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ON STOPPING TIMES OF SEQUENTIAL ESTIMATIONS OF THE MEAN OF A LOG-NORMAL DISTRIBUTION

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1. Introduction

Let $X, X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent random variables, identically distributed like $Y=\log X \sim N(\mu, \sigma^2)$, where $-\infty < \mu < \infty$ and $\sigma^2 > 0$. The expectation ξ of X is

(1.1) $$\xi = \exp\left[\mu + \frac{1}{2}\sigma^2\right] ,$$

and its variance is given by

(1.2) $$D^2 = \xi^2(\exp[\sigma^2]-1) .$$

We consider the following problem: Let h_n be any estimator of ξ based on $X_1, \cdots, X_n$. Then can we obtain n and h_n such that

(1.3) $$\mathrm{P}(|h_n - \xi| \leqq \delta\xi) \geqq \gamma$$

for all $\delta > 0$ and $0 < \gamma < 1$? As in Zacks [6], we can not get such fixed number n and h_n. So Zacks [6] considered a sequential approach to the problem. Using Anscombe [1] and Chow and Robbins [2], Zacks [6] investigated asymptotic properties of stopping times based on maximum likelihood estimator and sample mean for the mean ξ. Furthermore Zacks [6] gave the asymptotic relative efficiency for their two stopping times.

The purpose of this paper is, if anything, to give exact properties of their stopping times and to compare two procedures.

2. Preliminaries

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\{\mathcal{A}_j : \mathcal{A}_j \subset \mathcal{A},\ j \in J\}$ be a non-increasing sequence of σ-fields where J is a continuous sequence of integers including possibly $\pm\infty$. We say that a family $Z = \{Z_j, \mathcal{A}_j;\ j \in J\}$ is called a reverse submartingale if for all $j \in J$ (i) Z_j is an

$\mathcal{A}_j$-measurable random variable, (ii) $\mathrm{E}\,Z_j^+<\infty$ and (iii) $\mathrm{E}\,[Z_j|\mathcal{A}_k]\geqq Z_k$ for $j\leqq k$, $k\in J$. If $\mathrm{E}\,|Z_j|<\infty$ for (ii) and $\mathrm{E}\,[Z_j|\mathcal{A}_k]=Z_k$ for (iii), we call a family $Z=\{Z_j, \mathcal{A}_j;\ j\in J\}$ a reverse martingale. We say that a random variable M with values a.s. in J is a reverse stopping time if $\{M=j\}\in\mathcal{A}_j$ for all $j\in J$. We shall use the following lemmas in Chow, Robbins and Siegmund [3].

LEMMA 2.1. *Let Z be a reverse submartingale and M a reverse stopping time. If $M\geqq j_0$ ($j_0\in J$), then we have*

$$\mathrm{E}\,Z_M\leqq \mathrm{E}\,Z_{j_0}\,. \tag{2.1}$$

If Z is a reverse martingale, the equality in (2.1) establishes. A well known reverse martingale is $n^{-1}\sum_{i=1}^{n}X_i$ $(n=1, 2,\cdots)$, where $X_1, X_2, \cdots, X_n,\cdots$ is a sequence of i.i.d. random variables with $\mathrm{E}\,|X_i|<\infty$.

LEMMA 2.2. *Let $Z=\{Z_j, \mathcal{A}_j;\ j\in J\}$ be a reverse (sub)martingale and f be a real valued (increasing) convex function. If $\mathrm{E}\,\{f(Z_{j_0})\}^+<\infty$ for some $j_0\in J$, then*

$$\{f(Z_j), \mathcal{A}_j;\ j\in J,\ j_0\leqq j\} \tag{2.2}$$

is a reverse submartingale.

3. The sequential modified maximum likelihood estimation procedure

We consider the following modified maximum likelihood (M.M.L.) estimator as the estimation of $\xi=\mathrm{E}\,(X)$ based on $X_1, X_2,\cdots, X_n$:

$$\hat{\xi}_n=\exp\left[\bar{Y}_n+\frac{1}{2}S_n^2\right], \tag{3.1}$$

where $\bar{Y}_n=n^{-1}\sum_{i=1}^{n}Y_i$ and $S_n^2=(n-1)^{-1}\sum_{i=1}^{n}(Y_i-\bar{Y}_n)^2$ with $Y_i=\log X_i$. Zacks [6] considered the maximum likelihood estimator instead of $\hat{\xi}_n$ in (3.1). Putting $U_n=\sqrt{n}\,(\bar{Y}_n-\mu)/\sigma$ and $V_n=\sqrt{(n-1)/2}(S_n^2/\sigma^2-1)$, U_n is a standard normal distribution and the limiting distribution of V_n is a standard normal distribution. After expressing the statistic $\hat{\xi}_n$ with U_n and V_n, apply Taylor expansion to $\hat{\xi}_n$. Then the asymptotic distribution of $\hat{\xi}_n$ is $N(\xi, \xi^2\sigma^2(1+\sigma^2/2)/n)$. Then the probability in (1.3) can be written as follows.

$$\begin{aligned}\mathrm{P}\,(|\hat{\xi}_n-\xi|\leqq\delta\xi)&=\mathrm{P}\left(\sqrt{n}\,|\hat{\xi}_n-\xi|\Big/\left\{\xi^2\sigma^2\left(1+\frac{1}{2}\sigma^2\right)\right\}^{1/2}\right.\\&\qquad\left.\leqq\sqrt{n}\,\delta\Big/\left\{\sigma^2\left(1+\frac{1}{2}\sigma^2\right)\right\}^{1/2}\right).\end{aligned} \tag{3.2}$$

Therefore we define the following stopping time:

$$N=\inf\left\{n\geqq n_0 \mid n\geqq \chi_\gamma^2[1]\delta^{-2}S_n^2\left(1+\frac{1}{2}S_n^2\right)\right\}, \tag{3.3}$$

where $\chi_\gamma^2[1]$ denotes the lower $100\gamma\%$ point of chi-square with one degree of freedom and the fixed constant number n_0 is greater than 3. After N observations have been taken, we estimate ξ by $\hat{\xi}_N=\exp[\bar{Y}_N+S_N^2/2]$. The stopping time due to Zacks [6] is a little different from N in (3.3). However the author can not understand why he obtained $\log^{-2}(1+\delta)$ instead of δ^{-2} in (3.3) from the calculation (3.2). Here we show that

$$\lim_{\delta\to 0} \mathrm{P}(|\hat{\xi}_N-\xi|\leqq\xi\delta)=\gamma\,. \tag{3.4}$$

We note that $\mathrm{P}(|\hat{\xi}_N-\xi|\leqq\xi\delta)=\mathrm{P}((\hat{\xi}_N-\xi)^2\leqq\xi^2\delta^2)$, $N/c\to 1$ a.s. as in Lemma 3.1 of Zacks [6], where $c=\chi_\gamma^2[1]\delta^{-2}\sigma^2(1+\sigma^2/2)$ and the asymptotic distribution of $\hat{\xi}_n$ is a normal with mean ξ and variance $\xi^2\sigma^2(1+\sigma^2/2)/n$. Though $\hat{\xi}_n$ is not a maximum likelihood estimator for ξ, by the similar consideration with the proof in later Theorem 3.4, we have the desired conclusion (3.4).

A useful random variable related to N defined by (3.3) is the following variable:

$$M=\begin{cases} \sup\left\{m\geqq n_0 \mid m<\chi_\gamma^2[1]\delta^{-2}S_m^2\left(1+\frac{1}{2}S_m^2\right)\right\}, \\ n_0-1 \qquad \text{if } m\geqq\chi_\gamma^2[1]\delta^{-2}S_m^2\left(1+\frac{1}{2}S_m^2\right) \text{ for any } m\geqq n_0\,. \end{cases} \tag{3.5}$$

This is a reverse stopping time, which depends on the future and not on the past. Such consideration as in (3.5) was used by Simons [5]. He obtained the expectation of the stopping times to get the fixed-width confidence interval of mean of a normal distribution by using the theory of reverse martingale. In this place we give the evaluation of the stopping time N in (3.3) by applying reverse submartingale.

THEOREM 3.1. *For all large n, we have*

$$\mathrm{P}(N>n)\leqq\rho_1^{n-1}, \tag{3.6}$$

where $0<\rho_1<1$.

PROOF. We have

$$\mathrm{P}(N>n)\leqq\mathrm{P}\left(n<\chi_\gamma^2[1]\delta^{-2}S_n^2\left(1+\frac{1}{2}S_n^2\right)\right) \tag{3.7}$$

$$=\mathrm{P}(S_n^2>-1+(1+2n\delta^2/\chi_\gamma^2[1])^{1/2}) .$$

Since we can write $S_n^2=(n-1)^{-1}\sum_{i=1}^{n-1} Z_i$, where $Z_1,\cdots, Z_{n-1}$ are independent and identically distributed random variables, for some fixed $t>0$ we have

$$\text{(3.8)}\quad \mathrm{E}\left[\exp\left[t\left\{\sum_{i=1}^{n-1} Z_i-(n-1)([1+2\delta^2 n/\chi_\gamma^2[1]]^{1/2}-1)\right\}\right]\right]$$
$$=\{\exp[-t\{(1+2\delta^2 n/\chi_\gamma^2[1])^{1/2}-1\}]M(t)\}^{n-1}\geqq \text{R.H.S. of (3.7)} .$$

The above $M(t)$ is a moment generating function of Z_i. Therefore for some large n_1, $\exp[-t\{(1+2\delta^2 n_1/\chi_\gamma^2[1])^{1/2}-1\}]M(t)=\rho_1<1$. Thus for $n\geqq n_1$, we have (3.6).

Next we evaluate the expectations of M and N.

THEOREM 3.2. *For variables M and N, we have*

$$\text{(3.9)}\qquad \mathrm{E}\, M\leqq c+(n_0-1)+\frac{\chi_\gamma^2[1]\sigma^4\delta^{-2}}{(n_0-2)} ,$$

$$\text{(3.10)}\qquad \mathrm{E}\, N\leqq c+n_0+\frac{\chi_\gamma^2[1]\sigma^4\delta^{-2}}{(n_0-2)} ,$$

where $c=\chi_\gamma^2[1]\delta^{-2}\sigma^2(1+\sigma^2/2)$.

PROOF. By (3.5), we have

$$\text{(3.11)}\qquad M<\chi_\gamma^2[1]\delta^{-2}S_M^2\left(1+\frac{1}{2}S_M^2\right)I_{\{M\geqq n_0\}}+(n_0-1)I_{\{M=n_0-1\}} .$$

Since the statistic S_m^2 can be expressed as $\sigma^2\sum_{i=2}^{m} w_i/(m-1)$, where $w_2, \cdots, w_m,\cdots$ are independent chi-square random variables with one degree of freedom, $\{S_m^2, \mathcal{A}_m;\ m=n_0-1, n_0,\cdots\}$ is a reverse martingale, where $\mathcal{A}_m=\mathcal{A}(S_m^2, S_{m+1}^2,\cdots)$. Then by Lemma 2.2, $\{S_m^2(1+S_m^2/2), \mathcal{A}_m;\ m=n_0-1, n_0,\cdots\}$ is a reverse submartingale. Therefore by Lemma 2.1, we have

$$\text{(3.12)}\quad \mathrm{E}\, M\leqq\chi_\gamma^2[1]\delta^{-2}\,\mathrm{E}\, S_{n_0-1}^2\left(1+\frac{1}{2}S_{n_0-1}^2\right)+(n_0-1)\,\mathrm{P}\,(M=n_0-1)$$
$$=\chi_\gamma^2[1]\delta^{-2}\left(\sigma^2+\frac{\sigma^4}{2}+\frac{\sigma^4}{n_0-2}\right)+(n_0-1)\,\mathrm{P}\,(M=n_0-1) .$$

Thus we obtain (3.9). For (3.10) we get it because $N\leqq M+1$.

THEOREM 3.3. *For the expectations of powers of M and N, we have*

$$\text{(3.13)}\qquad \mathrm{E}\, M^\nu\leqq(2\chi_\gamma^2[1]\delta^{-2}(n_0-2)^{-1}\sigma^2)^\nu\sum_{k=0}^{\nu}\binom{\nu}{k}\{(n_0-2)^{-1}\sigma^2\}^k$$

$$\cdot \prod_{i=0}^{k+\nu-1} \{(n_0-2)/2+i\}+(n_0-1)^\nu ,$$

(3.14)
$$\mathrm{E}\, N^\nu \leqq \sum_{k=0}^{\nu} \binom{\nu}{k} (2\chi_r^2[1]\delta^{-2}\sigma^2(n_0-2)^{-1})^k \sum_{i=0}^{k} \binom{k}{i} \cdot \{(n_0-2)^{-1}\sigma^2\}^i \prod_{j=0}^{k+i-1} \{(n_0-2)/2+j\}+n_0^\nu ,$$

for $\nu=1, 2, \cdots$.

PROOF. Since $\{[S_m^2(1+S_m^2/2)]^\nu, \mathcal{A}_m;\ m=n_0-1, n_0, \cdots\}$ $(\nu\geqq 1)$ is a reverse submartingale by Lemma 2.2, we obtain (3.13) and (3.14) by Lemma 2.1.

Next we consider the property of the stopping time N as $\delta\to 0$.

THEOREM 3.4. *As* $\delta\to 0$, $\sqrt{c}\,\delta^2(N-c)/\sqrt{2}\,\sigma^2(1+\sigma^2)\chi_r^2[1]$ *converges in law to the standard normal distribution, where*

$$c=\chi_r^2[1]\delta^{-2}\sigma^2\left(1+\frac{1}{2}\sigma^2\right) .$$

PROOF. From (3.3), we have

(3.15)
$$N\geqq \chi_r^2[1]\delta^{-2}S_N^2\left(1+\frac{1}{2}S_N^2\right) \quad \text{and}$$
$$N-1<\chi_r^2[1]\delta^{-2}S_{N-1}^2\left(1+\frac{1}{2}S_{N-1}^2\right) .$$

Let $L_n^2=((n-1)/n)S_n^2$, then L_n^2 is a maximum likelihood estimator for σ^2. Therefore $L_n^2(1+L_n^2/2)$ is also a maximum likelihood estimator for $\sigma^2(1+\sigma^2/2)$. Since $N/c\to 1$ a.s. as $\delta\to 0$, we have

(3.16)
$$\begin{aligned} K_N &= c^{1/2}\left\{S_N^2\left(1+\frac{1}{2}S_N^2\right)-\sigma^2\left(1+\frac{1}{2}\sigma^2\right)\right\} \\ &= c^{1/2}\left\{L_N^2\left(1+\frac{1}{2}L_N^2\right)-\sigma^2\left(1+\frac{1}{2}\sigma^2\right)+O_p(c^{-1})\right\} \\ &= c^{1/2}\left\{L_N^2\left(1+\frac{1}{2}L_N^2\right)-\sigma^2\left(1+\frac{1}{2}\sigma^2\right)\right\}+O_p(c^{-1/2}) . \end{aligned}$$

By Theorems 1 and 4 in Anscombe [1], K_N and K_{N-1} converge in law to a normal distribution with mean zero and variance $2\sigma^4(1+\sigma^2)^2$. Therefore we have Theorem 3.4.

For the other problems in point estimation of the mean, Ghosh and Mukhopadhyay [4] considered the asymptotic normality of stopping times.

4. The sequential sample mean procedure

Let $\bar{X}_n=n^{-1}\sum_{i=1}^{n} X_i$, then the limiting distribution of sample mean (S.M.) $\bar{X}_n$ is a normal distribution with mean ξ and variance D^2/n by central limit theorem. Thus we define the following stopping time:

$$N^*=\inf\{n\geqq n_0 \mid n\geqq \chi_r^2[1]\delta^{-2}(\exp[S_n^2]-1)\}\ . \tag{4.1}$$

By the same consideration as in Section 3, we define the reverse stopping time M^*, that is,

$$M^*=\begin{cases}\sup\{m\geqq n_0 \mid m<\chi_r^2[1]\delta^{-2}(\exp[S_m^2]-1)\} & \\ n_0-1 & \text{if } m\geqq \chi_r^2[1]\delta^{-2}(\exp[S_m^2]-1) \text{ for all } m\geqq n_0\,.\end{cases} \tag{4.2}$$

By the same consideration as in Theorem 3.1,

THEOREM 4.1. *For all large n, we have*

$$\mathrm{P}\,(N^*>n)\leqq \rho_2^{n-1}\,, \tag{4.3}$$

where $0<\rho_2<1$.

Now put $f(x)=(\exp[x]-1)^{\alpha}$ for $x>0$ and positive integer α. Since $f'(x)=\alpha(\exp[x]-1)^{\alpha-1}\exp[x]>0$ and $f''(x)=\alpha(\alpha-1)(\exp[x]-1)^{\alpha-2}\exp[2x]+\alpha(\exp[x]-1)^{\alpha-1}\exp[x]>0$, the function $f(x)$ is increasing and convex. Then, by Lemma 2.2, $\{(\exp[S_m^2]-1)^{\alpha}, \mathcal{A}_m;\ m=n_0-1, n_0, \cdots\}$ is a reverse submartingale. Thus using Lemma 2.1, we obtain

THEOREM 4.2. *For N^* and M^* defined by* (4.1) *and* (4.2), *respectively, we have*

$$\mathrm{E}\,M^{*\nu}\leqq(\chi_r^2[1]\delta^{-2})^{\nu}\sum_{k=0}^{\nu}\binom{\nu}{k}(-1)^{\nu-k}\left(1-\frac{2k\sigma^2}{n_0-2}\right)^{-(n_0-2)/2}+(n_0-1)^{\nu}\,, \tag{4.4}$$

$$\mathrm{E}\,N^{*\nu}\leqq\sum_{i=0}^{\nu}\binom{\nu}{i}(\chi_r^2[1]\delta^{-2})^{i}\sum_{k=0}^{i}\binom{i}{k}(-1)^{i-k}\left(1-\frac{2k\sigma^2}{n_0-2}\right)^{-(n_0-2)/2}+n_0^{\nu}\,, \tag{4.5}$$

for $\nu=1, 2, \cdots$.

Next we have

THEOREM 4.3. *As $\delta\to 0$, $\sqrt{c'}\,\delta^2\{N^*-c'\}/\sqrt{2}\,\sigma^2\exp[\sigma^2]\chi_r^2[1]$ converges in law to the standard normal distribution, where $c'=\chi_r^2[1]\delta^{-2}(\exp[\sigma^2]-1)$.*

5. Comparison

Zacks [6] showed that the sequential maximum likelihood procedure is superior to the sequential sample mean procedure in an asymptotic sence as $\delta \to 0$. In this place we show that the sequential M.M.L. procedure is superior to the sequential S.M. procedure in an exact sence.

THEOREM 5.1. *For N and N^* defined by* (3.3) *and* (4.1), *respectively, we have $N \leqq N^*$ a.s.*

PROOF. We consider two event $\{N=n\}$ and $\{N^* \geqq n\}$. Then

$$\{N=n\}=\left\{i<\chi_r^2[1]\delta^{-2}S_i^2\left(1+\frac{1}{2}S_i^2\right),\ i=n_0,\cdots,n-1,\right. \\ \left. n \geqq \chi_r^2[1]\delta^{-2}S_n^2\left(1+\frac{1}{2}S_n^2\right)\right\} \tag{5.1}$$

and

$$\{N^* \geqq n\}=\{i<\chi_r^2[1]\delta^{-2}(\exp[S_i^2]-1),\ i=n_0,\cdots,n-1\}\ . \tag{5.2}$$

Since $\exp[S_i^2]-1 \geqq S_i^2(1+S_i^2/2)$, we have $\{N=n\} \subset \{N^* \geqq n\}$. Then we have

$$\mathrm{P}(N \leqq N^*)=\sum_{n=n_0}^{\infty} \mathrm{P}(n=N \leqq N^*)=\sum_{n=n_0}^{\infty} \mathrm{P}(N=n)=1\ . \tag{5.3}$$

Thus we obtain the desired conclusion.

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