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THE DISTRIBUTION AND QUANTILES OF A FUNCTION OF PARAMETER ESTIMATES

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Summary

Let $\hat{\omega}_n$ be an estimate of a parameter ω in R^p, n a known real parameter, and $t(\cdot)$ a real function on R^p. Suppose that the variance of $n^{1/2}(t(\hat{\omega}_n)-t(\omega))$ tends to $\sigma^2>0$ as $n\to\infty$, and that $\hat{\sigma}_n$ is an estimate of σ. We give asymptotic expansions for the distributions and quantiles of

$$Y_{n1}=n^{1/2}\sigma^{-1}(t(\hat{\omega}_n)-t(\omega)) \qquad \text{and} \qquad Y_{n2}=n^{1/2}\hat{\sigma}_n^{-1}(t(\hat{\omega}_n)-t(\omega))$$

to within $O(n^{-5/2})$. It is assumed that (i) $\mathrm{E}\,\hat{\omega}_n\to\omega$ as $n\to\infty$; (ii) $t(\cdot)$ is suitably differentiable at ω; (iii) for $r\geqq 1$ the rth order cross-cumulants of $\hat{\omega}_n$ have magnitude n^{1-r} as $n\to\infty$ and can be expanded as a power series in n^{-1}; (iv) that $\hat{\omega}_n$ has a valid Edgeworth expansion. (Bhattacharya and Ghosh [1] have given easily verifiable sufficient conditions for commonly used statistics like functions of sample moments and the m.l.e.)

As an application we investigate for what parameter ranges common confidence intervals for a linear combination of the means of normal samples are adequate.

1. Introduction

This paper applies the results of Withers [5] to obtain Edgeworth type expansions to within $O(n^{-5/2})$ for the distribution and quantiles of

$$Y_{n1}=n^{1/2}(t(\hat{\omega}_n)-t(\omega))/\sigma \qquad \text{and} \qquad Y_{n2}=n^{1/2}(t(\hat{\omega}_n)-t(\omega))/\hat{\sigma}_n$$

where n is a known real parameter, $t(\cdot)$ is a real function on R^p, $\hat{\omega}_n$ is an estimate of an unknown parameter ω in R^p, $\sigma^2>0$ is the asymptotic variance of $n^{1/2}(t(\hat{\omega}_n)-t(\omega))$, and $\hat{\sigma}_n$ is an estimate of σ. Thus Y_{n1}

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and Y_{n2} are *standardised* and *studentised* forms of $t(\hat{\omega})$, respectively.

When considering Y_{n2} we shall assume that the asymptotic covariance of $n^{1/2}(\hat{\omega}_n-\omega)$ is determined by ω—so that σ^2 is a function of ω, say $\sigma^2=\mathrm{V}(\omega)$, and that $\hat{\sigma}_n^2=\mathrm{V}(\hat{\omega}_n)$: if this is not so one may always augment $\hat{\omega}_n$ and ω to make it so. For example, if one requires an asymptotic expansion for the Behrens-Fisher statistic,

$$Y_{n2}=(\bar{X}_1-\bar{X}_2-\mu_1+\mu_2)(s_1^2/n_1+s_2^2/n_2)^{-1/2},$$

where for $i=1, 2$ $(\bar{X}_i, s_i^2)$ is the usual estimate of (μ_i, σ_i^2) based on a random sample of size n_i from $N(\mu_i, \sigma_i^2)$, then one could take $\omega=(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2)$, $t(\omega)=\mu_1-\mu_2$, $\hat{\omega}_n=(\bar{X}_1, \bar{X}_2, s_1^2, s_2^2)$ and $n=\min(n_1, n_2)$.

In Withers [5] the author gave formulas for Edgeworth-type expansions for the distribution and quantiles of random variables (r.v.s.) whose cumulants satisfy Cornish-Fisher type conditions and have power series expansions: suppose Y_n is a real r.v. whose rth cumulant satisfies

$$\kappa_r(Y_n)\simeq n^{r/2}\sum_{i=r-1}^{\infty} A_{ri}n^{-i}, \qquad r\geqq 1 \text{ with } A_{10}=0,\ A_{21}=1 \tag{1.1}$$

in the sense that as $n\to\infty$

$$\kappa_r(Y_n)=n^{r/2}\left\{\sum_{i=r-1}^{j} A_{ri}n^{-i}+O(n^{-j-1})\right\} \qquad \text{for } j\geqq r-1\geqq 0.$$

Let Φ and ϕ be the distribution and density of a standard normal r.v. Then $P_n(x)=\mathrm{P}(Y_n\leqq x)$ admits the formal asymptotic expansions

$$\begin{aligned} P_n(x)&\simeq\Phi(x)-\phi(x)\sum_{r=1}^{\infty} n^{-r/2}h_r(x), \\ \Phi^{-1}(P_n(x))&\simeq x-\sum_{r=1}^{\infty} n^{-r/2}f_r(x), \\ P_n^{-1}(\Phi(x))&\simeq x+\sum_{r=1}^{\infty} n^{-r/2}g_r(x) \end{aligned} \tag{1.2}$$

where $\{h_r(x), f_r(x), g_r(x)\}$ are polynomials in x and $\{A_{ij}\}$ given by Corollary 3.3 of Withers [5].

For example, $h_1(x)=f_1(x)=g_1(x)=A_{11}+A_{32}(x^2-1)/6$, and $h_2(x)=(A_{11}^2+A_{22})x/2+(4A_{11}A_{32}+A_{43})(x^3-3x)/24+A_{32}^2(x^5-10x^3+15x)/72$.

The coefficients needed to calculate h_r, f_r or g_r are

$$\begin{array}{llllll} A_{11} & & A_{32} & & & \text{for } r=1, \\ & A_{22} & & A_{43} & & \text{for } r=2, \\ A_{12} & & A_{33} & & A_{54} & \text{for } r=3, \\ & A_{23} & & A_{44} & \quad A_{65} & \text{for } r=4, \text{ and so on}. \end{array}$$

Thus the problem of obtaining the asymptotic expansions to within $O(n^{-5/2})$ for Y_{n1} and Y_{n2} reduces to verifying that they satisfy (1.1) for some coefficients $\{A_{ri}\}$ and finding expressions for $A_{11}, A_{32}, \cdots, A_{65}$.

In order to use the results of James and Mayne [4] we assume throughout that

$$\text{(1.3)} \qquad \mathrm{E}\,\hat{\omega}_n \to \omega \qquad \text{as } n \to \infty$$

and the rth order cross-cumulants of $\hat{\omega}_n$ have asymptotic expansions of the form

$$\text{(1.4)} \qquad \kappa^{\sigma_1 \cdots \sigma_r}(\hat{\omega}_n) \simeq \sum_{i=r-1}^{\infty} k_i^{\sigma_1 \cdots \sigma_r} n^{-i}, \qquad 1 \leqq \sigma_j \leqq p,\ 1 \leqq j \leqq r .$$

This condition holds for a very large class of estimates: c.f. Withers [6].

It then follows from James and Mayne [4] that for $t(\cdot)$ a function from R^p to R^q with finite derivatives at ω, the rth order cross-cumulants of $t(\hat{\omega}_n)$ have expansions of the form

$$\text{(1.5)} \qquad \kappa^{\sigma_1 \cdots \sigma_r}(t(\hat{\omega}_n)) \simeq \sum_{i=r-1}^{\infty} K_i^{\sigma_1 \cdots \sigma_r} n^{-i}, \qquad 1 \leqq \sigma_j \leqq q,\ 1 \leqq j \leqq r .$$

Expressions for the leading $\{K_i^{\sigma_1 \cdots \sigma_r}\}$ in terms of $\{k_i^{\sigma_1 \cdots \sigma_r}\}$ and the derivatives of $t(\cdot)$ at ω are given in the appendix.

Setting $q=1$ and $a_{ri} = K_i^{\sigma_1 \cdots \sigma_r}$ it follows that Y_{n1} satisfies the cumulant expansion (1.1) with

$$\text{(1.6)} \qquad A_{ri} = a_{21}^{-r/2} a_{ri} .$$

Expressions for $A_{11}, A_{32}, \cdots, A_{65}$ for Y_{n1} are obtainable thus from the appendix. That (1.1) is satisfied by Y_{n2} now follows by noting that Y_{n2} is just Y_{n1} with $\sigma=1$ and $t(\cdot)$ replaced by $t_{(0)}(\cdot)$ where

$$\text{(1.7)} \qquad t_{(0)}(\hat{\omega}_n) = \mathrm{V}(\hat{\omega}_n)^{-1/2}(t(\hat{\omega}_n) - t(\omega)) .$$

Consequently, $A_{11}, \cdots, A_{65}$ for Y_{n2} are also obtainable from the appendix in terms of the derivatives of $t_{(0)}$ at ω. Thus together with the formulas referred to above for the polynomials occurring in (1.2), this yields expressions for the distributions and quantiles of Y_{n1} and Y_{n2} to within $O(n^{-5/2})$. For further convenience explicit expressions for $A_{11}, A_{32}, \cdots, A_{43}$ for Y_{n1} and Y_{n2} in terms of the derivatives of t at ω, are given in Section 2, thus yielding explicit expressions for the distributions and quantiles of Y_{n1} and Y_{n2} to within $O(n^{-3/2})$ in terms of the derivatives of t at ω and the leading cross-cumulant coefficients of $\hat{\omega}_n$.

Section 3 gives expressions for the error in the tests and confidence intervals for $t(\omega)$ based on the C.L.T. approximation, and investigates the two most commonly used approximate confidence intervals for a linear combination of means of normal populations, $\{N(\mu_i, v_i)\}$ based on sample sizes $\{n_i\}$. In particular we show that

(i) when n_2/n_1 is large the C.I. for $\mu_1-\mu_2$ based on

$$(\mu_1-\mu_2-\hat{\mu}_1+\hat{\mu}_2)(\hat{v}_1/n_1+\hat{v}_2/n_2)^{-1/2} \mathrel{\dot{\sim}} N(0, 1)$$

is unsatisfactory unless v_2/v_1 is very large (>18 for $n_1=5$, $n_2=40$), while that based on

$$(\mu_1-\mu_2-\hat{\mu}_1+\hat{\mu}_2)(\hat{v}_1+\hat{v}_2)^{-1/2}(1/n_1+1/n_2)^{-1/2} \mathrel{\dot{\sim}} t_{n_1+n_2-2}$$

is unsatisfactory unless v_2/v_1 is very close to 1;

(ii) when n_2/n_1 is large the C.I. for the pooled mean $(n_1\mu_1+n_2\mu_2)/(n_1+n_2)$ based on the approximate normality of the studentised statistic is unsatisfactory unless v_2/v_1 is quite large (>2.2 for $n_1=5$, $n_2=40$), but that based on the t-distribution and the assumption of equal variances *is* satisfactory provided v_2/v_1 is not too small (>0.2 for $n_1=5$, $n_2=40$);

(iii) when $n_1=n_2$, the error in the C.I. for $\mu_1-\mu_2 \approx$ the error in the C.I. for the pooled mean and does not depend (to a first appoximation) on whether the statistic used assumes that $v_1=v_2$: the error in the level of the 2-sided C.I. of nominal level $2\Phi(x)-1=$ $2\times$the error in the level of the 1-sided C.I. of nominal level $\Phi(x)$
$$\dot{\approx} \begin{cases} -2(1+t^2)(1+t)^{-2}a(x)n_1^{-1} & \text{using } N(0,1) \\ -(1-t)^2(1+t)^{-2}a(x)n_1^{-1} & \text{using } t_{n_1+n_2-2} \end{cases}$$
where $t=v_2/v_1$ and $a(x)=\phi(x)(x^3+x)/4$ is given in Table 1; thus when $n_1=n_2$ the $t_{n_1+n_2-2}$ approximation is more than twice as accurate as the $N(0, 1)$ approximation and is satisfactory if $n_1>15$ (or if $n_1>8$ and the variances are not too unequal).

2. The cumulant coefficients of Y_{n1} and Y_{n2}

In Section 1 we saw that Y_{n1} satisfies (1.1) with $A_{ri}=a_{21}^{-r/2}a_{ri}$ where a_{ri} is given by setting $q=1$ and $\alpha_j\equiv 1$ in $K_i^{\sigma_1\cdots\sigma_r}$. The appendix then gives $a_{21}, a_{11}, \cdots, a_{65}$. These may be conveniently expressed by setting $t_{i_1\cdots i_r}=\{\partial_{i_1}\cdots\partial_{i_r}t(z)\}_{z=w}$ where $\partial_i=\partial/\partial z_i$, and using $I_{ab\cdots}\begin{pmatrix}\alpha_1\,\alpha_2\cdots\\ \beta_1\,\beta_2\cdots\end{pmatrix}$ to indicate the summation over a product of a first derivatives $\{t_i\}$, b second derivatives $\{t_{ij}\}, \cdots$ and cumulant coefficients with α_k superscripts and β_k (variable) subscripts, $k=1, 2, \cdots$. This notation does not define these terms uniquely. We do so as follows. Suppressing the summation of repeated suffixes over their range $1, \cdots, p$, define $I_2\begin{pmatrix}2\\0\end{pmatrix}=t_ik_1^{ij}t_j$,

$$I_1\begin{pmatrix}1\\0\end{pmatrix}=t_ik_1^i, \quad I_{01}\begin{pmatrix}2\\0\end{pmatrix}=t_{ij}k_1^{ij}, \quad I_3\begin{pmatrix}3\\0\end{pmatrix}=t_it_jt_kk_2^{ijk},$$

$$I_{21}\binom{22}{00}=t_i k_1^{ij} t_{jk} k_1^{kl} t_l\,, \quad \bar{I}_2\binom{2}{0}=t_i k_2^{ij} t_j\,,$$

$$I_{11}\binom{3}{0}=t_i k_2^{ijk} t_{jk}\,, \quad I_{02}\binom{22}{00}=t_{ij} k_1^{jk} t_{kl} k_1^{li}\,,$$

$$I_{101}\binom{22}{00}=t_i k_1^{ij} t_{jkl} k_1^{kl}\,, \quad I_4\binom{4}{0}=t_i t_j t_k t_l k_3^{ijkl}\,,$$

$$I_{31}\binom{23}{00}=t_i t_j k_2^{ijk} t_{kl} k_1^{lm} t_m\,, \quad I_{22}\binom{222}{000}=t_i k_1^{ij} t_{jk} k_1^{kl} t_{lm} k_1^{mn} t_n\,,$$

and $$I_{301}\binom{222}{000}=t_i k_1^{ij} t_{jkl} k_1^{km} t_m k_1^{ln} t_n\,.$$

The coefficients $\{a_{ij}\}$ needed for the first two polynomials in (1.2) are $a_{10}=t(\omega)$, $a_{21}=I_2\binom{2}{0}$,

$$a_{11}=I_1\binom{1}{0}+I_{01}\binom{2}{0}\Big/2\,, \quad a_{32}=I_3\binom{3}{0}+3I_{21}\binom{22}{00}\,,$$

$$a_{22}=\bar{I}_2\binom{2}{0}+I_{11}\binom{3}{0}+I_{02}\binom{22}{00}\Big/2+I_{101}\binom{22}{00}\,,$$

and $$a_{43}=I_4\binom{4}{0}+12I_{31}\binom{23}{00}+12I_{22}\binom{222}{000}+4I_{301}\binom{222}{000}\,.$$

Let $(a_{ri})_r$ denote a_{ri} with t replaced by $t_{(0)}$. These can be expressed directly in terms of the derivatives of t by using $k_{1,i_1\cdots i_r}^{ij}$ to denote $\partial/\partial\omega_{i_1}\cdots\partial/\partial\omega_{i_r}\; k_1^{ij}$ and setting

$$I_3\binom{22}{01}=t_i k_1^{ij} k_{1,j}^{kl} t_k t_l\,, \quad I_2\binom{12}{01}=k_1^i k_{1,i}^{jk} t_j t_k\,,$$

$$I_2\binom{22}{02}=k_1^{ij} k_{1,ij}^{kl} t_k t_l\,, \quad I_{11}\binom{22}{01}=k_1^{ij} t_{jk} k_{1,i}^{kl} t_l\,,$$

$$I_4\binom{23}{10}=t_i t_j k_{1,k}^{ij} k_2^{klm} t_l t_m\,, \quad I_4\binom{222}{020}=t_i k_1^{ij} k_{1,jk}^{lm} k_1^{kn} t_n t_l t_m\,,$$

$$I_4\binom{222}{101}=t_i t_j k_{1,k}^{ij} k_1^{kl} k_{1,l}^{mn} t_m t_n\,, \quad I_{31}\binom{222}{010}=t_i k_1^{ij} t_{jk} k_{1,m}^{kl} t_l k_1^{mn} t_n\,,$$

and $$I_{31}\binom{222}{001}=t_i k_1^{ij} t_{jk} k_1^{kl} k_{1,l}^{mn} t_m t_n\,.$$

Similar terms are easily distinguished by writing them in 'molecular' form, one 'bond' for each repeated suffix. For example

$$I_{02}\binom{22}{00} \text{ is } t_{ij}\left\langle\begin{array}{c} k_1^{jk} \\ k_1^{li} \end{array}\right\rangle t_{kl}\,, \quad I_2\binom{22}{02} \text{ is } k_1^{ij}=k_{1,ij}^{kl}\Big\langle\begin{array}{l} t_k \\ t_l \end{array}.$$

Differentiation yields

$$(a_{10})_1=0\,,\quad (a_{21})_2=1\,,$$

$$(a_{11})_1=I_2\binom{2}{0}^{-1/2}\left\{I_1\binom{1}{0}+I_{01}\binom{2}{0}\Big/2\right\}-I_2\binom{2}{0}^{-3/2}\left\{I_3\binom{22}{01}\Big/2+I_{21}\binom{22}{00}\right\}\,,$$

$$(a_{32})_3=I_2\binom{2}{0}^{-3/2}\left\{I_3\binom{3}{0}-3I_3\binom{22}{01}-3I_{21}\binom{22}{00}\right\}\,,$$

$$\begin{aligned}(a_{22})_2=I_2\binom{2}{0}^{-1}&\left\{\bar{I}_2\binom{2}{0}-I_2\binom{12}{01}-I_2\binom{22}{02}\Big/2+I_{11}\binom{3}{0}-2I_{11}\binom{22}{01}-I_{02}\binom{22}{00}\Big/2\right\}\\&-I_2\binom{2}{0}^{-2}\left[\left\{I_1\binom{1}{0}+I_{01}\binom{2}{0}\Big/2\right\}\left\{I_3\binom{22}{01}+2I_{21}\binom{22}{00}\right\}+I_4\binom{23}{10}\right.\\&+I_4\binom{222}{020}-I_4\binom{222}{101}+2I_{31}\binom{23}{22}+4I_{31}\binom{222}{010}-2I_{31}\binom{222}{001}+2I_{22}\binom{222}{000}\\&\left.+2I_{301}\binom{222}{000}\right]+7I_2\binom{2}{0}^{-3}\left\{I_3\binom{22}{01}+2I_{21}\binom{22}{00}\right\}^2\Big/4\,,\end{aligned}$$

and

$$\begin{aligned}(a_{43})_4=I_2\binom{2}{0}^{-2}&\left\{I_4\binom{4}{0}-6I_4\binom{23}{10}+3I_4\binom{222}{101}-6I_4\binom{222}{020}-12I_{22}\binom{222}{000}\right.\\&\left.-8I_{301}\binom{222}{000}-24I_{31}\binom{222}{010}\right\}-6I_2\binom{2}{0}^{-3}\left\{I_3\binom{22}{01}+2I_{21}\binom{22}{00}\right\}\\&\times\left\{I_3\binom{3}{0}-3I_3\binom{22}{01}-3I_{21}\binom{22}{00}\right\}\,.\end{aligned}$$

In particular when plugged into the expressions in Section 1 for $h_1(x)$ and $h_2(x)$, this gives the distribution of Y_{n1} and Y_{n2} to within $O(n^{-3/2})$.

3. Some examples

In this section we apply the above results to investigate the error in the level of commonly used confidence intervals for a linear combination of the means of normal populations, and discover for what parameter ranges they are adequate.

Tests and confidence intervals (C.I.s) for $t(\omega)$ are often based on $Y_n \stackrel{\cdot}{\sim} N(0,1)$ where $Y_n=Y_{n1}$ or Y_{n2}. Let Y_n be a continuous r.v. whose distribution satisfies (1.2). Given α in $(0,1)$, set $x=\Phi^{-1}(1-\alpha)$. Then the errors in the level of the one-sided test or C.I. of nominal level $1-\alpha$ corresponding to "$Y_n\leqq x$" or "$-x\leqq Y_n$" are

$$e_n(x)=\mathrm{P}(Y_n\leqq x)-(1-\alpha)\simeq-\phi(x)\sum_{r=1}^{\infty}n^{-r/2}h_r(x)\,,$$

and $\mathrm{P}(-x\leqq Y_n)-(1-\alpha)=-e_n(-x)$, respectively. Similarly, the error in the level of the two-sided test or C.I. or nominal level $1-2\alpha$ corresponding to "$|Y_n| \leqq x$" is

$$E_n(x) = P(|Y_n| \leqq x) - (1-2\alpha) = e_n(x) - e_n(-x)$$

$$\simeq 2\phi(x) \sum_{r=1}^{\infty} n^{-r} h_{2r}(x) , \qquad \text{since } h_r(x) \text{ is an even function}$$

for r odd.

Example 3.1. Suppose that random samples of sizes $n_1, \cdots, n_k$ are drawn from k normal populations with associated parameters (μ_j, v_j), $1 \leqq j \leqq k$. Independent estimates $\{\hat{\mu}_j, \hat{v}_j, 1 \leqq j \leqq k\}$ are obtained such that $\hat{\mu}_j \sim N(\mu_j, v_j/n_j)$ and $f_j \hat{v}_j / v_j \sim \chi^2_{f_j}$ where f_j is known. (Typically $f_j = n_j - 1$ but for regression models other values are possible.) To obtain a C.I. for $\sum_1^k a_j \mu_j$ for given weights $\{a_j\}$ it is common to use the approximation $Y_{n2} \sim N(0, 1)$ where

$$Y_{n2} = \hat{\sigma}_n^{-1} \sum_1^k a_j(\hat{\mu}_j - \mu_j) \text{—the generalised Behrens-Fisher statistic—}$$

and

$$\hat{\sigma}_n^2 = \sum_1^k a_j^2 \hat{v}_j / n_j \ .$$

Set $\hat{\omega}_n = (\hat{\mu}_1, \cdots, \hat{\mu}_k, \hat{v}_1, \cdots, \hat{v}_k)$, $\omega = (\mu_1, \cdots, \mu_k, v_1, \cdots, v_k)$, $t(\omega) = \sum_1^k a_j \mu_j$, $n = \sum_1^k n_j$, $\lambda_j = n_j/n$, $\nu_j = f_j/n$, $p = 2k$. (Several alternative parameterisations are available.) Using $\kappa_r(\hat{v}_i) = v_i^r (r-1)! (\nu_i n/2)^{r-1}$, one obtains for Y_{n2}

$$E_n(x)/2 = -e_n(-x) = e_n(x) = -V_{21} a(x)(1 + O(n^{-1}))$$

where $V_{21} = \left(\sum_1^k c_i\right)^{-2} \left(\sum_1^k c_i^2 / f_i\right)$, $c_i = a_i^2 v_i / n_i$, and $a(x) = (x + x^3)\phi(x)/4$ is tabled below.

Table 1

Nominal 2-sided level	50%	80%	90%	95%	98%	99%
Nominal 1-sided level	75%	90%	95%	97.5%	99%	99.5%
$a(x)$	.07796	.1486	.1572	.1387	.09939	.07109

Also $\left(\sum_1^k f_i\right)^{-1} \leqq V_{21} \leqq (\min_i f_i)^{-1}$, (the lower bound being attained when $c_i/f_i \equiv c_1/f_1$ and the upper bound when $c_i = 0$ for $i \neq I$ such that $f_I = \min_i f_i$). Thus from Table 1 we see that for the two-sided 95% C.I. to have error less than 1%, we need $\sum_1^k f_i > 2a(x)/.01 \simeq 27.7$. Similarly for the two-sided 99% C.I. to have error less than (1/2)%, we need $\sum_1^k f_i > 2a(x)/.005 \simeq 28.4$. Thus (i) if the total degrees of freedom is less than 28, this C.I. should be avoided, and (ii) if the minimum degree of freedom is greater than 28, this C.I. is acceptable—in the above sense. In the intermediate case, one may estimate the error by $\hat{\mathrm{E}}_n(x) = -2\hat{V}_{21}a(x)$ where $\hat{V}_{21}$ is V_{21} with $\{c_i\}$ replaced by $\{a_i^2 \hat{v}_i/n_i\}$. For example, if $k=2$, $f_j = n_j - 1$, $n_1 = 5$, $n_2 = 40$, then the two-sided 95% error is less than 1% if $c_2/c_1 > 2.24$; in particular this is so for $a_1 = -a_2 = 1$ if $v_2/v_1 > 18$, and for $a_1 = n_1/(n_1+n_2)$, $a_2 = n_2/(n_1+n_2)$ if $v_2/v_1 > 2.2$. Thus in this intermediate case, (iii) this C.I. is going to perform badly when the variances are fairly equal; and (iv) the error in the C.I. for the difference between means is much more sensitive than the error in the C.I. for the weighted mean.

If instead of $Y_{n2} \sim N(0,1)$, one used a C.I. based on $Y_{n2} \sim t_f$ (Student's distribution with f degrees of freedom), then the error in the level of the nominally $1-\alpha$ level one-sided C.I.s "$-t_{f,1-\alpha} \leqq Y_{n2}$" or "$Y_{n2} \leqq t_{f,1-\alpha}$" is

$$\tilde{e}_n(x) = e_n(t_{f,1-\alpha}) + \Phi(t_{f,1-\alpha}) - (1-\alpha) = (f^{-1} - V_{21})a(x) + O(f^{-2} + n^{-2})$$

and the error in the nominally $1-2\alpha$ level two-sided C.I.

"$|Y_{n2}| < t_{f,1-\alpha}$" is $2\tilde{e}_n(x)$; this error reduces to $O(n^{-2})$ if one chooses $f = \hat{V}_{21}^{-1}$, as defined above. (This uses the result $t_{f,1-\alpha} = x + f^{-1}(x^3+x)/4 + O(f^{-2})$.)

Example 3.2. Another common C.I. for $\sum_1^k a_i\mu_i$ for the situation described in Example 3.1 is that based on the equal variance assumption, i.e.

$$n^{1/2}t(\hat{\omega}_n) \sim t_f$$

where

$$t(\hat{\omega}_n) = V_0(\hat{v})^{-1/2} \sum_1^k a_i(\hat{\mu}_i - \mu_i)\ , \qquad f = \sum_1^k f_j$$

and

$$V_0(v) = f^{-1}\left(\sum_1^k f_i v_i\right)\left(\sum_1^k a_i^2/\lambda_i\right)\ .$$

This only gives an *exact* result when the variances are equal, though the corresponding C.I. is *consistent* when $a_i^2 n_i^{-1} f_i^{-1}$ does not depend on i. The error in the level of the one-sided C.I.s is $e_n^* = P_n(y_n) - (1-\alpha)$ where $y_n = a_{21}^{-1/2} t_{f,1-\alpha}$, $\{a_{ri}\}$ are the cumulant coefficients of $t(\hat{\omega}_n)$, and P_n is the distribution of $Y_n = n^{1/2} a_{21}^{-1/2} t(\hat{\omega}_n)$. Since $t(\hat{\omega}_n)$ is symmetrically distributed $a_{ri} = 0$ for r odd. Section 2 yields

$$a_{21}=\nu(\sum \nu_i v_i)^{-1}(\sum a_i^2/\lambda_i)^{-1}(\sum a_i^2 v_i/\lambda_i) \qquad \text{where } \nu=f/n ,$$

and

$$A_{43}/3=A_{22}=2(\sum \nu_i v_i)^{-2}(\sum \nu_i v_i^2) .$$

(This implies that to within $O(n^{-2})$ *the standardised distribution of* $t(\hat{\omega}_n)$ *does not depend on weights* $\{a_j\}$!). Finally, the one-sided error is

$$\begin{aligned} e_n^* &= \Phi(y_n)-n^{-1}\phi(y)h_2(y_n)+O(n^{-2})-(1-\alpha) \\ &= \Phi(y)+n^{-1}\phi(y)\{a_{21}^{-1/2}(x^3+x)/(4\nu)-A_{22}(y^3+y)/8\}+O(n^{-2})-(1-\alpha) \end{aligned}$$

where $y=a_{21}^{-1/2}x$. (The two-sided error of the nominally $1-2\alpha$ level C.I. is just twice this.) This 'second order' approximation is represented by the solid lines in Figures 1 and 2 below. The dotted lines represent the 'first order' approximation $e_n^* \simeq \Phi(y)-(1-\alpha)$. Figure 1 corresponds to

Case 1: $k=2$, $a_1=-a_2=1$, $f_j=n_j-1$ (a C.I. for $\mu_1-\mu_2$). (The above $\{a_{ri}\}$ agree with (2.28) of Geary [3] as corrected by Gayen [2]). Figures 1 a and 1 b correspond to the two-sided 95% C.I. and the two-sided 99% C.I., respectively, for the case $n_1=5$ and $n_2=5, 10, 20, 40$ as the variance ratio v_2/v_1 ranges from 1/10 to 10. Points to note are (i) the

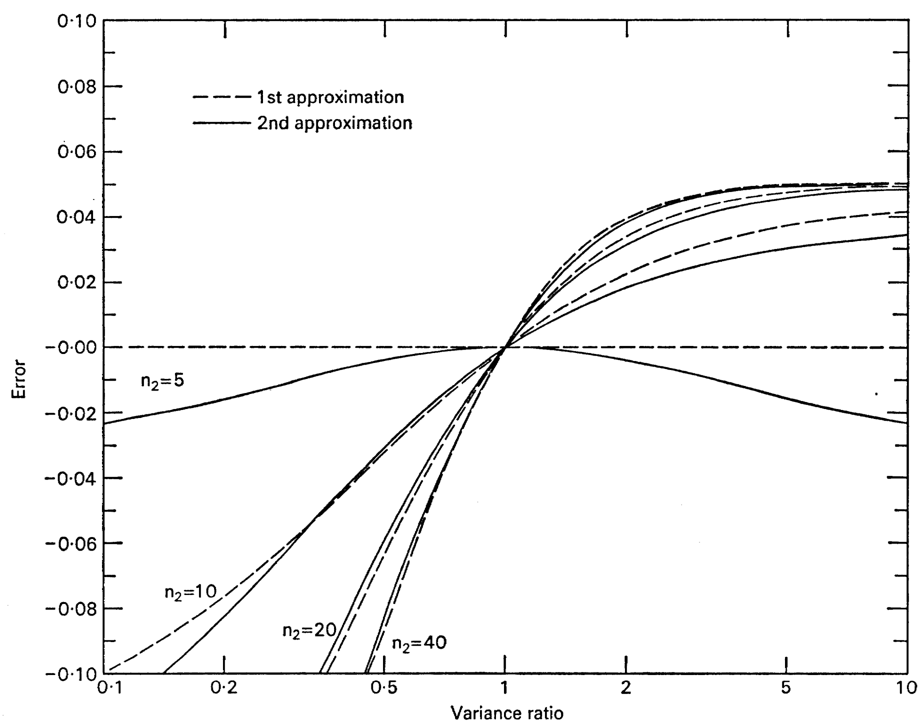

Fig. 1 a. Error in the crude 2-sided 95% Confidence Interval for $\mu_1-\mu_2$ when $n_1=5$

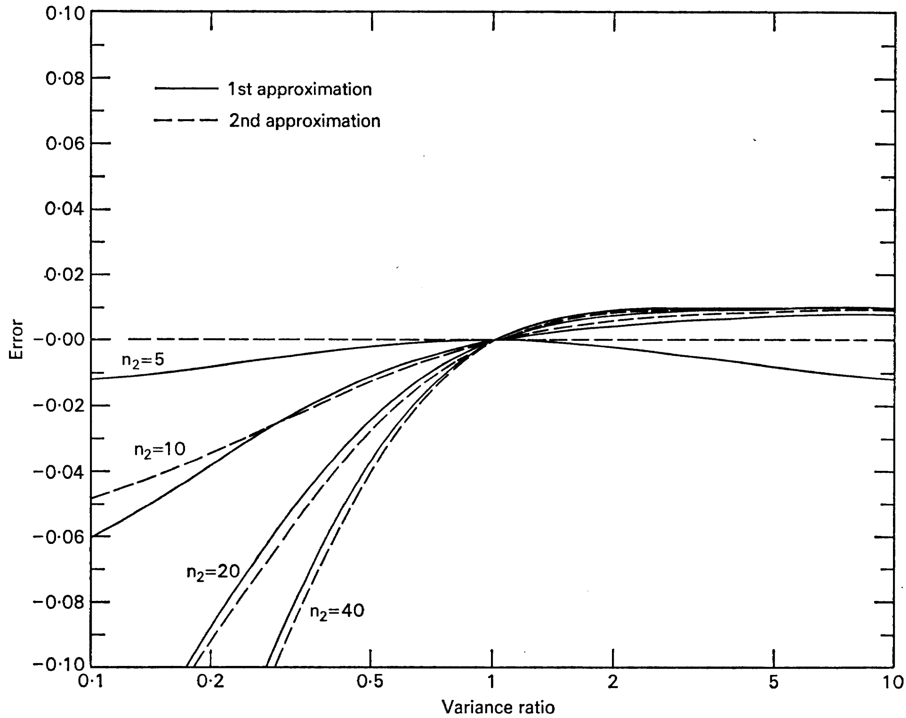

Fig. 1 b. Error in the crude 2-sided 99% Confidence Interval for $\mu_1-\mu_2$ when $n_1=5$.

error for the 99% C.I. is about half that for the 95% C.I.—as is true for the C.I. of Example 3.1; (ii) the first and second order approximations are reasonably close, and their difference decreases as n_2 increases; (iii) the error *increases* as n_2 increases from n_1; (iv) for a two-sided C.I. of level $.95\pm.01$ the variance ratio should lie in (1/3, 3) if $n_1=n_2=5$, in (3/4, 4/3) if $n_1=5$, $n_2=10$ and in (.9, 1.1) if $n_1=5$, $n_2=20$; (v) for a two-sided C.I. of level $.99\pm.005$ the variance ratio should lie in (1/3, 3) if $n_1=n_2=5$, in (2/3, 3) if $n_1=5$, $n_2=10$ and in (.8, 1.6) if $n_1=5$, $n_2=20$.

Case 2: $k=2$, $a_j=\lambda_j$, $f_j=n_j-1$ (a C.I. for the two sample pooled mean). Figures 2 a and 2 b give the error approximations for the two-sided 95% and 99% C.I.s. Points to note are

(i) the error is much smaller than for Case 1 (as noted for the C.I. of Example 3.1 in the 'intermediate' situation)—but the relative difference between the first and second order approximations (on the *different scale*) is much greater;

(ii) the error for the 99% C.I. is again about half that for the 95% C.I.;

(iii) the error is no longer monotonic in n_2 given n_1 and v_2/v_1; it tends to decrease as either n_1 or n_2 increase;

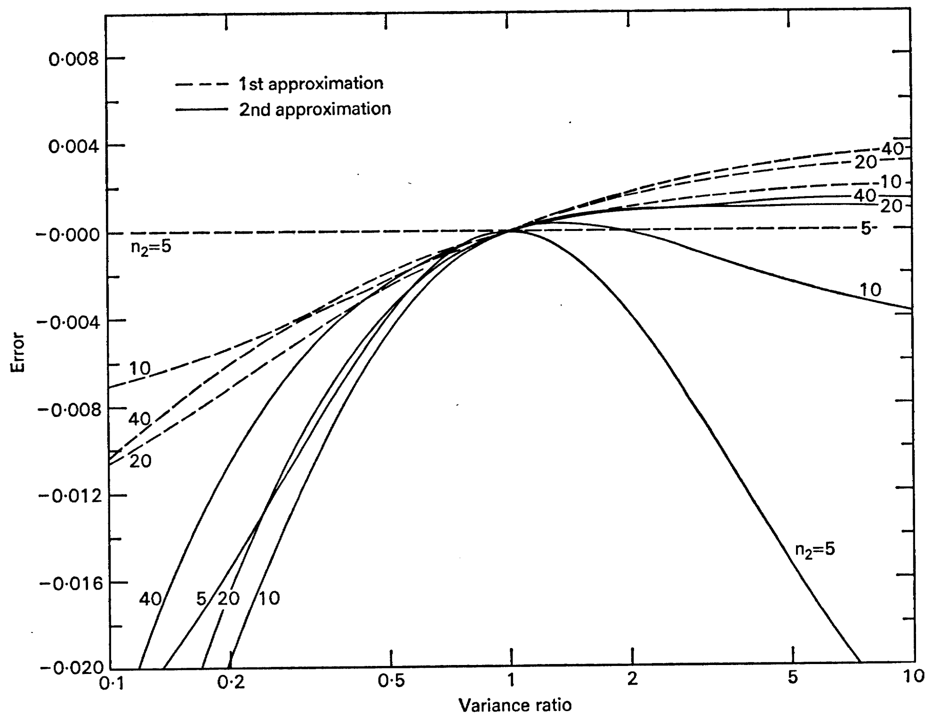

Fig. 2 a. Error in the crude 2-sided 95% Confidence Interval for $(n_1\mu_1+n_2\mu_2)/(n_1+n_2)$ when $n_1=5$.

(iv) for a C.I. of level $.95\pm.01$ the variance ratio should lie in (1/3, 4) if $n_1=n_2=5$ and should be greater than .35 if $n_1=5$, $n_2=10$ or .2 if $n_1=5$, $n_2=40$;

(v) for a C.I. of level $.99\pm.005$ the variance ratio should lie in (1/3, 3) if $n_1=n_2=5$ and should be greater than .3 if $n_1=5$, $n_2=20$ or .2 if $n_1=5$, $n_2=40$.

Example 3.3. It may be of interest to give the first order approximation for $\hat{\omega}_n$ as in Example 3.1 when $t(\omega)$ is an arbitrary function of $\{\mu_i\}$ but does not depend on $\{v_i\}$. In this case we have,

for Y_{n1},

$$h_1(x)=f_1(x)=g_1(x)$$
$$=I_2\binom{2}{0}^{-1/2}I_{01}\binom{2}{0}\Big/2-I_2\binom{2}{0}^{-3/2}I_{21}\binom{22}{00}(1-x^2)/2\ ,$$

and for Y_{n2},

$$h_1(x)=f_1(x)=g_1(x)$$
$$=I_2\binom{2}{0}^{-1/2}I_{01}\binom{2}{0}\Big/2-I_2\binom{2}{0}^{-3/2}I_{21}\binom{22}{00}(1+x^2)/2\ ,$$

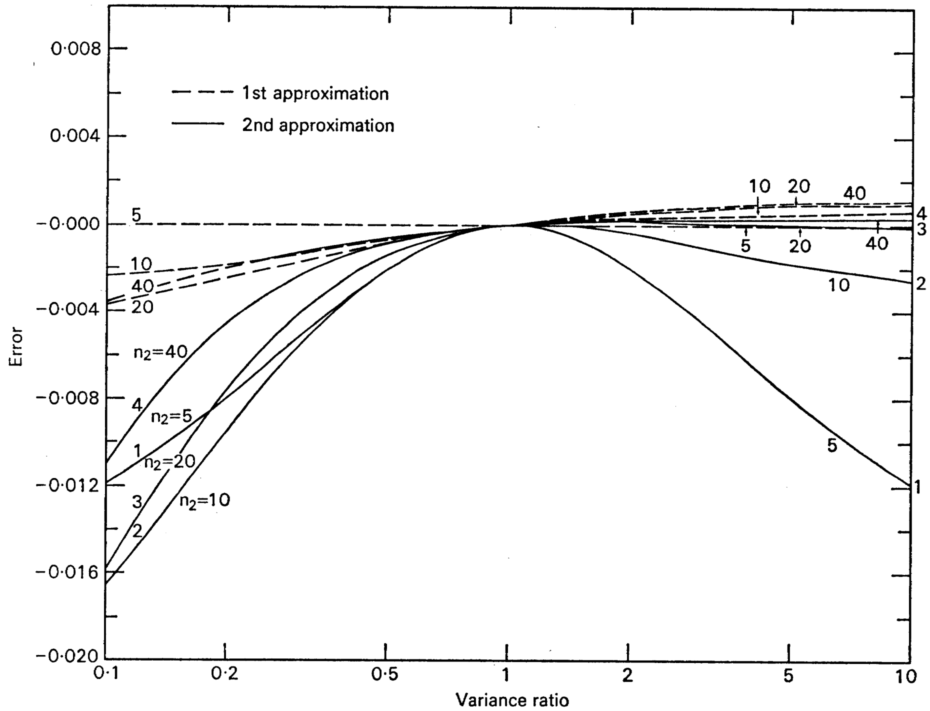

Fig. 2 b. Error in the crude 2-sided 99% Confidence Interval for $(n_1\mu_1+n_2\mu_2)/(n_1+n_2)$ when $n_1=5$.

where $I_2\binom{2}{0}=\sum_1^k t_j^2\tau_j$, $I_{01}\binom{2}{0}=\sum_1^k t_{jj}\tau_j$, $I_{21}\binom{22}{00}=\sum_{i,j=1}^k t_i\tau_i t_{ij}\tau_j t_j$, and $\tau_j=v_j/\lambda_j$.

APPENDIX

Let $t(\cdot)=(t^1(\cdot),\cdots,t^q(\cdot))$ be a q-dimensional function on R^p, and let $\hat{\omega}_n$ be a p-dimensional r.v. satisfying (1.3), (1.4). Then the rth order cross-cumulants of $t(\hat{\omega}_n)$ have the form

$$\text{(A1)}\qquad K^{a_1\cdots a_r}=\kappa^{a_1\cdots a_r}(t(\hat{\omega}_n))\simeq\sum_{i=r-1}^{\infty}K_i^{a_1\cdots a_r}n^{-i}\,,\qquad 1\leq a_j\leq q,\ 1\leq j\leq r\,.$$

Set $A^a=t^a(\omega)$, and $A^a_{i_1\cdots i_r}=[\partial_{i_1}\cdots\partial_{i_r}t^a(x)]_{x=\omega}/r!$ where $\partial_i=\partial/\partial x_i$. The coefficients needed for the rth polynomial in the Edgeworth expansion for the distribution of $t(\hat{\omega}_n)$, $1\leq r\leq 2$, are as follows:

$r=0:\quad K_0^a=A^a=t^a(\omega)\,,$

$\qquad\quad K_1^{ab}=A_i^aA_j^bk_1^{ij}\,,$

$r=1:\quad K_1^a=A_{ij}^ak_1^{ij}+A_i^ak_1^i\,,$

$$K_2^{abc}=A_i^aA_j^bA_k^ck_2^{ijk}+2\sum^3 A_{ik}^aA_j^bA_l^ck_1^{ij}k_1^{kl}\ ,$$

$$r=2:\quad K_2^{ab}=A_i^aA_j^bk_2^{ij}+\sum^2 A_{ij}^aA_k^bk_2^{ijk}+(3\sum^2 A_{ijk}^aA_l^b+2A_{ik}^aA_{jl}^b)k_1^{ij}k_1^{kl}$$
$$+2\sum^2 A_i^aA_{jk}^bk_1^{ik}k_1^j\ ,$$

$$K_3^{abcd}=A_i^aA_j^bA_k^cA_l^dk_3^{ijkl}+2\sum^{12} A_{il}^aA_j^bA_k^cA_m^dk_2^{ijk}k_1^{lm}+(6\sum^4 A_{ikm}^aA_j^bA_l^cA_n^d$$
$$+4\sum^{12} A_{ik}^aA_{jm}^bA_l^cA_n^d)k_1^{ij}k_1^{kl}k_1^{mn}\ .$$

For $r=3, 4$ the coefficients are given by

$$K_i^{i_1\cdots i_r}=\sum_{j=r-1}^{i}\ {}_jV_i^{i_1\cdots i_r}$$

where ${}_jV_i^{i_1\cdots i_r}$ is the coefficient of n^{-i} in the expansion of ${}_jV^{i_1\cdots i_r}$ in powers of n^{-1}, ${}_jV^{i_1\cdots i_r}={}_j\tilde{V}^{i_1\cdots i_r}+{}_j\Delta^{i_1\cdots i_r}$, ${}_j\tilde{V}^{i_1\cdots i_r}$ is the term ${}_j\{\cdot\}$ in the expression for $K^{i_1\cdots i_r}$ given by James and Mayne, ${}_{r-1}\Delta^{i_1\cdots i_r}=0$, and the other $\{{}_j\Delta^{i_1\cdots i_r}\}$ needed for the rth polynomial in the Edgeworth expansion of the distribution and quantiles of $t(\hat{\omega}_n)$, $1\leqq r\leqq 4$, are as follows:

$$r=1:\quad {}_1\Delta^a=A_i^ak^i\ ,$$

$$r=2:\quad {}_2\Delta^{ab}=2\sum^2 A_i^aA_{jk}^bk^{ik}k^j\ ,$$

$$r=3:\quad {}_2\Delta^a=A_{ij}^ak^ik^j+3A_{ijk}^ak^{ij}k^k\ ,$$

$${}_3\Delta^{abc}=2(A_{il}^aA_j^bA_k^c+A_i^aA_{jl}^bA_k^c+A_i^aA_j^bA_{kl}^c)k^{ijk}k^l$$
$$+2\sum^3 (3A_{ikm}^aA_j^bA_l^c+2A_{ik}^aA_{jm}^bA_l^c+2A_{ik}^aA_j^bA_{lm}^c)k^{ij}k^{kl}k^m\ ,$$

$$r=4:\quad {}_3\Delta^{ab}=(3\sum^2 A_i^aA_{jkl}^b+4A_{ik}^aA_{jl}^b)k^{ij}k^kk^l+(2\sum^2 A_{ij}^aA_{kl}^b+3\sum^2 A_{ijl}^aA_k^b)k^{ijk}k^l$$
$$+6\{\sum^2 (A_{ijk}^aA_{lm}^b+2A_{ijkm}^aA_l^b)+A_{ik}^aA_{jlm}^b+A_{ikm}^aA_{jl}^b\}k^{ij}k^{kl}k^m\ ,$$

$${}_4\Delta^{abcd}=2(A_{im}^aA_j^bA_k^cA_l^d+A_i^aA_{jm}^bA_k^cA_l^d+A_i^aA_j^bA_{km}^cA_l^d+A_i^aA_j^bA_k^cA_{lm}^d)k^{ijkl}k^m$$
$$+2\sum^{12} (3A_{ilp}^aA_j^bA_k^cA_m^d+2A_{il}^aA_{jp}^bA_k^cA_m^d+2A_{il}^aA_j^bA_{kp}^cA_m^d$$
$$+2A_{il}^aA_j^bA_k^cA_{mp}^d)k^{ijk}k^{lm}k^p+[12\sum^4 \{2A_{ikmp}^aA_j^bA_l^cA_n^d$$
$$+A_{ikm}^a(A_{jp}^bA_l^cA_n^d+A_j^bA_{lp}^cA_n^d+A_j^bA_l^cA_{np}^d)\}$$
$$+4\sum^{12} \{(3A_{ikp}^aA_{jm}^bA_l^c+3A_{ik}^aA_{jmp}^bA_l^c+2A_{ik}^aA_{jm}^bA_{lp}^c)A_n^d$$
$$+2A_{ik}^aA_{jm}^bA_l^cA_{np}^d\}]k^{ij}k^{kl}k^{mn}k^p\ .$$

(For, James and Mayne [4] give $K^{a_1\cdots a_r}$ in terms of $\{k^{a_1\cdots a_r}=\kappa^{a_1\cdots a_r}(\hat{\omega}_n-\omega)\}$ in the form

$$K^{i_1\cdots i_r}=\sum_{j=r-1}^{k}\ {}_j\tilde{V}^{i_1\cdots i_r}+O(n^{-k-1}) \tag{A2}$$

where ${}_j\tilde{V}^{i_1\cdots i_r}$ consists of the terms of magnitude n^{-j}, *under the assumption that* $\mathrm{E}\,\hat{\omega}_n=\omega$; for example, ${}_1\tilde{V}^a=A^a_{ij}k^{ij}$. To remove this assumption we simply replace $A^a_{i_1\cdots i_r}$ by

$$\tilde{A}^a_{i_1\cdots i_r}=[\partial_{i_1}\cdots\partial_{i_r}t^a(x)]_{x=\mathrm{E}\,\hat{\omega}_n}/r!$$

in their results and substitute the Taylor expansion

$$\text{(A3)}\qquad \tilde{A}^a_{i_1\cdots i_r}=A^a_{i_1\cdots i_r}+\binom{r+1}{1}A^a_{i_1\cdots i_r i}k^i+\binom{r+2}{r}A^a_{i_1\cdots i_r ij}k^ik^j+\cdots,\ r\geqq 0$$

to obtain

$$\text{(A4)}\qquad K^{i_1\cdots i_r}=\sum_{j=r-1}^{k}{}_jV^{i_1\cdots i_r}+O(n^{-k-1})\ .)$$

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