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SELECTION OF CERTAIN DICHOTOMOUS EXPERIMENTS

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Summary

The problem of allocating a single observation to one of the two available populations is considered. Suppose that a certain characteristic has density f in one population, and has density g in the other. On the basis of the value observed, one must specify which population has density f. It is assumed that when a wrong population is chosen, a certain known loss is incurred. The problem is to allocate the observation so as to minimize the expected loss. General conditions on f and g are derived to decide which population should be selected for taking the observation.

1. Introduction and summary

Let f and g be two given probability density functions (p.d.f.'s) with respect to some σ-finite measure μ on an arbitrary sample space. Consider a problem in which two experiments, or populations, X and Y are available to the experimenter, and it is known that one of the following two hypotheses must be true,

$$H_1: \quad X \text{ has density } f \text{ and } Y \text{ has density } g\,,$$

$$H_2: \quad X \text{ has density } g \text{ and } Y \text{ has density } f\,.$$

In this paper, it is assumed that a single observation on either X or Y is available to the experimenter. At first, the experimenter must decide which experiment to select, and after the experiment is performed he must decide which hypothesis should be accepted. In general, the experimenter is faced initially with the problem of deciding how many observations he should take, and in what order he should take them. Design problems of this type have been considered among

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others by Abramson [1], Bradt and Karlin [3], Chernoff [4], and Feldman [6].

For $i=1, 2$, let π_i denote the prior probability that H_i is true ($\pi_1+\pi_2=1$). It is assumed that the loss to the experimenter is 0 units if the correct decision is made and is a_i units ($a_i>0$) when H_i is true and H_{3-i} is accepted. Let $R_X(\pi_1)$ ($R_Y(\pi_1)$) denote the Bayes risk when the experiment X (Y) is performed. Therefore, the experimenter must select an experiment which yields the lower Bayes risk. In Section 2, expressions for $R_X(\pi_1)$ and $R_Y(\pi_1)$ are derived. Theorem 2.1 enables us to decide which experiment is optimal. In Section 3, various conditions on f and g are derived to decide which experiment yields a lower Bayes risk. In Section 4, some examples are presented to illustrate the results obtained.

For $\pi_1 a_1 \geqq \pi_2 a_2$, a common feature of the results given is that if the distribution represented by the p.d.f. f is more concentrated than the distribution represented by g then the optimal experiment is to select the experiment which under H_1 has p.d.f. f.

2. Basic result

For any two non-negative functions r and s that are integrable with respect to an arbitrary σ-finite measure μ, define

$$M(r, s)=\int \min (r(x), s(x))d\mu(x) . \tag{2.1}$$

DEFINITION. Two given p.d.f.'s r and s with respect to μ are said to be mutually disjoint if there exists a set A such that $P_r(A)=\int_A r(x)d\mu(x)=0$ and $P_s(A)=\int_A s(x)d\mu(x)=1$. The set A need not be unique.

A little reflection shows that this definition is indeed symmetric in r and s even though it doesn't look like it. If r and s are p.d.f.'s with respect to μ, the following properties of $M(r, s)$ are obvious and some of them will be used later.

(i) $M(r, s)=M(s, r)$,

(ii) $M(r, s)=1$ iff $r=s$,

(iii) for $k\geqq 0$, $M(kr, ks)=kM(r, s)$,

(iv) for $\alpha\geqq 0$, $0\leqq M(\alpha r, s)\leqq 1$,

(v) for $0\leqq \alpha_1\leqq \alpha_2$, $M(\alpha_1 r, s)\leqq M(\alpha_2 r, s)$,

(vi) $M(r, s)=0$ iff r and s are mutually disjoint, and

(vii) $\lim_{\alpha\to\infty} M(\alpha r, s)<1$ iff there exists a set A such that $P_r(A)=0$ and $P_s(A)>0$.

The following theorem enables us to compare the two experiments X and Y on the basis of their Bayes risks.

THEOREM 2.1. *If* $\pi_2>0$ *and* $\alpha=\pi_1 a_1/\pi_2 a_2$, *then*

$$R_X(\pi_1)\leqq R_Y(\pi_1) \quad \text{if and only if} \quad M(\alpha f, g)\leqq M(f, \alpha g) . \tag{2.2}$$

PROOF. Suppose that we allocate the observation to population X and observe $X=x$. For $i=1, 2$, let d_i denote the Bayes decision to accept the hypothesis H_i. Thus,

$$\begin{aligned} R_X(\pi_1) &= \pi_1 a_1 \,\mathrm{P}\,[d_2\,|\,H_1]+\pi_2 a_2 \,\mathrm{P}\,[d_1\,|\,H_2] \\ &= \pi_1 a_1 \int_{\{x:\, \pi_1 a_1 f(x)\leqq \pi_2 a_2 g(x)\}} f(x)d\mu(x)+\pi_2 a_2 \int_{\{x:\, \pi_1 a_1 f(x)>\pi_2 a_2 g(x)\}} g(x)d\mu(x) \\ &= M(\pi_1 a_1 f(x), \pi_2 a_2 g(x))=\pi_2 a_2 M(\alpha f, g) . \end{aligned} \tag{2.3}$$

Similarly, it can be shown that

$$R_Y(\pi_1)=\pi_2 a_2 M(f, \alpha g) . \tag{2.4}$$

The theorem now follows from (2.3) and (2.4).

Remarks. The optimal choice of the experiment depends only on f, g and the value of α. If $\alpha=1$, or if f and g are mutually disjoint, then $R_X(\pi_1)=R_Y(\pi_1)$. Also, if the experiments X and Y are sufficient for each other then the two experiments yield the same Bayes risk. The concept of sufficient experiments is due to Blackwell [2], and is discussed in DeGroot [5] and Lehmann [7]. Finally, if $M(\alpha_0 f, g)<M(f, \alpha_0 g)$ for some value of $\alpha_0>0$, then $M((1/\alpha_0)f, g)>M(f, (1/\alpha_0)g)$. This relation implies that if the experiment X should be performed when $\alpha=\alpha_0$ then the experiment Y should be performed when $\alpha=1/\alpha_0$. Hence, we shall assume from now on without loss of generality that $\alpha\geqq 1$.

In the next section, it is shown that if f and g satisfy certain conditions, then

$$M(\alpha f, g)\leqq M(f, \alpha g) \quad \text{for all values of } \alpha\geqq 1 . \tag{2.5}$$

Therefore, in such a case, experiment X will be selected by every experimenter whose subjective probabilities π_1 and π_2 satisfy the relation $\pi_1/\pi_2\geqq a_2/a_1$, regardless of the exact values of π_1 and π_2.

The following theorem provides an alternative method for the comparison of X and Y.

THEOREM 2.2. *Suppose that f and g are p.d.f.'s satisfying*

$$\mathrm{P}_f\,[x: g(x)=0]=\mathrm{P}_g\,[x: f(x)=0] . \tag{2.6}$$

Then $R_X(\pi_1)\leqq R_Y(\pi_1)$ *if and only if*

$$\text{(2.7)} \qquad \mathrm{E}_f\,[v_2(x,\alpha)] \geqq \mathrm{E}_g\,[v_1(x,\alpha)]\ ,$$

where $v_1(x,\alpha)=\max\,[f(x)/g(x)-\alpha,\ 0]$ *and* $v_2(x,\alpha)=\max\,[g(x)/f(x)-\alpha,\ 0]$.

PROOF. Define,

$$A_1=[x: \alpha f(x)\leqq g(x)] \qquad \text{and} \qquad A_2=[x: \alpha g(x)\leqq f(x)]\ .$$

By definition,

$$\begin{aligned} M(\alpha f, g) &= \int_{[x:\ \alpha f(x)\leqq g(x)]} \alpha f(x)d\mu(x)+\int_{[x:\ g(x)<\alpha f(x)]} g(x)d\mu(x) \\ &= \int_{A_1} \alpha f(x)d\mu(x)+1-\int_{A_1} g(x)d\mu(x)\ . \end{aligned}$$

Similarly,

$$M(f, \alpha g)=\int_{A_2} \alpha g(x)d\mu(x)+1-\int_{A_2} f(x)d\mu(x)\ .$$

Therefore,

$$\begin{aligned} d(\alpha) &= M(\alpha f, g)-M(f, \alpha g) \\ &= \int_{A_2} (f(x)-\alpha g(x))d\mu(x)-\int_{A_1} (g(x)-\alpha f(x))d\mu(x)\ , \end{aligned}$$

using (2.6), we get

$$\begin{aligned} &= \int_{A_2} \left(\frac{f(x)}{g(x)}-\alpha\right)g(x)d\mu(x)-\int_{A_1} \left(\frac{g(x)}{f(x)}-\alpha\right)f(x)d\mu(x)\ , \\ &= \mathrm{E}_g\,[v_1(x,\alpha)]-\mathrm{E}_f\,[v_2(x,\alpha)]\ . \end{aligned}$$

Since $R_X(\pi_1)\leqq R_Y(\pi_1)$ if and only if $d(\alpha)\leqq 0$, the theorem now follows.

3. General conditions on f and g which imply that $M(\alpha f, g)\leqq M(f, \alpha g)$

In this section, some conditions on f and g are obtained which imply the relation $M(\alpha f, g)\leqq M(f, \alpha g)$ and therefore the relation $R_X(\pi_1)\leqq R_Y(\pi_1)$. The following notation is developed for later use in this section. Define,

$$D=\{x: f(x)g(x)=0\}\ ,$$

$$B_{1\alpha}=\left[x: \frac{f(x)}{g(x)}\geqq\alpha,\ x\notin D\right]\ , \qquad B_{2\alpha}=\left[x: \frac{g(x)}{f(x)}\geqq\alpha,\ x\notin D\right]\ ,$$

$$B_{3\alpha}=\left[x: \frac{f(x)}{g(x)}<\alpha,\ x\notin D\right]\ , \qquad B_{4\alpha}=\left[x: \frac{g(x)}{f(x)}<\alpha,\ x\notin D\right]\ ,$$

$$r_1=\sup_{x\in B_{11}} \frac{f(x)}{g(x)}\ , \qquad r_2=\sup_{x\in B_{21}} \frac{g(x)}{f(x)}\ .$$

It can happen that B_{11} or B_{21} is an empty set. When both B_{11} and B_{21} are empty sets, then f and g are mutually disjoint. We shall define r_1 (r_2) to be zero when B_{11} (B_{21}) is an empty set.

In certain problems, it is easy to decide which experiment is optimal for large values of α. If $\alpha \geqq r_1$, then $\alpha g(x) \geqq f(x)$ for $x \in D^c$, where D^c is complement of the set D. Hence $M(f, \alpha g) = \int_{D^c} f(x) d\mu(x)$. Similarly, for $\alpha \geqq r_2$, $M(\alpha f, g) = \int_{D^c} g(x) d\mu(x)$. When $\alpha \geqq \max(r_1, r_2)$, the following result becomes obvious:

$$M(\alpha f, g) \leqq M(f, \alpha g) \qquad \text{iff } P_f(D) < P_g(D) .$$

THEOREM 3.1. *If $r_1 < r_2$ and $P_f(D) \leqq P_g(D)$ then X is optimal for any value of α in the interval $r_1 \leqq \alpha \leqq r_2$.*

PROOF. Since

$$M(\alpha f, g) \leqq M(r_2 f, g) = \int_{D^c} g(x) d\mu(x) \leqq \int_{D^c} f(x) d\mu(x) .$$

Also, since $M(f, \alpha g) = \int_{D^c} f(x) d\mu(x)$, the result now follows.

Given two sets A and B, let $A - B$ denote their difference set. The following result gives a set of sufficient conditions for experiment X to be optimal.

LEMMA 3.1. *Suppose that $\alpha > 1$ and suppose that the p.d.f.'s f and g satisfy the following relations:*

$$\int_{B_{2\alpha}} f(x) d\mu(x) \leqq \int_{B_{1\alpha}} g(x) d\mu(x) \tag{3.1}$$

and

$$\int_{B_{3\alpha} \cap B_{4\alpha}} g(x) d\mu(x) \leqq \int_{B_{3\alpha} \cap B_{4\alpha}} f(x) d\mu(x) . \tag{3.2}$$

Then $M(\alpha f, g) \leqq M(f, \alpha g)$.

PROOF. It can be easily shown that

$$M(\alpha f, g) - M(f, g) = (\alpha - 1) \int_{B_{2\alpha}} f(x) d\mu(x) + \int_{B_{4\alpha} - B_{41}} (g(x) - f(x)) d\mu(x)$$

and

$$M(f, \alpha g) - M(f, g) = (\alpha - 1) \int_{B_{1\alpha}} g(x) d\mu(x) + \int_{B_{3\alpha} - B_{31}} (f(x) - g(x)) d\mu(x) .$$

Since

$$(B_{3\alpha} - B_{31}) \cup (B_{4\alpha} - B_{41}) = B_{3\alpha} \cap B_{4\alpha} ,$$

it now follows from the given conditions that $d(\alpha)\leqq 0$.

In the remainder of this section, the sample space S is assumed to be the real line. Define,

$$H_f(\alpha)=\mathrm{P}_f\left[x:\frac{f(x)}{g(x)}\leqq\alpha,\ x\notin D\right]$$

and

$$H_g(\alpha)=\mathrm{P}_g\left[x:\frac{f(x)}{g(x)}\leqq\alpha,\ x\notin D\right].$$

In other words, $H_f(\alpha)$ and $H_g(\alpha)$ are analogous to the distribution function of the likelihood ratio f/g when X has densities f and g, respectively. Let $h_f(\alpha)$ and $h_g(\alpha)$ denote the derivatives of $H_f(\alpha)$ and $H_g(\alpha)$ when they exist. The following theorem gives a sufficient condition for the relation $M(\alpha f, g)\leqq M(f, \alpha g)$ to hold.

THEOREM 3.2. *Suppose that for some constant $c>1$, the p.d.f.'s f and g are such that $H_f(\alpha)$ and $H_g(\alpha)$ are differentiable functions of α in the interval $1/c<\alpha<c$, and suppose also that for all α in the interval $1<\alpha\leqq c$,*

$$\int_{B_{2\alpha}} f(x)dx\leqq\int_{B_{1\alpha}} g(x)dx\ . \tag{3.3}$$

Then $M(\alpha f, g)<M(f, \alpha g)$ for all α in the interval $1<\alpha\leqq c$.

PROOF. It is known that,

$$\begin{aligned} d(\alpha)&=\int_{B_{2\alpha}}\alpha f(x)dx+\int_{B_{4\alpha}} g(x)dx-\int_{B_{3\alpha}} f(x)dx-\int_{B_{1\alpha}}\alpha g(x)dx \\ &=\alpha H_f\left(\frac{1}{\alpha}\right)+1-P_g(D)-H_g\left(\frac{1}{\alpha}\right)-H_f(\alpha)-\alpha[1-P_g(D)-H_g(\alpha)]\ . \end{aligned} \tag{3.4}$$

Also,

$$\begin{aligned} h_f(\alpha)&=\lim_{\Delta\to 0}\mathrm{P}_f\left[x:\alpha\leqq\frac{f(x)}{g(x)}\leqq\alpha+\Delta,\ x\notin D\right]\frac{1}{\Delta} \\ &\leqq\lim_{\Delta\to 0}(\alpha+\Delta)\,\mathrm{P}_g\left[x:\alpha\leqq\frac{f(x)}{g(x)}\leqq\alpha+\Delta,\ x\notin D\right]\frac{1}{\Delta} \\ &=\alpha h_g(\alpha)\ . \end{aligned}$$

Similarly, it can be shown that $h_f(\alpha)\geqq\alpha h_g(\alpha)$, and therefore it follows that $h_f(\alpha)=\alpha h_g(\alpha)$. A similar argument also shows that $h_f(1/\alpha)=(1/\alpha)\cdot h_g(1/\alpha)$. Differentiating (3.4) and using the relations established above, we get

$$d'(\alpha)=H_f\left(\frac{1}{\alpha}\right)+H_g(\alpha)+P_g(D)-1 . \tag{3.5}$$

It follows from (3.3) that $d'(\alpha)\leqq 0$ for all α in the interval $1<\alpha\leqq c$. Therefore $d(\alpha)$ is a decreasing function and the theorem now follows.

Suppose now that the set D^c is an interval (a, b) of the real line, possibly infinite, and that $f(x)$ and $g(x)$ are p.d.f.'s such that $f(x)/g(x)$ is a strictly decreasing continuous function of x on D^c. If f and g are such that

$$\frac{f(x)}{g(x)}\geqq 1 \qquad \text{for all } x \notin D^c , \tag{3.6}$$

then it can be shown that (2.5) holds. Now, suppose that f and g are such that the relation (3.6) does not hold. Let x_0, x_1, and x_2 denote the points of intersection of $f(x)$ and $\alpha g(x)$ $(\alpha\leqq r_1)$, $f(x)$ and $g(x)$, and $\alpha f(x)$ and $g(x)$ $(\alpha\leqq r_2)$, respectively. It is clear that x_0, x_1, and x_2 exist and are unique. The following theorem gives a sufficient condition for the experiment X to be optimal.

THEOREM 3.3. *Suppose that D^c is an interval (a, b) of the real line and that $f(x)/g(x)$ is a strictly decreasing continuous function on D^c. Suppose also that $r_1<r_2$ and $P_f(D)\leqq P_g(D)$. Finally suppose that for all α such that $1<\alpha\leqq r_1$,*

$$f(x_2)\frac{dx_2}{d\alpha}+g(x_0)\frac{dx_0}{d\alpha}\leqq 0 . \tag{3.7}$$

Then $M(\alpha f, g)\leqq M(f, \alpha g)$ for all values of $\alpha\geqq 1$.

PROOF. By definition,

$$d(\alpha)=\int_a^{x_2} g(x)dx+\int_{x_2}^b \alpha f(x)dx-\int_a^{x_0} \alpha g(x)dx-\int_{x_0}^b f(x)dx .$$

Differentiating $d(\alpha)$ twice and using (3.7) we can show that $d''(\alpha)\geqq 0$ for all values of α in the interval $1<\alpha\leqq r_1$. Thus $d(\alpha)$ is convex function with $d(1)=0$ and $d(r_1)<0$. The theorem now follows.

The next lemma and theorem provide another set of conditions for (2.5) to hold.

LEMMA 3.2. *If $l(x)=\log(f(x)/g(x))$ is a decreasing continuous, concave and differentiable function of x $(x\geqq 0)$, then*

$$\frac{dx_2}{d\alpha}\leqq\frac{-dx_0}{d\alpha} \qquad (1\leqq\alpha\leqq r_1) .$$

PROOF. By definition, $l(x_0)=\log \alpha$ and $l(x_2)=-\log \alpha$. Given $\varepsilon>0$, define x_0' and x_2' such that $l(x_0')=\log(\alpha+\varepsilon)$ and $l(x_2')=-\log(\alpha+\varepsilon)$. It should also be noted that $x_0'<x_0<x_2<x_2'$, i.e., $dx_0/d\alpha$ and $dx_2/d\alpha$ are of opposite signs. To show that $dx_2/d\alpha \leqq -dx_0/d\alpha$, one need only show that $x_2'-x_2 \leqq x_0-x_0'$. If $x_2''=x_2+(x_0-x_0')$, then it suffices to show that $x_2''>x_2'$, i.e., $-l(x_2'')\geqq -l(x_2')=l(x_0')$. Now $-l(x_2'')\geqq l(x_0')$. By the mean value theorem, there exist points $\xi \in [x_0', x_0]$ and $\eta \in [x_2, x_2'']$ such that $l(x_0)-l(x_0')=(x_0-x_0')l'(\xi)$ and $l(x_2'')-l(x_2)=(x_2''-x_2)l'(\eta)$. Therefore, it follows that $-l(x_2'')\geqq l(x_0')$ if and only if $-l'(\eta)\geqq -l'(\xi)$.

Since $\xi>\eta$ and l is concave, it follows that $l'(x)$ is a decreasing function of x and hence $l'(\xi)\geqq l'(\eta)$. Therefore $x_2'-x_2\leqq x_0-x_0'$ and the result follows.

THEOREM 3.4. *Suppose that $f(x)$ and $g(x)$ are decreasing continuous functions of x on D^c. Suppose also that $r_1<r_2$, $P_f(D)\leqq P_g(D)$, and $\log(f(x)/g(x))$ is a decreasing, concave and differentiable function of x. Then $M(\alpha f, g)\leqq M(f, \alpha g)$ for all values of $\alpha\geqq 1$.*

PROOF. By Lemma 3.2, since $\log(f(x)/g(x))$ is concave we have $dx_2/d\alpha\leqq -dx_0/d\alpha$ $(1\leqq\alpha\leqq r_1)$. Therefore, in order to establish the relation (3.7) of Theorem 3.3, it is sufficient to show that $f(x_2)\leqq g(x_0)$, which is clearly true.

COROLLARY 3.1. *Suppose that $f(x)$ is a decreasing continuous function of x defined on R^+, the positive part of the real line. Suppose also that there exists a constant c $(0<c<1)$, such that $g(x)=cf(cx)$ for all $x>0$. If $\lim_{x\to\infty}(f(x)/f(cx))=0$ and $\log(f(x)/cf(cx))$ is a decreasing, concave and differentiable function of x $(x\geqq 0)$, then $M(\alpha f, g)<M(f, \alpha g)$ for all values of $\alpha\geqq 1$.*

COROLLARY 3.2. *Suppose that $f(x)$ is a decreasing continuous function of x defined on $[0, a]$ such that $f(a)=0$. Suppose also that there exists a constant c $(0<c<1)$ such that $g(x)=cf(cx)$ for all $x>0$. If $\log(f(x)/g(x))$ is a decreasing, concave and differentiable function of x, then $M(\alpha f, g)\leqq M(f, \alpha g)$ for all values of $\alpha\geqq 1$.*

PROOF. Since $r_2=\infty$ and r_1 is finite, it follows that $r_1<r_2$. Also we must have $0=P_f(D)\leqq P_g(D)$. Therefore the result follows by Theorem 3.3.

4. Examples

In this section we shall present some examples to illustrate some of the results presented in this paper.

Example 1. Suppose that f and g are uniform p.d.f.'s on $[0, a]$ and $[0, b]$, respectively, where $0<a<b$. Then for $\alpha>1$,

$$M(\alpha f, g)=\int_0^a \frac{1}{b}dx=\frac{a}{b} ,$$

and

$$M(f, \alpha g)=\begin{cases} \alpha\frac{a}{b} & \text{if } \alpha\leqq\frac{b}{a} , \\ 1 & \text{if } \alpha\geqq\frac{b}{a} . \end{cases}$$

It follows that (2.5) holds and that the experiment X should be performed.

It should be noted that in this example, $P_g[x: f(x)=0]=(b-a)/b>0$. It follows that assumption (2.6) does not hold and therefore (2.5) cannot be established by using Theorem 2.2.

However, since $B_{2\alpha}=\phi$, the empty set, for $\alpha\geqq1$, it follows that (3.3) holds for all values of $\alpha\geqq1$, and from Theorem 3.2 it can again be established that (2.5) holds.

Example 2. Suppose that f and g are two exponential p.d.f.'s defined as follows, for some constant $\beta>1$:

$$f(x)=\begin{cases} \beta e^{-\beta x} & x\geqq0 , \\ 0 & \text{otherwise} ; \end{cases}$$

$$g(x)=\begin{cases} e^{-x} & x\geqq0 , \\ 0 & \text{otherwise} . \end{cases}$$

It is possible to show with some effort that (2.5) holds by direct evaluation of $M(\alpha f, g)$ and $M(f, \alpha g)$. However, we shall show that condition (3.7) of Theorem 3.3 is satisfied for this example, and therefore (2.5) must hold.

Solving $f(x)=\alpha g(x)$ for $\alpha\leqq\beta$, and $\alpha f(x)=g(x)$, we obtain

$$x_0=\frac{1}{\beta-1}\log\left(\frac{\beta}{\alpha}\right)$$

and

$$x_2=\frac{1}{\beta-1}\log(\alpha\beta) .$$

Hence

$$\frac{dx_0}{d\alpha}=\frac{-dx_2}{d\alpha}=\frac{-1}{\alpha(\beta-1)} .$$

Also $D=[x: -\infty<x<0]$, $r_1=\beta$ and $r_2=\infty$. Thus $0=P_f(D)=P_g(D)$ and $r_1<r_2$.

It can now be shown that (3.7) holds, therefore, it follows from Theorem 3.3 that the experiment X yields a lower Bayes risk than the experiment Y for all values of $\alpha\geqq 1$.

Example 3. Suppose that f_1 and g_1 are two normal p.d.f.'s defined as follows, for some constant $\sigma^2>1$:

$$f_1(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \qquad -\infty<x<\infty ,$$

and

$$g_1(x)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-x^2/2\sigma^2} \qquad -\infty<x<\infty .$$

Define $f(x)$ and $g(x)$ by

$$f(x)=\begin{cases} \dfrac{2}{\sqrt{2\pi}}e^{-x^2/2} & 0<x<\infty , \\ 0 & \text{otherwise} ; \end{cases}$$

$$g(x)=\begin{cases} \dfrac{2}{\sqrt{2\pi}\,\sigma}e^{-x^2/2\sigma^2} & 0<x<\infty , \\ 0 & \text{otherwise} . \end{cases}$$

It is clear that $M(\alpha f_1, g_1)\leqq M(f_1, \alpha g_1)$ if and only if $M(\alpha f, g)\leqq M(f, \alpha g)$. As in Example 2, it can be shown that (3.7) holds for $1\leqq\alpha\leqq\sigma$, and it follows from Theorem 3.3 that (2.5) holds.

However, we shall obtain the same result using Corollary 3.1. Since $l=\log(f(x)/g(x))=\log\sigma-cx^2$, where $c=-(1-1/\sigma^2)/2$, it follows by differentiating twice that we have $d^2l/dx^2<0$. Therefore, l is concave. Hence the experiment X should be performed.

5. Conclusion

In each of the above examples, it is interesting to note that the distribution represented by the p.d.f. f is more concentrated than the distribution represented by g, and the optimal experiment is to select the experiment which under H_1 has p.d.f. f. Some other examples considered also exhibit the same feature. It is difficult to give a precise meaning to the concept of one distribution being more concentrated than another. An intuitive grasp can be obtained by noting that in each of the examples given here, the variance of the distribution represented by f is smaller than the variance of the distribution by g.

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