theoretical interest.

Section 2 deals with partitions of the set X. During this section the assumption of a Basu-Ghosh-structure can be dropped. The main result is that the (possibly more than countable) intersection of partition-inducible σ-fields is again partition-inducible.

Section 3 shows that the intersection of arbitrarily many sufficient subfields is sufficient again. However it is assumed here as in the following sections that $(X, \mathfrak{A}, \mathfrak{P})$ has Basu-Ghosh-structure. It should be pointed out that even in the dominated set-up there is no analogy for this property.

Section 4 links the notions of invariance and sufficiency. As we lack the existence of $\mathfrak{P}$-zero sets we can overcome some difficulties which arise in Basu's paper [2].

2. Partitions

In this section we do not impose any structural restrictions on X. As before the power set of X will be denoted by $\mathfrak{A}$.

By a partition of X we mean a family of mutually disjoint nonempty subsets of X which collectively cover X. In a natural way every partition can be understood as a family of equivalence classes induced by a statistic on (the sample space) X. The notions: partition of X and statistic on X can be identified. Henceforth we use the word partition only. Each system of subsets $\mathfrak{E}\subset\mathfrak{A}$ gives rise to a partition of X if we define an equivalence relation $\pi_{\mathfrak{E}}$ on X.

Let $x, y \in X$. Then we put $x\pi_{\mathfrak{E}}y$ if and only if each set in $\mathfrak{E}$ either contains both or neither of x and y. It is easily seen that $\pi_{\mathfrak{E}}$ is reflexive, symmetrical and transitive. The equivalence classes of $\pi_{\mathfrak{E}}$ yield a partition which will be denoted by $\mathfrak{T}(\mathfrak{E})$. (In Morimoto's article [7] this equivalence relation was considered only in the case of $\mathfrak{E}$ being a σ-field.) Let E_x be the member of $\mathfrak{T}(\mathfrak{E})$ containing x. E_x is the biggest set containing x with the property: for all $E\in\mathfrak{E}$ we have either $E_x\subset E$ or $E_x\subset\bar{E}$, where $\bar{E}=\{z\,|\,z\in X\wedge z\notin E\}$. If $\mathfrak{E}$ is a partition itself $\mathfrak{E}$ and $\mathfrak{T}(\mathfrak{E})$ coincide with each other.

The partitions of X will be ordered in the following way: Let $\mathfrak{T}$, $\mathfrak{T}'$ be partitions of X. They are written $\mathfrak{T}<\mathfrak{T}'$ if each set in $\mathfrak{T}$ is a union of some members in $\mathfrak{T}'$.

Every partition $\mathfrak{T}$ induces a σ-field given by

$$\mathfrak{B}(\mathfrak{T}):=\{\bigcup_{T\in\mathfrak{T}^*} T\,|\,\mathfrak{T}^*\subset\mathfrak{T}\}\ . \tag{2.1}$$

A σ-field $\mathfrak{C}$ on X will be called partition-inducible if one can find a partition $\mathfrak{T}$ of X such that

$$\mathfrak{C}=\mathfrak{B}(\mathfrak{T})\ .$$

Obviously

$$\mathfrak{F}_1 \subset \mathfrak{F}_2 \quad \text{implies} \quad \mathfrak{T}(\mathfrak{F}_1) < \mathfrak{T}(\mathfrak{F}_2)$$

$$\mathfrak{T}_1 < \mathfrak{T}_2 \quad \text{implies} \quad \mathfrak{B}(\mathfrak{T}_1) \subset \mathfrak{B}(\mathfrak{T}_2) .$$

Furthermore we have $\mathfrak{F} \subset \mathfrak{B}(\mathfrak{T}(\mathfrak{F}))$, and for every partition we can state

$$\mathfrak{T} = \mathfrak{T}(\mathfrak{B}(\mathfrak{T})) . \tag{2.2}$$

A very helpful characterization of partition-inducible σ-fields was given by Morimoto [7]. We present it here as a lemma.

LEMMA 1. *Let $\mathfrak{C}$ be a σ-field on X. The following assertions are equivalent:*

(2.3)
(i) *$\mathfrak{C}$ is partition-inducible.*
(ii) *$\mathfrak{C}$ is closed under the formation of (possibly more than countable) unions.*
(iii) $\mathfrak{C} = \mathfrak{B}(\mathfrak{T}(\mathfrak{C}))$.

For further considerations we need the following fundamental lemma stating that partition-inducibility is preserved against the formation of arbitrary intersections.

LEMMA 2. *Let $(\mathfrak{C}_i)_{i \in I}$ be a family of partition-inducible σ-fields on X. Then the σ-field*

$$\bigcap_{i \in I} \mathfrak{C}_i$$

is partition-inducible.

The proof follows directly from Lemma 1 (ii).

The preceding lemma justifies introducing the notion of the coarsest partition-inducible σ-field containing a family $\mathfrak{F}$ of subsets:

$$\sigma^I(\mathfrak{F}) := \bigcap_{\substack{\mathfrak{C} \supset \mathfrak{F} \\ \mathfrak{C} \text{ is partition-inducible}}} \mathfrak{C} .^{*)} \tag{2.4}$$

As $\mathfrak{A}$ is induced by the partition $\{\{x\} \mid x \in X\}$ and $\mathfrak{F} \subset \mathfrak{A}$ it can be seen that $\sigma^I(\mathfrak{F})$ is well defined. Generally, $\sigma^I(\mathfrak{F})$ contains $\sigma(\mathfrak{F})$, the smallest σ-field containing $\mathfrak{F}$. $\sigma^I(\mathfrak{F})$ can easily be calculated as the following theorem indicates.

THEOREM 1. *Let $\mathfrak{F}$ be a family of subsets of X. Then we have*

---

*) The I in $\sigma^I(\mathfrak{F})$ is an abbreviation for the word "induced". It should not be mixed up with the subsequent index set I.

$$\sigma^I(\mathfrak{F})=\mathfrak{B}(\mathfrak{T}(\mathfrak{F})) . \tag{2.5}$$

PROOF. Using the fact $\mathfrak{F}\subset\mathfrak{B}(\mathfrak{T}(\mathfrak{F}))$ we conclude that $\mathfrak{B}(\mathfrak{T}(\mathfrak{F}))$ is a partition-inducible σ-field containing $\mathfrak{F}$. $\mathfrak{B}(\mathfrak{T}(\mathfrak{F}))$ is moreover the coarsest partition-inducible σ-field containing $\mathfrak{F}$. Let $\mathfrak{C}$ be an arbitrarily chosen partition-inducible σ-field containing $\mathfrak{F}$. We see that $\mathfrak{T}(\mathfrak{F}) < \mathfrak{T}(\mathfrak{C})$. Hence we derive from Lemma 1:

$$\mathfrak{B}(\mathfrak{T}(\mathfrak{F}))\subset\mathfrak{C}=\mathfrak{B}(\mathfrak{T}(\mathfrak{C})) .$$

Thus $\sigma^I(\mathfrak{F})$ and $\mathfrak{B}(\mathfrak{T}(\mathfrak{F}))$ must be equal.

Especially if $\mathfrak{C}$ is a σ-field it is clear from the above theorem that

$$\sigma^I(\mathfrak{C})=\mathfrak{B}(\mathfrak{T}(\mathfrak{C})) . \tag{2.6}$$

We proceed now to the question whether we can generate a new partition from a given family of partitions. We need not assume that the family is countable.

LEMMA 3. *Let $(\mathfrak{T}_i)_{i\in I}$ be a family of partitions of X, then there exists a partition denoted by $\bigvee_{i\in I}\mathfrak{T}_i$ with the following properties:*

(i) $\mathfrak{T}_j<\bigvee_{i\in I}\mathfrak{T}_i$ *for all* $j\in I$.

(ii) *For every partition $\mathfrak{T}$ with $\mathfrak{T}_i<\mathfrak{T}$ for all $i\in I$ it follows that*

$$\bigvee_{i\in I}\mathfrak{T}_i<\mathfrak{T} .$$

The proof can be taken from Blackwell-Girshick [4], p. 219.

THEOREM 2. *Let $(\mathfrak{T}_i)_{i\in I}$ be a family of partitions of X, then there exists a partition denoted by $\bigwedge_{i\in I}\mathfrak{T}_i$ with the following properties:*

(i) $\bigwedge_{i\in I}\mathfrak{T}_i<\mathfrak{T}_j$ *for all* $j\in I$.

(ii) *For every partition $\mathfrak{T}$ with $\mathfrak{T}<\mathfrak{T}_i$ for all $i\in I$ it follows that*

$$\mathfrak{T}<\bigwedge_{i\in I}\mathfrak{T}_i .$$

PROOF. Put

$$\mathfrak{S}:=\{\mathfrak{T}\mid\mathfrak{T}<\mathfrak{T}_i \text{ for all } i\in I\} .$$

Obviously $\mathfrak{S}$ is not empty. Hence we can define

$$\bigwedge_{i\in I}\mathfrak{T}_i:=\bigvee_{\mathfrak{T}\in\mathfrak{S}}\mathfrak{T} .$$

If we choose an arbitrary $j\in I$ it follows that $\mathfrak{T}<\mathfrak{T}_j$ for all $\mathfrak{T}\in\mathfrak{S}$. Lemma 3 (ii) gives $\bigvee_{\mathfrak{T}\in\mathfrak{S}}\mathfrak{T}<\mathfrak{T}_j$, i.e. $\bigwedge_{i\in I}\mathfrak{T}_i<\mathfrak{T}_j$. This is exactly condition (i).

To prove assertion (ii) we choose a partition $\mathfrak{T}$ such that $\mathfrak{T}<\mathfrak{T}_i$ for all $i \in I$. This means $\mathfrak{T} \in \mathfrak{S}$. Using Lemma 3 (i) we get

$$\mathfrak{T}<\bigwedge_{i\in I} \mathfrak{T}_i .$$

The proof is complete.

If we wish to show that in the case of Basu-Ghosh-structure the intersection of sufficient σ-fields will give a sufficient σ-field again we need the following theorem. The first part of it is of special interest, because it constitutes an extension of the statement of Basu-Ghosh [3] p. 857 (Remark 3).

THEOREM 3. *Let us be given a family* $(\mathfrak{T}_i)_{i\in I}$ *of partitions of* X. *Then we have the following identities*:

(2.7) (i) $$\mathfrak{B}(\bigwedge_{i\in I} \mathfrak{T}_i)=\bigcap_{i\in I} \mathfrak{B}(\mathfrak{T}_i) .$$

(2.8) (ii) $$\mathfrak{B}(\bigvee_{i\in I} \mathfrak{T}_i)=\sigma^I(\bigcup_{i\in I} \mathfrak{B}(\mathfrak{T}_i)) .$$

PROOF. (i) Since $\bigwedge_{i\in I} \mathfrak{T}_i<\mathfrak{T}_j$ for all $j \in I$ we have

$$\mathfrak{B}(\bigwedge_{i\in I} \mathfrak{T}_i)\subset\mathfrak{B}(\mathfrak{T}_j) \qquad \text{for all } j \in I .$$

Hence it follows that

(2.9) $$\mathfrak{B}(\bigwedge_{i\in I} \mathfrak{T}_i)\subset\bigcap_{i\in I} \mathfrak{B}(\mathfrak{T}_i) .$$

From Lemma 1 and Lemma 2 we conclude that

(2.10) $$\bigcap_{i\in I} \mathfrak{B}(\mathfrak{T}_i)=\mathfrak{B}(\mathfrak{T}(\bigcap_{i\in I} \mathfrak{B}(\mathfrak{T}_i))) .$$

Furthermore we have

$$\bigcap_{i\in I} \mathfrak{B}(\mathfrak{T}_i)\subset\mathfrak{B}(\mathfrak{T}_j) \qquad \text{for all } j \in I$$

which gives

$$\mathfrak{T}(\bigcap_{i\in I} \mathfrak{B}(\mathfrak{T}_i))<\mathfrak{T}(\mathfrak{B}(\mathfrak{T}_j))=\mathfrak{T}_j \qquad \text{for all } j \in I .$$

Therefore it follows in connection with Theorem 2 (ii) that

(2.11) $$\mathfrak{T}(\bigcap_{i\in I} \mathfrak{B}(\mathfrak{T}_i))<\bigwedge_{i\in I} \mathfrak{T}_i .$$

Thus

(2.12) $$\mathfrak{B}(\mathfrak{T}(\bigcap_{i\in I}\mathfrak{B}(\mathfrak{T}_i)))\subset\mathfrak{B}(\bigwedge_{i\in I}\mathfrak{T}_i)$$

or

(2.13) $$\bigcap_{i\in I}\mathfrak{B}(\mathfrak{T}_i)\subset\mathfrak{B}(\bigwedge_{i\in I}\mathfrak{T}_i)\ .$$

Together with (2.9) we finally have the desired identity in (i).

To prove (ii) let $\mathfrak{T}_j<\bigvee_{i\in I}\mathfrak{T}_i$ for all $j\in I$. It follows that $\mathfrak{B}(\mathfrak{T}_j)\subset\mathfrak{B}(\bigvee_{i\in I}\mathfrak{T}_i)$ for all $j\in I$ and hence

(2.14) $$\bigcup_{i\in I}\mathfrak{B}(\mathfrak{T}_i)\subset\mathfrak{B}(\bigvee_{i\in I}\mathfrak{T}_i)$$

so that

(2.15) $$\sigma^I(\bigcup_{i\in I}\mathfrak{B}(\mathfrak{T}_i))\subset\mathfrak{B}(\bigvee_{i\in I}\mathfrak{T}_i)$$

is valid because $\mathfrak{B}(\bigvee_{i\in I}\mathfrak{T}_i)$ is a partition-inducible σ-field. Moreover $\mathfrak{B}(\bigvee_{i\in I}\mathfrak{T}_i)$ is also the smallest partition-inducible σ-field containing $\bigcup_{i\in I}\mathfrak{B}(\mathfrak{T}_i)$.

Suppose that $\mathfrak{C}$ is a partition-inducible σ-field satisfying

(2.16) $$\mathfrak{C}\supset\bigcup_{i\in I}\mathfrak{B}(\mathfrak{T}_i)\ .$$

Then one can find a partition $\mathfrak{T}$ such that $\mathfrak{B}(\mathfrak{T})=\mathfrak{C}$ giving

$$\mathfrak{B}(\mathfrak{T})=\mathfrak{C}\supset\mathfrak{B}(\mathfrak{T}_j)\qquad\text{for all } j\in I$$

and afterwards

$$\mathfrak{T}=\mathfrak{T}(\mathfrak{B}(\mathfrak{T}))>\mathfrak{T}(\mathfrak{B}(\mathfrak{T}_j))=\mathfrak{T}_j\qquad\text{for all } j\in I\ .$$

From the definition of $\bigvee_{i\in I}\mathfrak{T}_i$ it follows that $\bigvee_{i\in I}\mathfrak{T}_i<\mathfrak{T}$ finally implying

$$\mathfrak{B}(\bigvee_{i\in I}\mathfrak{T}_i)\subset\mathfrak{B}(\mathfrak{T})=\mathfrak{C}\ .$$

Thus (ii) is shown.

3. Sufficiency and Basu-Ghosh-structures

We turn now our attention to the notion of sufficiency. Henceforth $(X, \mathfrak{A}, \mathfrak{P})$ will always have Basu-Ghosh-structure. We start from the results of Basu-Ghosh [3] which will not be proved here. We shall need at first some definitions.

A σ-field $\mathfrak{C}$ on X will be called sufficient for $\mathfrak{P}$, if for every $A\subset X$ there exists a $\mathfrak{C}$-measurable function $f_A: X\rightarrow R$ such that for all $C\in\mathfrak{C}$ and all $P\in\mathfrak{P}$,

$$(3.1)\qquad P(A\cap C)=\int_C f_A(x)dP(x)\,.$$

A partition $\mathfrak{T}$ is sufficient*) for $\mathfrak{P}$ if the induced subfield $\mathfrak{B}(\mathfrak{T})$ is sufficient for $\mathfrak{P}$. We present now the main facts stemming from the just mentioned authors concerning sufficiency for the Basu-Ghosh-structure. We shall do this in the form of a lemma.

LEMMA 4. (i) *A partition $\mathfrak{T}$ of X is sufficient for $\mathfrak{P}$ if and only if there exists a function $g: X\to R$ such that*

$$(3.2)\qquad P(x)=g(x)\cdot P(E_x)$$

for all $x\in X$ and for all $P\in\mathfrak{P}$. (E_x is the member of $\mathfrak{T}$ containing x.)
(ii) *There exists a sufficient partition $\mathfrak{M}$ of X such that $\mathfrak{M}<\mathfrak{T}$ for all sufficient partitions $\mathfrak{T}$. $\mathfrak{B}(\mathfrak{M})$ is the minimal sufficient σ-field.*
(iii) *Every sufficient σ-field is partition-inducible.*
(iv) *For two partitions $\mathfrak{T}_1$, $\mathfrak{T}_2$, if $\mathfrak{T}_1<\mathfrak{T}_2$ and if $\mathfrak{T}_1$ is sufficient, then $\mathfrak{T}_2$ is also sufficient.*

From Burkholder's paper [5] one immediately derives that a countably infinite intersection of sufficient σ-fields gives a sufficient σ-field again. The next theorem deals with an index set I of arbitrary cardinality. For a similar result see Hasegawa-Perlman [6].

THEOREM 4. *Suppose $(\mathfrak{B}_i)_{i\in I}$ is a family of sufficient σ-fields for $\mathfrak{P}$. Then $\bigcap_{i\in I}\mathfrak{B}_i$ is also sufficient for $\mathfrak{P}$.*

PROOF. From Lemma 4(iii), there exist sufficient partitions $\mathfrak{T}_i$ such that $\mathfrak{B}_i=\mathfrak{B}(\mathfrak{T}_i)$ for all $i\in I$. Applying Lemma 4(ii), we get $\mathfrak{M}<\mathfrak{T}_i$ for all $i\in I$ and hence from Theorem 2 $\mathfrak{M}<\bigwedge_{i\in I}\mathfrak{T}_i$. Lemma 4(iv) implies that $\bigwedge_{i\in I}\mathfrak{T}_i$ is a sufficient partition.

Finally we have from Theorem 3(i):

$$(3.3)\qquad \mathfrak{B}(\bigwedge_{i\in I}\mathfrak{T}_i)=\bigcap_{i\in I}\mathfrak{B}(\mathfrak{T}_i)\,.$$

Thus $\bigcap_{i\in I}\mathfrak{B}_i$ is a sufficient σ-field for $\mathfrak{P}$.

It is now possible to introduce the notion of the sufficient σ-field generated by a family $\mathfrak{E}\subset\mathfrak{A}$. It is characterized by the formula

$$(3.4)\qquad \mathfrak{B}^{\text{suff}}(\mathfrak{E}):=\bigcap_{\substack{\mathfrak{B}\supset\mathfrak{E}\\ \mathfrak{B}\text{ is sufficient}}}\mathfrak{B}\,.$$

Observe that $\mathfrak{B}^{\text{suff}}(\mathfrak{E})$ is well defined as $\mathfrak{A}$ is sufficient for $\mathfrak{P}$.

*) We sometimes omit "for $\mathfrak{P}$".

Since $\mathfrak{B}^{\text{suff}}(\mathfrak{F})$ is a partition-inducible σ-field we should have a look at the explicit form of the partition inducing it. The result of the investigation is somewhat surprising. We shall see that

$$\mathfrak{T}(\mathfrak{B}^{\text{suff}}(\mathfrak{F}))=\mathfrak{T}(\mathfrak{F})\vee\mathfrak{M}\,. \tag{3.5}$$

(If $\mathfrak{T}_1$, $\mathfrak{T}_2$ are partitions, $\mathfrak{T}_1\vee\mathfrak{T}_2$ clearly is understood in the sense of Lemma 3.)

THEOREM 5. *Let* $\mathfrak{F}$ *be a family of subsets of* X. *Then*

$$\mathfrak{B}^{\text{suff}}(\mathfrak{F})=\mathfrak{B}(\mathfrak{M}\vee\mathfrak{T}(\mathfrak{F}))\,. \tag{3.6}$$

PROOF. From $\mathfrak{M}<\mathfrak{M}\vee\mathfrak{T}(\mathfrak{F})$ and Lemma 4 (iii), it follows that $\mathfrak{M}\vee\mathfrak{T}(\mathfrak{F})$ is a sufficient partition. Furthermore we have $\mathfrak{F}\subset\mathfrak{B}(\mathfrak{T}(\mathfrak{F}))\subset\mathfrak{B}(\mathfrak{M}\vee\mathfrak{T}(\mathfrak{F}))$ which means that $\mathfrak{B}(\mathfrak{M}\vee\mathfrak{T}(\mathfrak{F}))$ is a sufficient subfield containing $\mathfrak{F}$. Let $\mathfrak{C}$ be an arbitrary sufficient σ-field containing $\mathfrak{F}$. We assert that $\mathfrak{C}\supset\mathfrak{B}(\mathfrak{M}\vee\mathfrak{T}(\mathfrak{F}))$. Since $\mathfrak{C}$ is sufficient it is partition-inducible i.e. there is a partition $\mathfrak{T}$ such that $\mathfrak{C}=\mathfrak{B}(\mathfrak{T})$. From the condition $\mathfrak{F}\subset\mathfrak{C}$ it follows that $\mathfrak{T}(\mathfrak{F})<\mathfrak{T}(\mathfrak{C})$ implying $\mathfrak{M}\vee\mathfrak{T}(\mathfrak{F})<\mathfrak{M}\vee\mathfrak{T}(\mathfrak{C})$. Now we have $\mathfrak{T}(\mathfrak{C})=\mathfrak{T}$ giving $\mathfrak{M}\vee\mathfrak{T}(\mathfrak{F})<\mathfrak{M}\vee\mathfrak{T}$. $\mathfrak{M}$ is the minimal sufficient partition from which one obtains $\mathfrak{M}\vee\mathfrak{T}=\mathfrak{T}$. Finally we get $\mathfrak{C}\supset\mathfrak{B}(\mathfrak{M}\vee\mathfrak{T}(\mathfrak{F}))$. Hence we can state: $\mathfrak{B}(\mathfrak{M}\vee\mathfrak{T}(\mathfrak{F}))$ is the coarsest sufficient σ-field containing $\mathfrak{F}$. Thus $\mathfrak{B}^{\text{suff}}(\mathfrak{F})$ and $\mathfrak{B}(\mathfrak{M}\vee\mathfrak{T}(\mathfrak{F}))$ must coincide.

By definition $\mathfrak{B}^{\text{suff}}(\mathfrak{F})$ is always sufficient. We should now raise the question: Under which conditions is the σ-field $\sigma^I(\mathfrak{F})$ sufficient? A necessary and sufficient criterion is given by the following theorem.

THEOREM 6. *Let* $\mathfrak{F}$ *be a family of subsets of* X. $\sigma^I(\mathfrak{F})$ *is sufficient if and only if there exists a sufficient partition* $\mathfrak{T}$ *such that*

$$\mathfrak{T}<\mathfrak{T}(\mathfrak{F})\,.$$

PROOF. To prove necessity we assume that $\sigma^I(\mathfrak{F})$ is sufficient. Then we have $\sigma^I(\mathfrak{F})\supset\mathfrak{B}(\mathfrak{M})$. Using Theorem 1 one obtains

$$\mathfrak{T}(\mathfrak{F})=\mathfrak{T}(\mathfrak{B}(\mathfrak{T}(\mathfrak{F})))=\mathfrak{T}(\sigma^I(\mathfrak{F}))>\mathfrak{T}(\mathfrak{B}(\mathfrak{M}))=\mathfrak{M}\,.$$

The reverse direction is shown as follows. Let there exist a sufficient partition such that $\mathfrak{T}<\mathfrak{T}(\mathfrak{F})$. From Lemma 3 (iv) it follows that $\mathfrak{T}(\mathfrak{F})$ is also sufficient. Finally we have from Theorem 1 that $\sigma^I(\mathfrak{F})=\mathfrak{B}(\mathfrak{T}(\mathfrak{F}))$. This implies the sufficiency of $\sigma^I(\mathfrak{F})$.

4. Invariance and sufficiency

Although "the invariance principle usually falls to pieces when faced with a discrete model" (Basu [2], p. 83) we shall give some attention to that field. In this section f is always a one-to-one mapping from X onto X, i.e. $f: X \to X$ is a bijective transformation. For each $P \in \mathfrak{P}$ let $f(P)$ denote the measure on $\mathfrak{A}$ induced by f and given by the equation:

$$(4.1) \qquad f(P)(A) := P(f^{-1}(A)) \quad \text{for } A \subset X .$$

f is called model-preserving if we have $f(P)=P$ for all $P \in \mathfrak{P}$. In a natural way every bijective f yields a partition-inducible σ-field which is given by

$$(4.2) \qquad \mathfrak{A}(f) := \{A \mid A \subset X \text{ and } f^{-1}(A)=A\} .$$

The members of $\mathfrak{T}(\mathfrak{A}(f))$ can be described as sets of the form $\{f^n(x) \mid n \in Z\} = E_x$.

Suppose $\mathfrak{F}$ is a nonempty class of bijective mappings from X to X. Then

$$(4.3) \qquad \mathfrak{A}(\mathfrak{F}) := \bigcap_{f \in \mathfrak{F}} \mathfrak{A}(f)$$

is also partition-inducible by Lemma 2.

The following theorem has been proved already by Basu [2]. We will show here that one can simplify the proof excluding the principles of ergodic theory if the underlying structure is Basu-Ghosh.

THEOREM 7. *Let f be model-preserving. Then $\mathfrak{A}(f)$ is sufficient for $\mathfrak{P}$.*

PROOF. We have to construct a mapping $g: X \to R$ such that

$$(4.4) \qquad P(x) = g(x) \cdot P(E_x)$$

for all $x \in X$ and all $P \in \mathfrak{P}$. Let $x \in X$, $P \in \mathfrak{P}$ be arbitrarily chosen. As mentioned before E_x can be written in the form $\{f^n(x) \mid n \in Z\}$. The Basu-Ghosh-structure of $(X, \mathfrak{A}, \mathfrak{P})$ implies the existence of some $P_0 \in \mathfrak{P}$ such that $P_0(x) > 0$.

As we have

$$(4.5) \qquad P_0(f^n(x)) = P_0(x) \quad \text{for all } n \in Z ,$$

it is clear that there is only a finite number $m(x)$ of different points in E_x. Put

$$g(x) := \frac{1}{m(x)} . \tag{4.6}$$

Obviously $g(x)$ does not depend on P. (4.5) is also valid for P which gives immediately

$$P(x) = g(x) \cdot P(E_x) . \tag{4.7}$$

It follows at once from Theorem 4 that $\mathfrak{A}(\mathfrak{G})$ is sufficient if $\mathfrak{G}$ is a nonempty class of model-preserving transformations. A weaker statement can be read in the paper of Basu [2] p. 65, Theorem 2. Especially if $\mathfrak{G}^*$ is the class of all model-preserving transformations then $\mathfrak{A}(\mathfrak{G}^*)$ is sufficient. But generally $\mathfrak{B}(\mathfrak{M})$ is coarser than $\mathfrak{A}(\mathfrak{G}^*)$. Also in the case of Basu-Ghosh-structure one can easily construct examples showing that $\mathfrak{B}(\mathfrak{M})$ and $\mathfrak{A}(\mathfrak{G}^*)$ need not coincide.

Acknowledgement

The author is very grateful to R. Schlittgen from the Technical University of Berlin. Thanks to his helpful suggestions some proofs could be decisively shortened.

UNIVERSITY OF HANNOVER

REFERENCES

[1] Bahadur, R. R. (1954). Sufficiency and statistical decision functions, *Ann. Math. Statist.*, **25**, 423-462.

[2] Basu, D. (1970). On sufficiency and invariance, *Essays in Probability and Statistics*, Univ. of North Carolina Press, Chapel Hill, N.C.

[3] Basu, D. and Ghosh, J. K. (1969). Sufficient statistics in sampling from a finite universe, *Proc. 36th Session Internat. Statist. Inst.*, 850-859.

[4] Blackwell, D. and Girshick, A. A. (1954). *Theory of Games and Statistical Decisions*, Wiley, New York.

[5] Burkholder, D. L. (1961). Sufficiency in the undominated case, *Ann. Math. Statist.*, **32**, 1191-1200.

[6] Hasegawa, M. and Perlman, M. D. (1974). On the existence of a minimal sufficient subfield, *Ann. Statist.*, **2**, 1049-1055.

[7] Morimoto, H. (1972). Statistical structure of the problem of sampling from finite populations, *Ann. Math. Statist.*, **43**, 490-497.