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ASYMPTOTIC NORMALITY OF A QUADRATIC MEASURE OF ORTHOGONAL SERIES TYPE DENSITY ESTIMATE

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Abstract

Let $X_1, \cdots, X_n$ be i.i.d. random variable with a common density f. Let $f_n(x)=\sum_{k=0}^{q_n} \hat{a}_k \phi_k(x)$ be an estimate of $f(x)$ based on a complete orthonormal basis $\{\phi_k\colon k\geqq 0\}$ of $L_2[a, b]$. A Martingale central limit theorem is used to show that $(\sqrt{2}\,\sigma_n)^{-1}\left[n\int (f_n(x)-f(x))^2dx-\mu_n\right]\xrightarrow{\mathcal{L}} N(0, 1)$, where $\mu_n=\sum_{k=0}^{q_n} \mathrm{Var}\,[\phi_k(X)]$ and $\sigma_n^2=\sum_{k=0}^{q_n}\sum_{k'=0}^{q_n} [\mathrm{Cov}\,(\phi_k(X), \phi_{k'}(X))]^2$.

1. Introduction

Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed random variables with a common density f. It is assumed that f is continuous and square integrable with respect to Lebesgue measure. Let $\{\phi_n\colon n\geqq 0\}$ be a complete orthonormal basis for $L_2[a, b]$, i.e.;

$$(1) \qquad \int_a^b \phi_i(x)\phi_j(x)dx=\delta_{ij}\ , \qquad i, j\geqq 0\ ,$$

where δ_{ij} is the Kronecker delta and the interval $[a, b]$ could be infinite. Asymptotic unbiasedness and various kinds of consistency properties of the estimate $f_n(x)$ of $f(x)$, defined by

$$(2) \qquad f_n(x)=\sum_{k=0}^{q_n} \hat{a}_k\phi_k(x)$$

where $\hat{a}_k=\frac{1}{n}\sum_{j=1}^{n}\phi_k(X_j)$, $q_n\to\infty$ and $q_n/n\to 0$ as $n\to\infty$, have been studied in the literature. The object of this paper is to study the asymptotic distribution of the quadratic functional

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$$S_n=\int (f_n(x)-f(x))^2dx \tag{3}$$

where $f_n(x)$ is the estimate defined in (2). Bickel and Rosenblatt [2] have studied the asymptotic distribution of the functionals S_n and T_n where S_n is given by (3),

$$T_n=\int \{[f_n(x)-f(x)]^2/f(x)\}dx \tag{4}$$

and $f_n(x)$ is the kernel type density estimate. Lii [4] has derived the asymptotic distribution of T_n when $f_n(x)$ is the density estimate based on spline functions.

2. Notations and assumptions

We will use the following notations and assumptions. Assume that all the random variables are defined on the probability space (Ω, F, P).

$$\begin{aligned} &a_k=\mathrm{E}\,[\phi_k(X)]\ , \qquad \sigma_{kk'}=\mathrm{Cov}\,[\phi_k(X), \phi_{k'}(X)]\ , \\ &\mu_n=\sum_{k=0}^{q_n}\sigma_{kk} \qquad \text{and} \qquad \sigma_n^2=\sum_{k=0}^{q_n}\sum_{k'=0}^{q_n}\sigma_{kk'}^2\ . \end{aligned} \tag{5}$$

Also define $\bar{\phi}_k(X)=\phi_k(X)-a_k$,

$$z_{jj'}^{(n)}=\frac{\sqrt{2}}{n\sigma_n}\sum_{k=0}^{q_n}\bar{\phi}_k(X_j)\bar{\phi}_k(X_{j'})\ ,$$

$$W_{nj}=\begin{cases}\sum_{i=1}^{j-1} Z_{ij}^{(n)} & \text{for } j=2,\cdots,n\\ 0 & \text{for } j=1, \text{ and } j>n\ ,\end{cases}$$

$$V_{nj}=\sum_{i=1}^{j} W_{ni} \qquad \text{for all } n \text{ and } j \text{ and}$$

$$F_{nj}=\mathscr{B}(X_1,\cdots,X_j) \qquad j=1,\cdots,n \text{ and } n\geqq 1\ ,$$

where $\mathscr{B}(X_1,\cdots,X_j)$ is the σ-algebra generated by $X_1,\cdots,X_j$; and $F_{n0}=\sigma\{\phi, \Omega\}$. The following assumptions are used in the proof of asymptotic normality.

(A1) $\frac{n}{\sigma_n}\sum_{k>q_n} a_k^2\rightarrow 0$ as $n\rightarrow\infty$,

(A2) $\sigma_n^2=\sum_{k=0}^{q_n}\sum_{k'=0}^{q_n}\sigma_{kk'}^2\rightarrow\infty$ as $n\rightarrow\infty$,

(A3) $\mathrm{E}\,\phi_k^4(X)\leqq M$ for some M and for all k, (A4) $q_n = O(n^\delta)$ where $\frac{1}{4} \leqq \delta < \frac{1}{3}$.

3. Results

LEMMA 1. *Let V_{nj} and F_{nj} be as defined in* (6). *Then* $\{(V_{nj}, F_{nj}), j \geqq 1\}$ *is a Martingale for each* $n = 1, 2, \cdots$.

The proof of Lemma 1 follows directly from definition of V_{nj} and F_{nj}.

LEMMA 2. *Let W_{nj} be as defined in* (6). *Then*

(i) $\sum_{j \geqq 1} \mathrm{E}\, W_{nj}^2 \leqq 1$ *for all* n,

(ii) $\sum_{j \geqq 1} W_{nj}^2 \xrightarrow{P} 1$ *as* $n \to \infty$ *and*

(iii) $\sup_j |W_{nj}| \xrightarrow{P} 0$ *as* $n \to \infty$.

The proof of Lemma 2 is given in the Appendix.

DEFINITION 1. Suppose $\{(Z_n, \mathcal{B}_n), n \geqq 1\}$ is a Martingale. Then $\{(Z_n - Z_{n-1}, \mathcal{B}_n), n \geqq 1\}$ is called a Martingale difference array.

DEFINITION 2. A double sequence $\{(W_{nj}, F_{nj}), n \geqq 1, j \geqq 1\}$ is called Martingale difference array if $\{(W_{nj}, F_{nj}), j \geqq 1\}$ is a Martingale difference array for each fixed n.

DEFINITION 3 (Conditional Weak Convergence). Let $\{Z_n\}$ be a sequence of random variables defined on Ω and let $\{B_n\}$ be a sequence of sub-σ-fields of F. Then we say $Z_n | B_n$ converges weakly to a random variable Y on (Ω, F, P) iff

$$\mathrm{E}\,(f(Z_n) | \mathcal{B}_n) \to \mathrm{E}\, f(Y)$$

for every bounded continuous f. We will denote this convergence by $Z_n | B_n \xrightarrow{\mathcal{L}} Y$.

Remark. In particular when $\mathcal{B}_n = \sigma\{\phi, \Omega\}$, the trivial σ-field, then this conditional convergence is equivalent to the usual unconditional convergence. (See Remark 1 on page 12 of [1].)

The main result of this paper is given in Theorem 2 below. The proof of Theorem 2 is based on a conditional central limit theorem for Martingales due to Adnan [1]. We state this result (without proof) as Theorem 1 for the sake of completeness. Let $F_{n0} \subset F_{n1}$ be as defined in Section 2.

THEOREM 1 (Adnan). *If* (i) $\{(W_{nj}, F_{nj}), n \geqq 1, j \geqq 1\}$ *is a Martingale*

difference array, (ii) $\sum_j \mathrm{E}\, W_{nj}^2 \leqq K < \infty$ *for some* K *and for all* n, (iii) $\sum_j W_{nj}^2 \xrightarrow{\mathrm{P}} 1$ *and*
(iv) $\sup_j |W_{nj}| \xrightarrow{\mathrm{P}} 0$ *then* $\sum_{j=1}^{\infty} W_{nj} | F_{n0} \xrightarrow{\mathcal{L}} N(0, 1)$.

We now state the main Theorem.

THEOREM 2. *If the assumptions* (A1), (A2) *and* (A3) *hold, then*

$$\frac{1}{\sqrt{2}\,\sigma_n}\left[n \int (f_n(x)-f(x))^2 dx - \mu_n\right] \xrightarrow{\mathcal{L}} N(0, 1) .$$

PROOF. It is easy to see that

$$n \int (f_n(x)-f(x))^2 dx$$
$$= \frac{1}{n} \sum_{j=1}^{n} \sum_{k=0}^{q_n} \bar{\phi}_k^2(X_j) + \frac{1}{n} \sum_{j \neq j'} \sum \sum_{k=0}^{q_n} \phi_k(X_j)\phi_k(X_{j'}) + n \sum_{k > q_n} a_k^2$$

and

$$\mathrm{E}\, n \int (f_n(x)-f(x))^2 dx = \sum_{k=0}^{q_n} \sigma_{kk} + n \sum_{k>q_n} a_k^2 = \mu_n + n \sum_{k>q_n} a_k^2 .$$

Therefore

$$(7) \quad \frac{1}{\sqrt{2}\,\sigma_n}\left[n \int (f_n(x)-f(x))^2 dx - \mu_n\right]$$
$$= \frac{1}{\sqrt{2}\,\sigma_n} \frac{1}{n} \sum_{j=1}^{n} \sum_{k=0}^{q_n} (\bar{\phi}_k^2(X_j) - \sigma_{kk}) + \frac{1}{\sqrt{2}\,\sigma_n} \frac{1}{n} \sum_{j \neq j'} \sum \sum_{k=0}^{q_n} \bar{\phi}_k(X_j)\bar{\phi}_k(X_{j'})$$
$$+ \frac{1}{\sqrt{2}\,\sigma_n} \sum_{k>q_n} a_k^2 = A_n + B_n + C_n \qquad \text{(say)} .$$

Now by assumption (A1), $C_n \to 0$ as $n \to \infty$. Also $\mathrm{E}\, A_n = 0$ and $\mathrm{Var}\,[A_n] = \frac{1}{2n\sigma_n^2} \mathrm{E}\left[\sum_{k=0}^{q_n} (\bar{\phi}_k^2(X) - \sigma_{kk})\right]^2 = O\left(\frac{q_n^2}{n\sigma_n^2}\right)$ if (A3) holds. Consequently $\mathrm{Var}\,(A_n) \to 0$ if (A4) holds. Hence it is sufficient to show that $B_n \xrightarrow{\mathcal{L}} N(0, 1)$. But

$$B_n = \frac{1}{\sqrt{2}\,\sigma_n} \frac{1}{n} \sum_{j \neq j'} \sum \sum_{k=0}^{q_n} \bar{\phi}_k(X_j)\bar{\phi}_k(X_{j'})$$
$$= \sum_{j<j'} \sum \frac{\sqrt{2}}{n\sigma_n} \sum_{k=0}^{q_n} \bar{\phi}_k(X_j)\bar{\phi}_k(X_{j'})$$
$$= \sum_{j<j'} \sum Z_{jj'}^{(n)} = \sum_{j=2}^{n} W_{nj} ,$$

where $Z_{jj'}^{(n)}$ and W_{nj} are defined in (6) and the asymptotic normality of B_n follows from Lemma 2. This completes the proof of Theorem 2.

Next we give two examples of $\{\phi_n\}$ for which assumptions (A1), (A2) and (A3) are satisfied.

Example 1. Let $\phi_k(x)=\sqrt{2}\cos(k\pi x)$, $k=0, 1, \cdots$, and $f^{(2)} \in L_2[0, 1]$. Then $f(x)=\sum_{k=0}^{\infty} a_k\phi_k(x)=\sum_{k=0}^{\infty} a_k \cos(k\pi x)$ implies $f^{(2)}(x)=-\sum_{k=0}^{\infty} a_k(k\pi)^2\cos(k\pi x)$ $=\sum b_k \cos(k\pi x)$. But $f^{(2)} \in L_2[0, 1]$ implies $\sum_{k>q_n} b_k^2=\sum_{k>q_n} \pi^4k^4a_k^2 \to 0$. Therefore

$$\sum_{k>q_n} a_k^2 \leqq \sum_{k>q_n} \frac{k^4a_k^2}{q_n^4}=O(q_n^{-4}) . \tag{8}$$

For some $\varepsilon>0$ let $[\alpha, \beta]$ be such that $f(x)\geqq\varepsilon$ ($\forall\, x \in [\alpha, \beta]$), then

$$\mathrm{E}\,\phi_k^2(X)\geqq\varepsilon\int_\alpha^\beta \cos^2(k\pi x)dx=\frac{\varepsilon}{k\pi}\int_{k\pi\alpha}^{k\pi\beta}\cos^2(y)dy\geqq\frac{\varepsilon}{k\pi}\int_{\theta_1}^{\theta_2}\cos^2 y dy ,$$

$$\text{where } \theta_1=([2k\alpha]+1)\frac{\pi}{2} \text{ and } \theta_2=([2k\beta]-1)\frac{\pi}{2},$$

$$=\frac{\varepsilon}{k\pi}([2k\beta]-1-[2k\alpha]-1)\int_0^{\pi/2}\cos^2 y dy$$

$$=\frac{\varepsilon}{4k}[2k(\beta-\alpha)-2+([2k\beta]-(2k\beta))-([2k\alpha]-2k\alpha)]$$

$$=\frac{\varepsilon(\beta-\alpha)}{2}-\frac{\gamma}{k} \qquad \text{where } 0<\gamma\leqq 1 .$$

Therefore

$$\sigma_{kk}=\mathrm{E}\,\phi_k^2-a_k^2\geqq\frac{\varepsilon(\beta-\alpha)}{2}-\frac{\gamma}{k}-a_k^2 . \tag{9}$$

Since $\sum_{k=0}^{\infty} a_k^2<\infty$, there exists k_0 such that $\sigma_{kk}>\frac{\varepsilon(\beta-\alpha)}{4}$ for $k\geqq k_0$ and hence $\sigma_n^2=\sum_k^{q_n}\sum_{k'}^{q_n}\sigma_{kk'}^2\geqq\sum_{k=0}^{q_n}\sigma_{kk}^2\to\infty$ as $q_n\to\infty$. This shows that assumptions (A1) and (A2) are satisfied. (A3) is trivially satisfied.

Example 2. Let $\phi_k(x)=\frac{e^{-x^2/2}H_n(x)}{(\sqrt{\pi}\,2^n n!)^{1/2}}$; $k=0, 1, \cdots$, and $(x-D)^2f \in L_2(-\infty, \infty)$, where $D=\frac{d}{dx}$ and $H_n(x)$ is the Hermite polynomial of degree n. Then from (14) in [6] we get $\sum_{k>q_n} a_k^2=O\left(\frac{1}{n}\right)$. To show that $\sigma_n^2\to\infty$ we proceed as follows. Let α, β be defined in Example 1. Then $\mathrm{E}\,\phi_k^2\geqq\varepsilon\int_\alpha^\beta\phi_k^2(x)dx$. Since $\phi_k^2(x)=\phi_k^2(-x)$ we can assume $\alpha, \beta>0$ without a loss of generality. Now using (8.65.1) in [5] we get

(10)
$$\phi_k^2(x)=\frac{1}{\sqrt{\pi}\,2^k k!}\Big[\lambda_k \cos\Big(\sqrt{N}x-\frac{k\pi}{2}\Big)+\frac{1}{\sqrt{N}}\int_0^x \sin(\sqrt{N}(x-t))t^2 e^{-t^2/2}H_k^{(t)}dt\Big]^2$$
$$=\frac{1}{\sqrt{\pi}\,2^k k!}\Big[\lambda_k A_k(x)+\frac{B_k(x)}{\sqrt{N}}\Big]^2, \qquad \text{say},$$

where $N=2k+1$ and

$$\lambda_k=\begin{cases}\dfrac{(2m)!}{m!} & \text{if } k=2m\\[2ex] \dfrac{(2m+2)!}{(m+1)!}\dfrac{1}{\sqrt{N}} & \text{if } k=2m+1.\end{cases}$$

Expanding the right-hand side of (10) and integrating we get

(11)
$$\int_\alpha^\beta \phi_k^2(x)dx=\frac{\lambda_k^2}{\sqrt{\pi}\,2^k k!}\int_\alpha^\beta A_k^2(x)dx+\frac{1}{N}\int_\alpha^\beta \frac{B_k^2(x)dx}{\sqrt{\pi}\,2^k k!}+\frac{2\lambda_k}{\sqrt{N}(\sqrt{\pi}\,2^k k!)^{1/2}}\int_\alpha^\beta \frac{A_k(x)B_k(x)}{(\sqrt{\pi}\,2^k k!)^{1/2}}dx\,.$$

Since $|\phi_k(x)|\leqq C_M(k+1)^{-1/4}$ for all $x\in(-M, M)$, $j=0, 1, \cdots$, it follows from the definition λ_k and N that the second term in (11) is $O\Big(\frac{1}{k}\Big)$, the third term is $O(k^{-3/4})$ and $\frac{\lambda_k^2}{\sqrt{\pi}\,2^k k!}\geqq\frac{d}{\sqrt{k}}$ for some $d>0$. Also

(12)
$$\int_\alpha^\beta A_k^2(x)dx=\int_\alpha^\beta \cos^2\Big(\sqrt{N}x-\frac{K\pi}{2}\Big)dx$$
$$=\frac{1}{\sqrt{N}}\int_{\alpha\sqrt{N}-(K\pi/2)}^{\beta\sqrt{N}-(K\pi/2)}\cos^2 y\,dy\geqq\frac{(\beta-\alpha)}{2}-\frac{\pi\gamma}{\sqrt{N}}$$

where $0<\gamma\leqq 4$. Now substituting (12) in (11) we get

$$\int_{-\infty}^{\infty}\phi^2(x)f(x)\geqq\varepsilon\int_\alpha^\beta \phi_k^2(x)dx\geqq\varepsilon\Big[\frac{(\beta-\alpha)d}{2\sqrt{k}}-\frac{\pi\gamma d}{\sqrt{N}\sqrt{k}}+O(k^{-3/4})\Big]\,.$$

Also $\sum_{k>q_n} a_k^2=O\Big(\frac{1}{n}\Big)$ implies that there exists a K_0 such that

$$\sigma_{kk}=\mathrm{E}\,\phi_k^2-a_k^2>\frac{\varepsilon(\beta-\alpha)d}{4\sqrt{k}} \qquad \text{if } k>K_0\,,$$

and hence

$$\sigma_n^2=\sum_k^{q_n}\sum_{k'}^{q_n}\sigma_{kk'}^2\geqq\sum_{k=0}^{q_n}\sigma_{kk}^2>\sum_{k=k_0}^{q_n}\frac{\varepsilon(\beta-\alpha)d}{4k}\to\infty \qquad \text{as } q_n\to\infty\,.$$

This shows that for Hermite functions assumptions (A1) and (A2) hold.

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Appendix

PROOF OF LEMMA 2. For simplicity we will write Z_{ij} for $Z_{ij}^{(n)}$.
(i) $\mathrm{E}\, W_{nj}^2 = \mathrm{E}\,[Z_{1j}+\cdots+Z_{j-1,j}]^2 = 2n^{-2}(j-1)$.
Hence

$$\sum_{j=1}^{n} \mathrm{E}\, W_{nj}^2 = \frac{n(n-1)}{n^2} = 1-\frac{1}{n} \leqq 1 \qquad \text{for all } n\,. \tag{a1}$$

(ii) To prove (ii) it is enough to show that $\mathrm{E}\,(\sum W_{nj}^2-1)^2 \to 0$ as $n\to\infty$, in view of Chebychev's inequality. Observe that

$$\begin{aligned}
\mathrm{E}\left(\sum_{j=1}^{n} W_{nj}^2-1\right)^2 &= \mathrm{E}\left(\sum_{j=1}^{n} W_{nj}^2\right)^2 + 1 - 2\,\mathrm{E}\sum_{j=1}^{n} W_{nj}^2 \\
&= \mathrm{E}\left(\sum_{j=1}^{n} W_{nj}^2\right)^2 - 1 + \frac{2}{n}\,, \qquad \text{using (a1)}\,.
\end{aligned}$$

Hence it is enough to show that $\mathrm{E}\left(\sum_{j=1}^{n} W_{nj}^2\right)^2 \to 1$ as $n\to\infty$. But

$$\mathrm{E}\left(\sum_{j=1}^{n} W_{nj}^2\right)^2 = \sum_{j=1}^{n} \mathrm{E}\, W_{nj}^4 + 2\sum_{j<j'}\sum \mathrm{E}\, W_{nj}^2 W_{nj'}^2\,. \tag{a2}$$

$$\begin{aligned}
\mathrm{E}\, W_{nj}^4 &= \mathrm{E}\,[Z_{1j}+\cdots+Z_{j-1,j}]^4 \\
&= \mathrm{E}\left\{\frac{\sqrt{2}}{n\sigma_n}\sum_{k=0}^{q_n}\left[\sum_{i=1}^{j-1}\bar\phi_k(X_i)\right]\bar\phi_k(X_j)\right\}^4 \\
&\leqq \frac{4(q_n+1)^3}{n^4\sigma_n^4}\sum_{k=0}^{q_n}\mathrm{E}\left[\sum_{i=1}^{j-1}\bar\phi_k(X_i)\right]^4 \mathrm{E}\,\bar\phi_k^4(X_j) \\
&= \frac{4q_n^3}{n^4\sigma_n^4}\left[\sum_{k=0}^{q_n}\mathrm{E}\,\bar\phi_k^4\{(j-1)\,\mathrm{E}\,\bar\phi_k^4+3(j-1)(j-2)\sigma_{kk}^2\}\right]\,.
\end{aligned}$$

Therefore

$$\sum_{j=1}^{n}\mathrm{E}\, W_{nj}^4 \leqq \frac{4q_n^3}{n^4\sigma_n^4}\left(\frac{n(n-1)}{2}\sum_{k=0}^{q_n}(\mathrm{E}\,\bar\phi_k^4)^2 + n(n-1)(n-3)\sum_{k=0}^{q_n}\sigma_{kk}^2\,\mathrm{E}\,\bar\phi_k^4\right) \to 0 \tag{a3}$$

as $n\to\infty$ if (A2)-(A4) hold.

Now consider

(a4)
$$W_{nj}^2 W_{nj'}^2 = [Z_{1j}+\cdots+Z_{j-1,j}]^2[Z_{1j'}+\cdots+Z_{j'-1,j'}]^2$$
$$= \Big(\sum_{i=1}^{j-1} Z_{ij}^2\Big)\Big(\sum_{i=1}^{j'-1} Z_{ij'}^2\Big) + \Big(\sum_{i=1}^{j-1} Z_{ij}^2\Big)\Big(\sum_{l\neq l'=1}^{j'-1}\sum^{j'-1} Z_{lj'}Z_{l'j'}\Big)$$
$$+ \Big(\sum_{l=1}^{j'-1} Z_{lj'}^2\Big)\Big(\sum_{i\neq i'}^{j-1}\sum^{j-1} Z_{ij}Z_{i'j}\Big)$$
$$+ \Big(\sum_{i\neq i'=1}^{j-1}\sum^{j-1} Z_{ij}Z_{i'j}\Big)\Big(\sum_{l\neq l'=1}^{j'-1}\sum^{j'-1} Z_{lj'}Z_{l'j'}\Big)$$
$$= A_n(jj')+B_n(jj')+C_n(jj')+D_n(jj') .$$

It is easy to show $\mathrm{E}\, A_n(jj') = 2(j-1)\,\mathrm{E}\, Z_{12}^2 Z_{23}^2 + (j-1)(j'-3)(\mathrm{E}\, Z_{12}^2)^2$. Therefore

$$2\sum_{j<j'}\sum \mathrm{E}\, A_n(jj')$$
$$= 4\,\mathrm{E}\, Z_{12}^2 Z_{23}^2 \sum_{j<j'}\sum (j-1) + 2(\mathrm{E}\, Z_{12}^2)^2 \sum_{j<j'}\sum (j-1)(j'-3)$$
$$= \frac{16}{n^4\sigma_n^4}\Big(\sum_k\sum_{k'}\sum_l\sum_{l'} \sigma_{kk'}\sigma_{ll'}\,\mathrm{E}\,\bar\phi_k\bar\phi_{k'}\bar\phi_l\bar\phi_{l'}\Big)\Big(\frac{n(n-1)(n-2)}{6}\Big)$$
$$+2\Big(\frac{4}{n^4}\Big)\Big[\frac{1}{2}\Big\{\frac{n(n-1)(n-2)(n-3)}{4}\Big\}\Big]$$
$$= 1+O(1) .$$

Now

$$|\mathrm{E}\, B_n(jj')| = \Big|2\sum_{i=1}^{j-1} \mathrm{E}\, Z_{ij}^2 Z_{ij'} Z_{jj'}\Big|$$
$$\leqq \frac{8(j-1)}{n^4\sigma_n^4}\Big\{\sum_k^{q_n}\sum_{k'}^{q_n}\sum_l^{q_n}\sum_{l'}^{q_n} |\sigma_{ll'}\,\mathrm{E}\,[\bar\phi_k\bar\phi_{k'}\bar\phi_l]\,\mathrm{E}\,[\bar\phi_k\bar\phi_{k'}\bar\phi_{l'}]|\Big\}$$
$$\leqq \frac{8(j-1)q_n^2}{n^4\sigma_n^2} .$$

Therefore $2\sum_{j<j'}\sum |\mathrm{E}\, B_n(jj')| = O\Big(\frac{q_n^2}{n\sigma_n^2}\Big)$. Also it is easy to show that $\mathrm{E}\, C_n(jj')=0$ and hence $2\sum_{j<j'}\sum \mathrm{E}\, C_n(jj')=0$. Finally

(a5)
$$\mathrm{E}\, D_n(jj') = \mathrm{E}\Big[\sum_{i\neq i'}\sum Z_{ij}Z_{i'j}\Big]\Big[\sum_{i\neq i'}\sum Z_{ij'}Z_{i'j'}\Big]$$
$$= 4\sum_{i<i'}^{j-1}\sum^{j-1} \mathrm{E}\, Z_{ij}Z_{i'j}Z_{ij'}Z_{i'j'}$$
$$= 4\sum_{i<i'}^{j-1}\sum^{j-1} \mathrm{E}\,[Z_{i'j}Z_{i'j'}\,\mathrm{E}\,(Z_{ij}Z_{ij'}\,|\,X_{i'}, X_j, X_{j'})]$$
$$= \frac{16}{n^4\sigma_n^4}\,\mathrm{E}\Big(\sum_k^{q_n}\sum_{k'=0}^{q_n} \sigma_{kk'}\bar\phi_k(X_1)\bar\phi_{k'}(X_2)\Big)^2 (j-1)(j-2) .$$

Now we will show that $\frac{1}{\sigma_n^2}\,\mathrm{E}\Big(\sum_k^{q_n}\sum_{k'=0}^{q_n} \sigma_{kk'}\bar\phi_k(X_1)\bar\phi_{k'}(X_2)\Big)^2 \leqq M$ for some M.

Observe that

$$\sum_{k}^{q_n}\sum_{k'}^{q_n}\sigma_{kk'}\bar{\phi}_k(X_1)\bar{\phi}_{k'}(X_2)=\sum_{k}^{q_n}\sum_{k'}^{q_n}\sigma_{kk'}\phi_k(X_1)\phi_{k'}(X_2)-\sum_{k}^{q_n}\sum_{k'}^{q_n}\sigma_{kk'}a_k\phi_{k'}(X_2)$$
$$-\sum_{k}^{q_n}\sum_{k'}^{q_n}\sigma_{kk'}a_{k'}\phi_k(X_1)+\sum_{k}^{q_n}\sum_{k'}^{q_n}\sigma_{kk'}a_k a_{k'} .$$

Squaring we get

(a6) $$\mathrm{E}\left(\sum_{k}^{q_n}\sum_{k'}^{q_n}\sigma_{kk'}\bar{\phi}_k(X_1)\bar{\phi}_{k'}(X_2)\right)^2$$
$$\leqq 4\left(\mathrm{E}\left(\sum_{k}^{q_n}\sum_{k'}^{q_n}\sigma_{kk'}\phi_k(X_1)\phi_{k'}(X_2)\right)^2+2\,\mathrm{E}\left(\sum_{k}\sum_{k'}\sigma_{kk'}a_k\phi_{k'}(X)\right)^2\right.$$
$$\left.+\left(\sum_{k}\sum_{k'}\sigma_{kk'}a_k a_{k'}\right)^2\right) .$$

First term on the right-hand side of (a6) is

(a7) $$\mathrm{E}\left(\sum_{k}^{q_n}\sum_{k'}^{q_n}\sigma_{kk'}\phi_k(X_1)\phi_{k'}(X_2)\right)^2$$
$$=\int\int\left(\sum_{k}\sum_{k'}\sigma_{kk'}\phi_k(x)\phi_{k'}(y)\right)^2 f(x)f(y)dxdy$$
$$\leqq L^2\int\int\left[\sum\sigma_{kk}\phi_k(x)\phi_k(y)+\sum_{k\neq k'}\sum\sigma_{kk'}\phi_k(x)\phi_{k'}(y)\right]^2 dxdy$$
$$=L^2\sum_{k}^{q_n}\sum_{k'=0}^{q_n}\sigma_{kk'}^2 ,$$

where $L=\sup_n f(x)$. The second term in (a6) is

(a8) $$\mathrm{E}\left[\sum_{k}\sum_{k'}\sigma_{kk'}a_k\phi_{k'}(x)\right]^2=\int\left(\sum_{k}\sum_{k'}\sigma_{kk'}a_k\phi_{k'}(x)\right)^2 f(x)dx$$
$$\leqq L\int\left[\sum_{k'}\left(\sum_{k}a_k\sigma_{kk'}\right)\phi_{k'}(x)\right]^2 dx$$
$$=L\sum_{k'}\left(\sum_{k}a_k\sigma_{kk'}\right)^2$$
$$\leqq L\left(\sum_{k=0}^{q_n}a_k^2\right)\left(\sum_{k}^{q_n}\sum_{k'=0}^{q_n}\sigma_{kk'}^2\right) .$$

The last term in (a6) is

(a9) $$\left(\sum_{k}\sum_{k'}\sigma_{kk'}a_k a_{k'}\right)^2\leqq\left(\sum a_k^2\right)^2\left(\sum_{k}\sum_{k'}\sigma_{kk'}^2\right) .$$

Using (a7)–(a9) in (a6) we get

$$\frac{1}{\sigma_n^2}\,\mathrm{E}\left[\sum_{k}\sum_{k'}\sigma_{kk'}\bar{\phi}_k(X_1)\bar{\phi}_{k'}(X_2)\right]^2\leqq 4\left(L^2+2L\sum_{k=0}^{q_n}a_k^2+\left(\sum_{k=0}^{q_n}a_k^2\right)^2\right)=M \qquad \text{(say)} .$$

Since $\sigma_n^2 \to \infty$ as $n \to \infty$, $L < \infty$ and $\sum_{k=0}^{\infty} a_k^2 < \infty$, we conclude from (a5) that $\sum_{j<j'}^{n}\sum^{n} |\mathrm{E}\, D_n(jj')| \leqq \frac{16M}{n^4\sigma_n^2} \sum_{j<j'}^{n}\sum^{n} (j-1)(j-2) = O\left(\frac{1}{\sigma_n^2}\right) \to 0$ as $n \to \infty$. Therefore from (a4) we conclude that

$$2 \sum_{j<j'}\sum \mathrm{E}\, W_{nj}^2 W_{nj'}^2 \to 1 \qquad \text{as } n \to \infty\,.$$

This implies that $\mathrm{E}\,(\sum W_{nj}^2 - 1)^2 \to 0$ as $n \to \infty$.

(iii) To show $\sup_j |W_{nj}| \xrightarrow{\mathrm{P}} 0$.

$$\mathrm{P}\,[\sup_j |W_{nj}| > \varepsilon] \leqq \sum_{j=1}^{n} \mathrm{P}\,[W_{nj}^2 > \varepsilon^2] \leqq \frac{1}{\varepsilon^4} \sum_{j=1}^{n} \mathrm{E}\, W_{nj}^4 \to 0\,, \qquad \text{by (a3)}\,.$$

This completes the proof of Lemma 2.

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