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ASYMPTOTIC OPTIMALITY OF ESTIMATORS IN NON-REGULAR CASES*

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Summary

The bound of the asymptotic distributions of $n|\hat{\theta}_n-\theta|$ for all asymptotically median unbiased (AMU) estimators $\hat{\theta}_n$ is given in non-regular cases. It provides us with a powerful criterion for an AMU estimator to be two-sided asymptotically efficient and also useful in the cases when there may not exist a two-sided asymptotically efficient estimator since we may find an AMU estimator whose asymptotic distribution attains at least at a point, or an AMU estimator whose asymptotic distribution is uniformly "close" to it. Some examples are given.

1. Introduction

In the asymptotic theory of statistical estimation the concepts of asymptotic efficiency and asymptotic sufficiency play an important part. The asymptotic efficiency including higher order in regular cases has been extensively studied by Akahira and Takeuchi [5], [9] and Pfanzagl and Wefelmeyer [7]. The asymptotic sufficiency in non-regular cases has been discussed by Akahira [3] and recently extended by Weiss [10]. The asymptotic efficiency in non-regular cases has been discussed by Takeuchi [8], Akahira [2], [4] and Akahira and Takeuchi [6] in special cases.

In this paper we obtain the bound of the asymptotic distributions of $n|\hat{\theta}_n-\theta|$ for all AMU estimators $\hat{\theta}_n$ in non-regular cases. It provides us with a powerful criterion for an AMU estimator to be two-sided asymptotically efficient, which can be used for constructing an AMU estimator with this property. And it is also useful in the cases when there may not exist a two-sided asymptotically efficient estimator, i.e., an AMU estimator whose asymptotic distribution uniformly attains it.

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In fact we may find, even in those cases, an AMU estimator whose asymptotic distribution attains it at least at a point, or, an AMU estimator whose asymptotic distribution is uniformly "close" to it.

2. Definitions

Let $\mathcal{X}$ be an abstract sample space whose generic point is denoted by x, $\mathcal{B}$ a σ-field of subsets of $\mathcal{X}$ and $\{P_\theta\colon \theta \in \Theta\}$ a set of probability measures on $\mathcal{B}$, which Θ is called a parameter space. We assume that Θ is an open set of R^1. Consider n-fold direct products $(\mathcal{X}^n, \mathcal{B}^n)$ of $(\mathcal{X}, \mathcal{B})$ and the corresponding product measure P_θ^n of P_θ. An estimator of θ is defined to be a sequence $\{\hat{\theta}_n\}$ of $\mathcal{B}^n$-measurable functions $\hat{\theta}_n$ on $\mathcal{X}^n$ into Θ. For simplicity we denote $\{\hat{\theta}_n\}$ by $\hat{\theta}_n$.

For an increasing sequence of positive numbers $\{c_n\}$ (c_n tending to infinity) an estimator $\hat{\theta}_n$ is called consistent with order $\{c_n\}$ (or $\{c_n\}$-consistent for short) if for every $\varepsilon>0$ and every $\vartheta \in \Theta$ there exist a sufficiently small positive number δ and a sufficiently large positive number L satisfying the following:

$$\varlimsup_{n\to\infty} \sup_{\theta:|\theta-\vartheta|<\delta} P_\theta^n\{c_n|\hat{\theta}_n-\theta|\geqq L\} < \varepsilon$$

(Akahira [1]). For a $\{c_n\}$-consistent estimator $\hat{\theta}_n$ a distribution function $F_\theta(\cdot)$ is called to be the asymptotic distribution function of $c_n(\hat{\theta}_n-\theta)$ of order $C=\{c_n\}$ if for each real number t, $F_\theta(t)$ is continuous in θ and for any $\vartheta \in \Theta$ there exists a positive number d such that for any continuity point t of $F_\theta(t)$

$$\lim_{n\to\infty} \sup_{\theta:|\theta-\vartheta|<d} |P_\theta^n\{c_n(\hat{\theta}_n-\theta)<t\}-F_\theta(t)|=0 .$$

A $\{c_n\}$-consistent estimator is called to be asymptotically median unbiased (AMU) if for every $\vartheta \in \Theta$ there exists a positive number δ such that

$$\lim_{n\to\infty} \sup_{\theta:|\theta-\vartheta|<\delta} \left|P_\theta^n\{\hat{\theta}_n\leqq\theta\}-\frac{1}{2}\right|=0 ;$$

$$\lim_{n\to\infty} \sup_{\theta:|\theta-\vartheta|<\delta} \left|P_\theta^n\{\hat{\theta}_n\geqq\theta\}-\frac{1}{2}\right|=0 .$$

DEFINITION. For an AMU estimator $\hat{\theta}_n^*$ it is called two-sided asymptotically efficient if for any AMU estimator $\hat{\theta}_n$ and any $t>0$

$$\varliminf_{n\to\infty} [P_\theta^n\{c_n|\hat{\theta}_n^*-\theta|<t\}-P_\theta^n\{c_n|\hat{\theta}_n-\theta|<t\}]\geqq 0 .$$

We may have different definitions of asymptotic efficiency as follows. An AMU estimator $\hat{\theta}_n^{**}$ is called right-hand side (left-hand side) asymptotically efficient if for any AMU estimator $\hat{\theta}_n$

$$\varliminf_{n\to\infty} [P_\theta^n\{c_n(\hat{\theta}_n^{**}-\theta)<t\}-P_\theta^n\{c_n(\hat{\theta}_n-\theta)<t\}]\geqq 0 \qquad \text{for all } t>0$$

$$(\varliminf_{n\to\infty} [P_\theta^n\{c_n(\theta_n-\theta)<t\}-P_\theta^n\{c_n(\hat{\theta}_n^{**}-\theta)<t\}]\geqq 0 \qquad \text{for all } t<0)\,.$$

In non-regular cases it is shown in [8] that there are right-hand side and left-hand side asymptotically efficient estimators but not generally an asymptotically efficient estimator, that is, an AMU estimator satisfying simultaneously the above inequalities.

3. Two-sided asymptotic efficiency

We shall obtain the upper bound of the asymptotic distributions of $c_n|\hat{\theta}_n-\theta|$ in the class $\boldsymbol{A}$ of the all AMU estimators $\hat{\theta}_n$. Let $\hat{\theta}_n$ be an AMU estimator and t be any positive number. We have

$$P_\theta^n\{c_n|\hat{\theta}_n-\theta|<t\}=P_\theta^n\{c_n(\hat{\theta}_n-\theta)<t\}-P_\theta^n\{c_n(\hat{\theta}_n-\theta)<-t\}\,. \tag{3.1}$$

Let θ_0 be any fixed in Θ. Put $\theta_1=\theta_0+(t/c_n)$ and $\theta_2=\theta_0-(t/c_n)$. Suppose that there exists the asymptotic distribution of $c_n(\hat{\theta}_n-\theta)$. Since

$$P_{\theta_1}^n\{\hat{\theta}_n<\theta_0\}=P_{\theta_1}^n\{c_n(\hat{\theta}_n-\theta_1)<-t\}\ ; \qquad P_{\theta_2}^n\{\hat{\theta}_n<\theta_0\}=P_{\theta_2}^n\{c_n(\hat{\theta}_n-\theta_2)<t\}\ ,$$

it follows by the uniformity of the neighbourhood of θ_0 that

$$\lim_{n\to\infty}|P_{\theta_1}^n\{\hat{\theta}_n<\theta_0\}-P_{\theta_0}^n\{c_n(\hat{\theta}_n-\theta_0)<-t\}|=0\ ; \tag{3.2}$$

$$\lim_{n\to\infty}|P_{\theta_2}^n\{\hat{\theta}_n<\theta_0\}-P_{\theta_0}^n\{c_n(\hat{\theta}_n-\theta_0)<t\}|=0\ . \tag{3.3}$$

In order to obtain the upper bound of (3.1), from (3.1), (3.2) and (3.3) it is enough to get the AMU estimator maximizing

$$P_{\theta_2}^n\{\hat{\theta}_n<\theta_0\}-P_{\theta_1}^n\{\hat{\theta}_n<\theta_0\} \tag{3.4}$$

in the class $\boldsymbol{A}$. Suppose that every $P_\theta(\cdot)$ $(\theta\in\Theta)$ is absolutely continuous with respect to a σ-finite measure μ. We denote the density $dP_\theta/d\mu$ by $f(x,\theta)$. Let $\phi_n(\tilde{x}_n)=\chi_{\{\hat{\theta}_n<\theta_0\}}(\tilde{x}_n)$, where $\chi_{\{\hat{\theta}_n<\theta_0\}}(\tilde{x}_n)$ is an indicator of the set $\{\hat{\theta}_n<\theta_0\}$ with $\tilde{x}_n=(x_1,\cdots,x_n)$. Then we have

$$\int\cdots\int\phi_n(\tilde{x}_n)\prod_{i=1}^n f(x_i,\theta_0)\prod_{i=1}^n d\mu(x_i)=\mathrm{E}_{\theta_0}^n(\phi_n)\to\frac{1}{2} \qquad (n\to\infty)\,. \tag{3.5}$$

If the maximum value of

$$(3.6)\quad \int\cdots\int \phi_n(\tilde{x}_n)\left\{\prod_{i=1}^n f(x_i, \theta_2)-\prod_{i=1}^n f(x_i, \theta_1)\right\}\prod_{i=1}^n d\mu(x_i)=\mathrm{E}_{\theta_2}^n(\phi_n)-\mathrm{E}_{\theta_1}^n(\phi_n)$$

is obtained for every ϕ_n satisfying (3.5) and $0\leqq\phi_n(\tilde{x}_n)\leqq 1$, it is the upper bound of (3.1) in the class $\boldsymbol{A}$. By the similar way as the proof of the fundamental lemma of Neyman and Pearson it is seen that the ϕ_n^* maximizing (3.6) under the condition (3.5) is given by

$$(3.7)\quad \phi_n^*(\tilde{x}_n)=\begin{cases} 1 & \text{for } \prod_{i=1}^n f(x_i, \theta_2)-\prod_{i=1}^n f(x_i, \theta_1)>\lambda\prod_{i=1}^n f(x_i, \theta_0)\,; \\ \gamma & \text{for } \prod_{i=1}^n f(x_i, \theta_2)-\prod_{i=1}^n f(x_i, \theta_1)=\lambda\prod_{i=1}^n f(x_i, \theta_0)\,; \\ 0 & \text{for } \prod_{i=1}^n f(x_i, \theta_2)-\prod_{i=1}^n f(x_i, \theta_1)<\lambda\prod_{i=1}^n f(x_i, \theta_0)\,, \end{cases}$$

where λ is some constant and γ is a 0-1 valued function of $\tilde{x}_n$.

In the subsequent discussions we deal only with the case when $c_n=n$. Suppose that $\mathcal{X}=\Theta=R^1$. Let $Y_1, Y_2, \cdots, Y_n, \cdots$, be a sequence of independent and identically distributed (i.i.d) real random variables with a density

$$(3.8)\quad g(y, \theta)=\begin{cases} m(\theta)\,, & \underline{k}(\theta)<y<\bar{k}(\theta)\,; \\ 0\,, & \text{otherwise} \end{cases}$$

where $m(\theta)=1/\{\bar{k}(\theta)-\underline{k}(\theta)\}$ and $\underline{k}(\theta)$ and $\bar{k}(\theta)$ are differentiable in θ and satisfy $\bar{k}'(\theta)\leqq\underline{k}'(\theta)<0$. When $\bar{k}'(\theta)=\underline{k}'(\theta)$, it follows from the results of Weiss and Wolfowitz [11] that the estimator $\underline{k}^{-1}\{[\min X_i+\underline{k}(\bar{k}^{-1}(\max X_i))]/2\}$ is two-sided asymptotically efficient†. It is easily verified that this estimator is AMU. Examples 1 and 3 below illustrate this. From (3.8) it follows that for each $\theta\in\Theta$ and each t there exists

$$\lim_{n\to\infty}\left\{m\left(\theta-\frac{t}{n}\right)\Big/m(\theta)\right\}^n=\exp\{-m'(\theta)/m(\theta)\}t$$
$$=\exp\left\{\frac{\bar{k}'(\theta)-\underline{k}'(\theta)}{\bar{k}(\theta)-\underline{k}(\theta)}\right\}t=\exp\{\alpha(\theta)-\beta(\theta)\}t\,,$$

where $\alpha(\theta)=\bar{k}'(\theta)/(\bar{k}(\theta)-\underline{k}(\theta))$ and $\beta(\theta)=\underline{k}'(\theta)/(\bar{k}(\theta)-\underline{k}(\theta))$.

Remark. Consider more general density $f(x, \theta)$ of the following form than (3.8)

$$\begin{aligned} f(x, \theta)&>0 \quad \text{for } a(\theta)<x<b(\theta)\,; \\ &=0 \quad \text{otherwise}\,. \end{aligned}$$

† This is pointed out by the referee.

Then there may be a measurable transformation φ such that $f(x, \theta)$ is (asymptotically) considered to be changed to a density of the type (3.8) by $y=\varphi(x)$. Such examples will be given later.

We consider three cases in (3.7), i.e. $\lambda>0$, $\lambda=0$ and $\lambda<0$.

(a) *Case* $\lambda>0$. If $0\leqq u\leqq t$, then $\bar{k}(\theta_1)<\bar{k}(\theta_0+u/n)<\bar{k}(\theta_0)$. We have for sufficiently large n

$$\begin{aligned}
\mathrm{E}_{\theta_0}(\phi_n^*)&=P_{\theta_0}^n\left\{\underline{k}(\theta_2)<\min y_i,\ \bar{k}\left(\theta_0+\frac{u}{n}\right)<\max y_i<\bar{k}(\theta_0)\right\}\\
&=\{m(\theta_0)\}^n\{\bar{k}(\theta_0)-\underline{k}(\theta_2)\}^n-\{m(\theta_0)\}^n\left\{\bar{k}\left(\theta_0+\frac{u}{n}\right)-\underline{k}(\theta_2)\right\}^n\\
&\sim\left\{1+\frac{\underline{k}'(\theta_0)t}{(\bar{k}(\theta_0)-\underline{k}(\theta_0))^n}\right\}^n-\left\{1+\frac{\bar{k}'(\theta_0)u+\underline{k}'(\theta_0)t}{(\bar{k}(\theta_0)-\underline{k}(\theta_0))^n}\right\}^n\\
&\sim e^{\beta(\theta_0)t}\{1-e^{\alpha(\theta_0)u}\}\ .
\end{aligned}$$

If we take as u

$$\frac{1}{\alpha(\theta_0)}\log\left\{1-\frac{1}{2}e^{-\beta(\theta_0)t}\right\}$$

for all t satisfying

$$e^{\beta(\theta_0)t}\{1-e^{\alpha(\theta_0)t}\}\geqq\frac{1}{2}\ ,$$

then

$$\mathrm{E}_{\theta_0}^n(\phi_n^*)\to\frac{1}{2}\qquad(n\to\infty)\ .$$

We also obtain for sufficiently large n

$$\begin{aligned}
\mathrm{E}_{\theta_2}^n(\phi_n^*)-\mathrm{E}_{\theta_1}^n(\phi_n^*)&=1-P_{\theta_2}^n\left\{\underline{k}(\theta_2)<\min y_i,\ \max y_i<\bar{k}\left(\theta_0+\frac{u}{n}\right)\right\}\\
&=1-\{m(\theta_2)\}^n\left\{\bar{k}\left(\theta_0+\frac{u}{n}\right)-\underline{k}(\theta_2)\right\}^n\\
&\sim1-e^{\alpha(\theta_0)t}+\frac{1}{2}e^{\{\alpha(\theta_0)-\beta(\theta_0)\}t}\ .
\end{aligned}$$

Hence we have for any $\hat{\theta}_n\in\boldsymbol{A}$

$$\varlimsup_{n\to\infty}P_\theta^n\{n|\hat{\theta}_n-\theta|<t\}=1-e^{\alpha(\theta)t}+\frac{1}{2}e^{\{\alpha(\theta)-\beta(\theta)\}t}\tag{3.9}$$

for all t satisfying $e^{\beta(\theta)t}\{1-e^{\alpha(\theta)t}\}\geqq1/2$.

(b) *Case* $\lambda=0$. We consider the case when $e^{\beta(\theta_0)t}\{1-e^{\alpha(\theta_0)t}\}<1/2$. If If $0\leqq u\leqq t$, then $\underline{k}(\theta_0)<\underline{k}(\theta_0-u/n)<\underline{k}(\theta_2)$. We have for sufficiently large n

$$\begin{aligned}
\mathrm{E}_{\theta_0}^n(\phi_n^*)&=P_{\theta_0}^n\left\{\underline{k}\left(\theta_0-\frac{u}{n}\right)<\min y_i,\ \bar{k}(\theta_1)<\max y_i<\bar{k}(\theta_0)\right\}\\
&=\{m(\theta_0)\}^n\left\{\bar{k}(\theta_0)-\underline{k}\left(\theta_0-\frac{u}{n}\right)\right\}^n\\
&\quad-\{m(\theta_0)\}^n\left\{\bar{k}\left(\theta_0+\frac{t}{n}\right)-\underline{k}\left(\theta_0-\frac{u}{n}\right)\right\}^n\\
&\sim e^{\beta(\theta_0)u}\{1-e^{\alpha(\theta_0)t}\}\ .
\end{aligned}$$

If we take as u

$$-\frac{1}{\beta(\theta_0)}\log 2\{1-e^{\alpha(\theta_0)t}\}$$

for all $t\geqq-(1/\alpha(\theta_0))\log 2$, then

$$\mathrm{E}_{\theta_0}(\phi_n^*)\to\frac{1}{2}\qquad(n\to\infty)\ .$$

We also obtain for sufficiently large n

$$\begin{aligned}
\mathrm{E}_{\theta_2}^n(\phi_n^*)-\mathrm{E}_{\theta_1}^n(\phi_n^*)&=P_{\theta_2}^n\{\underline{k}(\theta_2)<\min y_i,\ \bar{k}(\theta_1)<\max y_i<\bar{k}(\theta_2)\}\\
&=1-P_{\theta_0}^n\{\underline{k}(\theta_2)<\min y_i,\ \max y_i<\bar{k}(\theta_2)\}\\
&\sim 1-e^{2\alpha(\theta_0)t}\ .
\end{aligned}$$

Hence we have for any $\hat{\theta}_n\in\boldsymbol{A}$

$$\varlimsup_{n\to\infty}P_\theta^n\{n|\hat{\theta}_n-\theta|<t\}\leqq 1-e^{2\alpha(\theta)t} \tag{3.10}$$

for all t satisfying $e^{\beta(\theta)t}\{1-e^{\alpha(\theta)t}\}<1/2$ and $t\geqq-(1/\alpha(\theta))\log 2$.

(c) *Case* $\lambda<0$. Consider the case when $e^{\beta(\theta_0)t}\{1-e^{\alpha(\theta_0)t}\}<1/2$ and $t<-(1/\alpha(\theta_0))\log 2$. Let $u\geqq t$ and $\underline{k}(\theta_2)<\bar{k}(\theta_0+u/n)$. Note that if for sufficiently large n, $\bar{k}(\theta_0+u/n)-\underline{k}(\theta_2)\sim 0$, then $u=O(n)$. Indeed since

$$\bar{k}\left(\theta_0+\frac{u}{n}\right)-\underline{k}(\theta_2)\sim(\bar{k}(\theta_0)-\underline{k}(\theta_0))\left\{1+\alpha(\theta_0)\frac{u}{n}+\beta(\theta_0)\frac{t}{n}\right\}\ ,$$

it follows that for sufficiently large n

$$u\sim-\frac{n}{\alpha(\theta_0)}-\frac{\beta(\theta_0)}{\alpha(\theta_0)}t\ .$$

Hence we have

$$u=O(n)\ .$$

We have for sufficiently large n

$$\begin{aligned}E^n_{\theta_0}(\phi^*_n)&=P^n_{\theta_0}\left\{\underline{k}(\theta_2)<\min y_i,\ \bar{k}\left(\theta_0+\frac{u}{n}\right)<\max y_i<\bar{k}(\theta_1)\right\}\\&\quad+P^n_{\theta_0}\{\underline{k}(\theta_0)<\min y_i,\ \bar{k}(\theta_1)<\max y_i<\bar{k}(\theta_0)\}\\&=\{m(\theta_0)\}^n\left\{\bar{k}\left(\theta_0+\frac{t}{n}\right)-\underline{k}\left(\theta_0-\frac{t}{n}\right)\right\}^n\\&\quad-\{m(\theta_0)\}^n\left\{\bar{k}\left(\theta_0+\frac{u}{n}\right)-\underline{k}\left(\theta_0-\frac{t}{n}\right)\right\}^n\\&\quad+P^n_{\theta_0}\{\underline{k}(\theta_0)<\min y_i,\ \bar{k}(\theta_1)<\max y_i<\bar{k}(\theta_0)\}\\&\sim e^{\{\alpha(\theta_0)+\beta(\theta_0)\}t}-e^{\alpha(\theta_0)u+\beta(\theta_0)t}+1-e^{\alpha(\theta_0)t}\ .\end{aligned}$$

If we take as u

$$\frac{1}{\alpha(\theta_0)}\log\left\{e^{\alpha(\theta_0)t}-e^{-\beta(\theta_0)t}\left(e^{\alpha(\theta_0)t}-\frac{1}{2}\right)\right\}$$

for all t satisfying $e^{\alpha(\theta_0)t}\{1-e^{\beta(\theta_0)t}\}\leqq 1/2$, then

$$E_{\theta_0}(\phi^*_n)\to\frac{1}{2}\qquad(n\to\infty)\ .$$

We also obtain for sufficiently large n

$$E^n_{\theta_2}(\phi^*_n)-E^n_{\theta_1}(\phi^*_n)\sim 1-e^{2\alpha(\theta_0)t}+\{1-2e^{\alpha(\theta_0)t}\}\sinh\left[\{\alpha(\theta_0)-\beta(\theta_0)\}t\right]\ .$$

Hence we have for any $\theta_n\in\boldsymbol{A}$

$$\varlimsup_{n\to\infty}P^n_\theta\{n|\hat{\theta}_n-\theta|<t\}\leqq 1-e^{2\alpha(\theta)t}+\{1-2e^{\alpha(\theta)t}\}\sinh[\{\alpha(\theta)-\beta(\theta)\}t] \tag{3.11}$$

for all t satisfying $e^{\beta(\theta)t}\{1-e^{\alpha(\theta)t}\}<1/2$, $t<-(1/\alpha(\theta))\log 2$ and $e^{\alpha(\theta)}\{1-e^{\beta(\theta)t}\}\leqq 1/2$.

Next we consider the case when $e^{\beta(\theta_0)t}\{1-e^{\alpha(\theta_0)t}\}<1/2$, $t<-(1/\alpha(\theta_0))\cdot\log 2$ and $e^{\alpha(\theta_0)t}\{1-e^{\beta(\theta_0)t}\}>1/2$. Let $u\geqq t$ and $\underline{k}(\theta_2)<\bar{k}(\theta_0+u/n)$. We have for sufficiently large n

$$\begin{aligned}E^n_{\theta_0}(\phi^*_n)&=1-P^n_{\theta_0}\left\{\underline{k}(\theta_0)<\min y_i<\underline{k}(\theta_2),\ \max y_i<\bar{k}\left(\theta_0+\frac{u}{n}\right)\right\}\\&=1-\left[\{m(\theta_0)\}^n\left\{\bar{k}\left(\theta_0+\frac{u}{n}\right)-\underline{k}(\theta_0)\right\}^n\right.\\&\quad\left.-\{m(\theta_0)\}^n\left\{\bar{k}\left(\theta_0+\frac{u}{n}\right)-\underline{k}(\theta_2)\right\}^n\right]\\&\sim 1-e^{\alpha(\theta_0)u}+e^{\alpha(\theta_0)u+\beta(\theta_0)t}\ .\end{aligned}$$

If we take as u

$$-\frac{1}{\alpha(\theta_0)}\log\left[2\{1-e^{\beta(\theta_0)t}\}\right],$$

then

$$\mathrm{E}_{\theta_0}(\phi_n^*)\to\frac{1}{2}\qquad(n\to\infty).$$

We also obtain for sufficiently large n

$$\begin{aligned}\mathrm{E}_{\theta_2}^n(\phi_n^*)-\mathrm{E}_{\theta_1}^n(\phi_n^*)=1-\Big[1-P_{\theta_1}^n\Big\{\underline{k}(\theta_0)<\min y_i<\underline{k}(\theta_2),\ \max y_i<\bar{k}\Big(\theta_0+\frac{u}{n}\Big)\Big\}\\ -P_{\theta_1}\{\underline{k}(\theta_1)<\min y_i<\underline{k}(\theta_0),\ \max y_i<\bar{k}(\theta_1)\}\Big]\\ \sim e^{-\{\alpha(\theta_0)-\beta(\theta_0)\}t}[e^{\alpha(\theta_0)u}-e^{\{\alpha(\theta_0)+\beta(\theta_0)\}t}]\\ +P_{\theta_1}^n\{\underline{k}(\theta_1)<\min y_i,\ \max y_i<\bar{k}(\theta_1)\}\\ -P_{\theta_1}^n\{\underline{k}(\theta_0)<\min y_i,\ \max y_i<\bar{k}(\theta_1)\}\\ \sim e^{-\{\alpha(\theta_0)-\beta(\theta_0)\}t}[e^{\alpha(\theta_0)u}-e^{\alpha(\theta_0)u+\beta(\theta_0)t}]+1-e^{\beta(\theta_0)t}\\ =1-e^{\beta(\theta_0)t}+\frac{1}{2}e^{-\{\alpha(\theta_0)-\beta(\theta_0)\}t}.\end{aligned}$$

Hence we have for any $\hat{\theta}_n\in\boldsymbol{A}$

$$\varlimsup_{n\to\infty}P_\theta^n\{n|\hat{\theta}-\theta|<t\}\leqq 1-e^{\beta(\theta)t}+\frac{1}{2}e^{-\{\alpha(\theta)-\beta(\theta)\}t}\tag{3.12}$$

for all t satisfying $e^{\beta(\theta)t}\{1-e^{\alpha(\theta)t}\}<1/2$, $t<-(1/\alpha(\theta))\log 2$ and $e^{\alpha(\theta)t}\{1-e^{\beta(\theta)t}\}>1/2$. In order to summarize the above discussion we divide the range $(0,\infty)$ into the sets of t satisfying following cases (I)–(IV):

(I) $e^{\beta(\theta)t}\{1-e^{\alpha(\theta)t}\}\geqq\frac{1}{2}$;

(II) $e^{\beta(\theta)t}\{1-e^{\alpha(\theta)t}\}<\frac{1}{2}$, $\quad t\geqq-\frac{1}{\alpha(\theta)}\log 2$;

(III) $e^{\beta(\theta)t}\{1-e^{\alpha(\theta)t}\}<\frac{1}{2}$, $\quad t<-\frac{1}{\alpha(\theta)}\log 2$, $\quad e^{\alpha(\theta)t}\{1-e^{\beta(\theta)t}\}\leqq\frac{1}{2}$;

(IV) $e^{\beta(\theta)t}\{1-e^{\alpha(\theta)t}\}<\frac{1}{2}$, $\quad t<-\frac{1}{\alpha(\theta)}\log 2$, $\quad e^{\alpha(\theta)t}\{1-e^{\beta(\theta)t}\}>\frac{1}{2}$;

where

$$\alpha(\theta)=\frac{\bar{k}'(\theta)}{\bar{k}(\theta)-\underline{k}(\theta)};\qquad\beta(\theta)=\frac{\underline{k}'(\theta)}{\bar{k}(\theta)-\underline{k}(\theta)}$$

(see Figure 3.1).

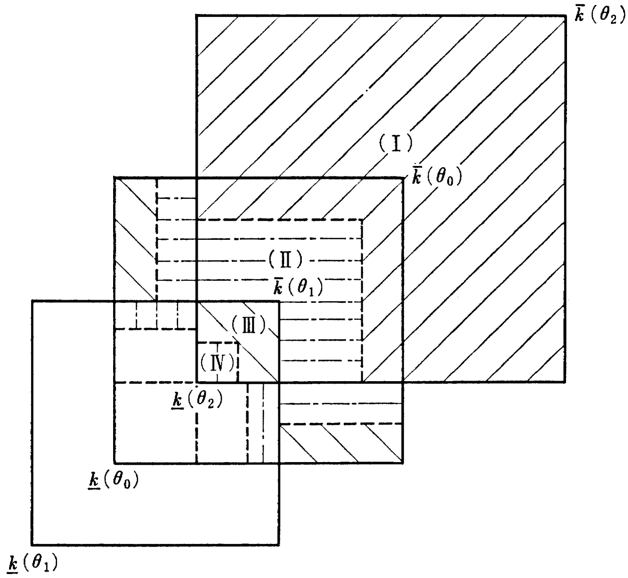

Fig. 3.1. The relation between $\{\phi_n^*=1\}$ and the cases (I)-(IV) in the supports of $g(y, \theta_i)$ $(i=0, 1, 2)$ when one consider the AMU condition.

Remark. If

$$\frac{1}{\alpha(\theta)}\log 2 \geqq \frac{1}{\beta(\theta)}\log\left\{1+\frac{\beta(\theta)}{\alpha(\theta)}\right\}, \tag{3.13}$$

then the following holds:

$$e^{\alpha(\theta)t}\{1-e^{\beta(\theta)t}\}\leqq\frac{1}{2}$$

for all $t<-(1/\alpha(\theta))\log 2$.

From (3.9)-(3.12) we have established the following:

THEOREM. *If there exists an AMU estimator $\hat{\theta}_n^*$ such that*

$$\overline{\lim_{n\to\infty}} P_\theta^n\{n|\hat{\theta}_n^*-\theta|<t\} \tag{3.14}$$

$$=\begin{cases} 1-e^{\alpha(\theta)t}+\dfrac{1}{2}e^{[\alpha(\theta)-\beta(\theta)]t} & \textit{for the case (I)};\\ 1-e^{2\alpha(\theta)t} & \textit{for the case (II)};\\ 1-e^{2\alpha(\theta)t}+\{1-2e^{\alpha(\theta)t}\}\sinh[\{\alpha(\theta)-\beta(\theta)\}t] & \\ & \textit{for the case (III)};\\ 1-e^{\beta(\theta)t}+\dfrac{1}{2}e^{-[\alpha(\theta)-\beta(\theta)]t} & \textit{for the case (IV)}, \end{cases}$$

then $\hat{\theta}_n^*$ *is two-sided asymptotically efficient.*

From the theorem it is seen that the right-hand side of (3.14) is the bound of the asymptotic distributions of $n|\hat{\theta}_n-\theta|$ for all AMU estimators $\hat{\theta}_n$. The bound is also useful in the cases when there may not exist a two-sided asymptotically efficient estimator, i.e. an AMU estimator whose asymptotic distribution uniformly attains it. In fact we may find, even in those cases, an AMU estimator whose asymptotic distribution attains at least at a point, or an AMU estimator whose asymptotic distribution is uniformly "close" to it (see Examples 2 and 4 in Section 4).

4. Examples

In this section we shall apply the theorem to several examples.

Example 1 (Uniform distribution case). Suppose that $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of i.i.d. random variables with a density

$$f(x, \theta)=\begin{cases} 1 & \text{for } -\theta-\frac{1}{2}<x<-\theta+\frac{1}{2}\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

This density is of the form of (3.8) with

$$\text{(i)}\quad m(\theta)\equiv 1\,; \qquad \text{(ii)}\quad \underline{k}(\theta)=-\theta-\frac{1}{2}\,,\quad \bar{k}(\theta)=-\theta+\frac{1}{2}\,.$$

Since this falls in only cases (II) and (III), it follows by the theorem that for any AMU estimator $\hat{\theta}_n$

$$\overline{\lim_{n\to\infty}}\, P_\theta^n\{n|\hat{\theta}_n-\theta|<t\}\leqq 1-e^{-2t}$$

for all $t>0$. Let $\hat{\theta}_n^*=(\min X_i+\max X_i)/2$. Since $\text{mix}\, X_i$ and $\max X_i$ are asymptotically independent, it follows that the asymptotic distribution of $n|\hat{\theta}_n^*-\theta|$ is given by $1-e^{-2t}$. Hence it is seen that $\hat{\theta}_n^*$ is two-sided asymptotically efficient.

Example 2 (Truncated exponential distribution case). Suppose that $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of i.i.d. random variables with a density

$$f(x, \theta)=\begin{cases} ce^{-(x-\theta)} & \text{for } \theta<x<\theta+1\,; \\ 0 & \text{otherwise}\,, \end{cases}$$

where $c=(1-e^{-1})^{-1}$. Transforming $f(x, \theta)$ by $y=ce^{-x}$, we have

$$g(y,\theta)=\begin{cases} e^{\theta} & \text{for } ce^{-(\theta+1)}<y<ce^{-\theta}\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

This density is of the form of (3.8) with

(i) $m(\theta)=e^{\theta}$; (ii) $\underline{k}(\theta)=ce^{-(\theta+1)}$, $\bar{k}(\theta)=ce^{-\theta}$.

Since this falls in only cases (II) and (III) by (3.13), it follows by the theorem that for any AMU estimator $\hat{\theta}_n$

$$\text{(4.1)}\qquad \overline{\lim_{n\to\infty}}\, P_{\theta}^{n}\{n|\hat{\theta}_n-\theta|<t\} \leqq \begin{cases} 1-e^{-2ct}-(1-2e^{-ct})\sinh t & \text{for } 0<t<\dfrac{1}{c}\log 2\,; \\ 1-e^{-2ct} & \text{for } t\geqq \dfrac{1}{c}\log 2\,. \end{cases}$$

Then there may not exist a two-sided asymptotically efficient estimator. Let $\hat{\theta}_{ML}$ be the maximum likelihood estimator. Since $\hat{\theta}_{ML}=\min X_i$ is not AMU, we modify $\hat{\theta}_{ML}$ to be AMU and denote it by $\hat{\theta}_{ML}^{*}$. Then it follows that

$$\hat{\theta}_{ML}^{*}=\hat{\theta}_{ML}-\left(\frac{1}{c}\log 2\right)\frac{1}{n}\,.$$

Since

$$\lim_{n\to\infty} P_{\theta}^{n}\{n|\hat{\theta}_{ML}^{*}-\theta|<t\}=\begin{cases} \sinh ct & \text{for } 0<t<\dfrac{1}{c}\log 2 \\ 1-\dfrac{1}{2}e^{-ct} & \text{for } t\geqq\dfrac{1}{c}\log 2\,, \end{cases}$$

$\hat{\theta}_{ML}^{*}$ is not two-sided asymptotically efficient but the asymptotic distribution of $n|\hat{\theta}_{ML}^{*}-\theta|$ attains at $t=(1/c)\log 2$ the bound of the asymptotic distributions of all AMU estimators, i.e. the right-hand side of (4.1). Let $\hat{\theta}_n^{*}=\max\{\max_i X_i-1, \min_i X_i\}-((1/c)\log 2)/n$. $\hat{\theta}_n^{*}$ is an AMU estimator and the asymptotic distribution of $n|\hat{\theta}_n^{*}-\theta|$ also attains the bound at $t=(1/c)\log 2$. Let $\hat{\theta}_n^{**}=\max\{\max_i X_i-1+v/n, \min_i X_i-t/n\}$, where $v=(e/c)\log 2(1-e^{-ct})$. We consider the case when $t\geqq(1/c)\log 2$. Since $t>v$, it follows that the asymptotic distribution $F(x,t)$ of $n(\hat{\theta}_n^{**}-\theta)$ is given by

$$F(x, t)=\begin{cases} e^{ce^{-1}(x-v)}\{1-e^{-c(x+t)}\} & \text{for } -t<x<v\,; \\ 1-e^{-c(x+t)} & \text{for } x\geqq v\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

It is easily seen that $\hat{\theta}_n^{**}$ is AMU. We also have for every $t\geqq(1/c)\log 2$

$$\text{(4.2)}\quad \lim_{n\to\infty} P_\theta^n\{n|\hat{\theta}_n^{**}-\theta|<x\}$$

$$=\begin{cases} e^{ce^{-1}(x-v)}\{1-e^{-c(x+t)}\}-e^{-ce^{-1}(x+v)}\{1-e^{-c(t-x)}\} & \text{for } 0<x<v\,; \\ 1-e^{-c(x+t)}-e^{ce^{-1}(x+v)}\{1-e^{-c(t-x)}\} & \text{for } v\leqq x<t\,; \\ 1-e^{-c(x+t)} & \text{for } x\geqq t\,, \end{cases}$$

where $v=(e/c)\log 2(1-e^{-ct})$. From (4.1) and (4.2) it follows that the asymptotic distribution of $n|\theta_n^{**}-\theta|$ attains the bound (4.1) of the asymptotic distribution of AMU estimators at an arbitrary point t in $[(1/c)\log 2, \infty)$. Let $\hat{\theta}_{1/2}=(\min X_i+\max X_i-1)/2$ and $\hat{\theta}_{e/(1+e)}=(e/(1+e))\cdot \min X_i+(1/(1+e))(\max X_i-1)$. Then it is easily shown that $\hat{\theta}_{e/(1+e)}$ is asymptotically better than $\hat{\theta}_{1/2}$ uniformly in t, i.e.

$$\lim_{n\to\infty} P_\theta^n\{n|\hat{\theta}_{1/2}-\theta|<t\}<\lim_{n\to\infty} P_\theta^n\{n|\hat{\theta}_{e/(1+e)}-\theta|<t\}$$

for all $t>0$.

Example 3 (Symmetric truncated normal density case). Suppose that $X_1, X_2,\cdots, X_n,\cdots$ is a sequence of i.i.d. random variables with a density

$$f(x-\theta)=\begin{cases} ce^{-(x-\theta)^2/2} & \text{for } \theta-1<x<\theta+1\,; \\ 0 & \text{otherwise}\,, \end{cases}$$

where c is some constant. We consider the likelihood function

$$\begin{aligned} L_n(\tilde{x}_n, \theta)=\prod_{i=1}^n \frac{f(x_i-\theta)}{f(x_i)} &=\exp\left[-\frac{1}{2}\left\{\sum_{i=1}^n (x_i-\theta)^2-\sum_{i=1}^n x_i^2\right\}\right] \\ &=\exp\left(\theta\sum_{i=1}^n x_i-\frac{n}{2}\theta^2\right) \\ &=\exp\left(n\theta\bar{x}-\frac{n\theta^2}{2}\right) \end{aligned}$$

for $\max_i x_i-1<\theta<\min_i x_i+1$, where $\bar{x}=\frac{1}{n}\sum_{i=1}^n x_i$ and $\tilde{x}_n=(x_1,\cdots, x_n)$. Since $\bar{X}$ converges in probability to 0 under P_0 as $n\to\infty$, if $\theta=O(1/n)$, then $L_n(\tilde{X}_n, \theta)$ converges in probability to 1 for some interval under

P_0 as $n \to \infty$. We may consider that the case is asymptotically equivalent to the uniform distribution case. Hence the case is essentially (asymptotically) reduced to that of Example 1.

Example 4 (Asymmetric truncated normal density case). Suppose that $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of i.i.d. random variables with a density

$$f(x-\theta)=\begin{cases} c'e^{-(x-\theta)^2/2} & \text{for } \theta<x<\theta+1\,; \\ 0 & \text{otherwise}\,, \end{cases}$$

where c' is some constant. We consider the likelihood function

$$L_n(\tilde{x}_n, \theta)=\prod_{i=1}^{n} \frac{f(x_i-\theta)}{f(x_i)}=\exp\left(n\theta\bar{x}-\frac{n\theta^2}{2}\right)$$

for $\max_i x_i-1<\theta<\min x_i$. Since $\bar{X}$ converges in probability to constant K under P_0 as $n\to\infty$, where $K=c'(1-e^{-1/2})$, if $\theta=O(1/n)$, then $L_n(\tilde{X}_n, \theta)$ converges in probability to e^{Ku} for some interval of u under P_0 as $n\to\infty$, where u is a real number. We may consider that the case is asymptotically equivalent to the truncated exponential distribution case. Hence the case is essentially (asymptotically) reduced to that of Example 2.

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