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PERTURBATIONS OF COUNTABLE MARKOV CHAINS AND PROCESSES

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Abstract

If P is a transition matrix of a Markov chain, and $\tilde{P}$ is derived by perturbing the elements of P, then we find conditions such that $\tilde{P}$ is also positive recurrent when P is, and relate the invariant probability measures for the two. Similar results are found for recurrence of chains, and the methods then yield analogues for continuous time processes also.

1. Introduction

Suppose that $P=(p_{ij})$ is the transition matrix of a Markov chain $\{X_n\}$ on the countable state space $\{0, 1, 2, \cdots\}$; and that $\{X_n\}$ is irreducible (cf. Chung [1] for terminology). When P is positive recurrent, we denote its invariant probability measure by (π_j).

In this note we indicate some consequences of perturbing the matrix P: specifically, if $\tilde{P}=(\tilde{p}_{ij})$ is another transition matrix with

$$\tilde{p}_{ij}=p_{ij}(1+\Delta_{ij}) ,$$

then we find conditions on the perturbations (Δ_{ij}) which ensure that $\{\tilde{X}_n\}$ is positive recurrent when $\{X_n\}$ is, and we find bounds on the differences between π_j and $\tilde{\pi}_j$, where $(\tilde{\pi}_j)$ is the invariant measure for $\{\tilde{X}_n\}$. We also investigate similar questions concerning recurrence.

The question of preserving recurrence and positive recurrence under perturbation of P is examined in Tweedie [9]; there it is shown that for countable space chains, positivity is preserved provided $\Delta_{ij}\leqq 0$ for all but a finite number of pairs (i, j). Here we improve this by allowing positive Δ_{ij} for all pairs (i, j). The more important quantitative question of comparing the two stationary measures is considered in Schweitzer [4] and Takahashi [7]; in Section 2 we use a matrix result of the latter (Lemma 1 below), together with truncation techniques of

Seneta [5], [6], to extend Takahashi's results to the countable case. We also demonstrate that similar results hold in the preservation of recurrence, and show how the unique (possibly infinite) invariant measures of the chains are related when the transition matrix is perturbed.

In Section 3 we show that analogues of the results hold for continuous time processes also, using truncation techniques of Tweedie [8], and Takahashi's result applied to the Q-matrix of the process. For ease of reference we conclude the introduction by giving

LEMMA 1 (Takahashi [7], Lemma 9). *Let* $X=(x_{ij})$, $i, j=0, 1, \cdots, n$ *be any matrix, and define* y_{ij} *by setting* $y_{ij}=-x_{ij}$, $i\neq j$, *and* $x_{ii}=y_{ii}+\sum_{\substack{j\neq i\\ j=0}}^{n} y_{ij}$. *Then the determinant of* X *can be written*

$$\sum_{J} y_{0k_0}\cdots y_{nk_n} ,$$

where J *is some set of* $(n+1)$*-tuples* $(k_0, \cdots, k_n)$.

The key point of this result is that it expresses the determinant in terms of a sum of products with every term having a plus sign; this enables us to deduce inequalities on determinants from inequalities on the elements (y_{ij}).

2. Results for recurrent chains

We shall first prove the following result, which may be compared with Theorem 1 or (3.36) of Takahashi [7].

THEOREM 1. *Suppose that, for each state* i, *there exists* $\varepsilon_i \geqq 0$ *such that*

$$(1) \qquad (1+\varepsilon_i)^{-1}p_{ij} \leqq \tilde{p}_{ij} \leqq p_{ij}(1+\varepsilon_i) , \qquad j\neq i$$

and that the ε_i *satisfy*

$$(2) \qquad 1+\varepsilon \equiv \prod_{i=0}^{\infty} (1+\varepsilon_i) < \infty .$$

Then provided P *is positive recurrent, so is* $\tilde{P}$, *and* $\tilde{P}$ *has invariant measure* $(\tilde{\pi}_j)$, *satisfying*

$$(3) \qquad (1+\varepsilon)^{-2}\pi_j \leqq \tilde{\pi}_j \leqq (1+\varepsilon)^2 \pi_j .$$

PROOF. Clearly the left-hand side of (1) implies that $\tilde{P}$ is irreducible, since P is irreducible. Thus to prove $\tilde{P}$ positive recurrent, it suffices to prove that, if $\tilde{\tau}=\inf(n: \tilde{X}_n=0)$, then

$$\tilde{T}_0 = \mathrm{E}(\tilde{\tau} \mid \tilde{X}_0=0) < \infty .$$

Define τ, T_0 similarly for P. We need the following relations, valid for $\tilde{P}$ as well as for P, with suitable notational changes. First, let $T_i = \mathrm{E}(\tau | X_0 = i)$; then

$$T_0 = \sum_{i \neq 0} p_{0i}[1 + T_i] . \tag{4}$$

We write $A = (a_{ij})$ for the (substochastic) matrix formed from P by deleting the first row and column of P, i.e., $a_{ij} = p_{ij}$, for $i, j = 1, 2, \cdots$. Then, if the nth power of A has elements $(a_{ij}^{(n)})$, we have

$$T_i = \sum_{n=1}^{\infty} \Pr(\tau \geqq n | X_0 = i) = \sum_{m=0}^{\infty} \sum_{j=1}^{\infty} a_{ij}^{(m)} , \tag{5}$$

setting $a_{ij}^{(0)} = 1$ if $i = j$ and 0 otherwise. Now let ${}_{(n)}A$ denote the $n \times n$ "northwest truncation" of A; i.e. ${}_{(n)}A$ has elements a_{ij}, $i, j = 1, \cdots, n$. If $C_{ij}(n)$ is the cofactor of the (i, j) element of $[I_n - {}_{(n)}A]$ and $\Delta(n)$ is the determinant of $[I_n - {}_{(n)}A]$, where I_n is the $n \times n$ identity matrix, then from Seneta [5], [6], $\Delta(n) > 0$ and

$$C_{ji}(n)/\Delta(n) \uparrow \sum_{m=0}^{\infty} a_{ij}^{(m)} . \tag{6}$$

By using (4)–(6) we can reduce the problem of comparing $\tilde{T}_0$ with T_0 to a finite matrix problem. In Lemma 1, if we take $X = I_n - {}_{(n)}A$, then for $i \neq j$ we have $y_{ij} = p_{ij}$ and for $i = j$, $y_{ii} = \sum_{j=n+1}^{\infty} p_{ij}$. Hence

$$\Delta(n) = \sum_{J} p_{1k_1} \cdots p_{nk_n} \tag{7}$$

where the sum is over a set J of ordered n-tuples $(k_1, \cdots, k_n)$; note that $k_l \neq l$ for any l and that since y_{ll} may occur, the k_l may be larger than n. Similarly, by taking X as $I_n - {}_{(n)}A$ with the jth row and ith column deleted, we find that

$$C_{ji}(n) = \sum_{J_{ji}} p_{1k_1} \cdots p_{j-1k_{j-1}} p_{j+1k_{j+1}} \cdots p_{nk_n} \tag{8}$$

where the summation set J_{ji} of ordered $(n-1)$-tuples is again such that $k_l \neq l$. Now from (7), (8) and our assumption (1),

$$\prod_{\substack{k=1 \\ k \neq j}}^{n} [1 + \varepsilon_k]^{-1} C_{ji}(n) \leqq \tilde{C}_{ji}(n) \leqq \prod_{\substack{k=1 \\ k \neq j}}^{n} [1 + \varepsilon_k] C_{ji}(n) \tag{9}$$

$$\prod_{k=1}^{n} [1 + \varepsilon_k]^{-1} \Delta(n) \leqq \tilde{\Delta}(n) \leqq \prod_{k=1}^{n} [1 + \varepsilon_k] \Delta(n) . \tag{10}$$

Using (9) and (10) in (6), we have, letting $n \to \infty$,

$$\prod_{k=1}^{\infty} [1 + \varepsilon_k]^{-2} \sum_{m=0}^{\infty} a_{ij}^{(m)} \leqq \sum_{m=0}^{\infty} \tilde{a}_{ij}^{(m)} \leqq \prod_{k=1}^{\infty} [1 + \varepsilon_k]^2 \sum_{m=0}^{\infty} a_{ij}^{(m)} , \tag{11}$$

and so from (5)

$$\prod_{k=1}^{\infty}[1+\varepsilon_k]^{-2}[1+T_i]\leqq 1+\tilde{T}_i\leqq\prod_{k=1}^{\infty}[1+\varepsilon_k]^2[1+T_i]\ .$$

Putting this into (4), and using (1) again on p_{0i} and $\tilde{p}_{0i}$, we see that

$$[1+\varepsilon_0]^{-1}\prod_{k=1}^{\infty}[1+\varepsilon_k]^{-2}T_0\leqq\tilde{T}_0\leqq[1+\varepsilon_0]\prod_{k=1}^{\infty}[1+\varepsilon_k]^2T_0\ . \tag{12}$$

Since $T_0<\infty$ by the positive recurrence of P, from (12) we have $\tilde{T}_0<\infty$ and so $\tilde{P}$ is also positive recurrent.

This method of proof also yields the inequalities (3) as a straightforward byproduct; we have only to observe that $\pi_0=T_0^{-1}$ to see that (3) for $j=0$ follows immediately from (12), and since the state 0 could be replaced by an arbitrary state throughout the above working, (3) holds in full generality.

The same sort of result holds for recurrent chains. If P is recurrent, let $\{\alpha_j\}$ denote its unique (possibly infinite) invariant measure, normalised so that $\alpha_0=1$.

THEOREM 2. *Suppose that P and $\tilde{P}$ satisfy* (1) *and* (2). *Then provided P is recurrent, so is $\tilde{P}$; and $\tilde{P}$ has invariant measure $\{\tilde{\alpha}_j\}$ satisfying*

$$(1+\varepsilon)^{-2}\alpha_j\leqq\tilde{\alpha}_j\leqq(1+\varepsilon)^2\alpha_j\ . \tag{13}$$

PROOF. To prove $\tilde{P}$ recurrent, it suffices to prove the divergence of $\sum_m \tilde{p}_{00}^{(m)}$. But again we can use finite approximations (see Seneta [5], [6]) to find

$$\sum_0^{\infty} p_{00}^{(m)}=\lim_{n\to\infty}\Delta'_{00}(n)/\Delta'(n)$$

where $\Delta'(n)$ denotes the determinant and $\Delta'_{ij}(n)$ the cofactor of the (i, j)-element of $[I_n-{}_{(n)}P]$; hence from (1) and (2), $\sum_0^{\infty} p_{00}^{(m)}$ and $\sum_0^{\infty}\tilde{p}_{00}^{(m)}$ converge or diverge together, by applying Lemma 1 with $X=I_n-{}_{(n)}P$ as in (7), (8).

To compare invariant measures, we note that we can again construct α_j (see Seneta [5], [6]) by

$$\alpha_j=\lim_{n\to\infty}\Delta'_{j0}(n)/\Delta'(n)\ . \tag{14}$$

To deduce (13) we compare the approximations of α_j and $\tilde{\alpha}_j$, given by (14), using (2), as in the proof of Theorem 1.

Remarks. (a) Many of the other results of Takahashi [7] could be extended to the countable case using the truncation techniques above, as they rely purely on positive term expansions analogous to (7) and (8). We should also note that our results are not optimally expressed. The inequalities in (3) can clearly be tightened by using (11) more precisely in (4), giving the bounds

$$(15)\qquad \begin{aligned}&(1+\varepsilon_0)(1+\varepsilon)^{-2}\{T_0+(1-p_{00})(\varepsilon-\varepsilon_0)(\varepsilon+\varepsilon_0+2)\}\leqq\tilde{T}_0\\ &\leqq(1+\varepsilon_0)^{-1}(1+\varepsilon)^2\{T_0-(1-p_{00})(\varepsilon-\varepsilon_0)(\varepsilon+\varepsilon_0+2)/(1+\varepsilon)^2\}\ .\end{aligned}$$

These bounds are, in particular, better than those in (3) when ε_0 is close to ε and p_{00} is small.

(b) If P is positive recurrent, then we can use (14) to compare π_j/π_0 and $\tilde{\pi}_j/\tilde{\pi}_0$; however, this direct approximation of the invariant measure clearly gives worse bounds (of order $(1+\varepsilon)^4$) to the probabilistically normalised measure $\tilde{\pi}_j$ than does the indirect calculation in the proof of Theorem 1.

Comparing the results of Theorems 1 and 2 with those of Tweedie [9], it is plausible to conjecture that positive recurrence or recurrence should be preserved provided only the right-hand inequality of (1) holds. The next two examples show that this is not in fact true.

Example 1. Let P be defined by

$$p_{00}=p_{01}=1/2\ ;$$

$$p_{n,n+1}=p_{n,n-1}=[1-p_{n0}]/2\ ,\qquad n\geqq 1\ .$$

A sufficient condition for P to be positive recurrent is that, for all sufficiently large n, and some $\varepsilon>0$

$$(16)\qquad -n^2p_{n0}+1-p_{n0}\leqq-\varepsilon\ ;$$

see Tweedie [10], Proposition 7.2.

Now form $\tilde{P}$ by adding all the mass of p_{n0} equally to $p_{n,n+1}$ and $p_{n,n-1}$; that is, set

$$\tilde{p}_{n,n+1}=\tilde{p}_{n,n-1}=1/2\ .$$

Clearly $\tilde{P}$ is a null-recurrent random walk, and

$$\tilde{p}_{ij}\leqq p_{ij}[1-p_{n0}]^{-1}\ .$$

Hence provided p_{n0} satisfies (16) and

$$(17)\qquad \sum[1-p_{n0}]^{-1}<\infty\ ,$$

we have the counterexample we seek. Such a sequence is given by, for example, $p_{n0}=[n\log n]^{-1}$, $n\geqq 2$.

Example 2. To show that recurrence need not be preserved by maintaining only the right-hand inequality in (1), let us take the same chain as in Example 1, but with $p_{n,n+1}=c_n+1/n$, $p_{n,n-1}=c_n$ where c_n is determined by additivity; and change P to give $\tilde{P}$ exactly as before. The resulting chain is transient provided, for some $\theta>1$, and all large enough n,

$$2/[1-p_{n0}]-\theta\geqq 0\ ; \tag{18}$$

see Tweedie [10], p. 770. Hence when $p_{n0}\to 0$ as $n\to\infty$, this suffices for $\tilde{P}$ to be transient. The original chain P is recurrent provided, for large enough n,

$$-np_{n0}+1/n\leqq 0\ ; \tag{19}$$

see Tweedie [10], Section 10. Again, a choice of p_{n0} satisfying (17)–(19) is $p_{n0}=[n\log n]^{-1}$.

3. Results for recurrent processes

We now extend these results to Markov processes in continuous time. Let (X_t) be a process on the integers, with standard transition probabilities $p_{ij}(t)$, and let $Q=(q_{ij})$ be its Q-matrix, given by

$$q_{ij}=\lim_{t\downarrow 0}\,[\delta_{ij}-p_{ij}(t)]\ ;$$

see Chung [1] for nomenclature. We assume $q_{ii}<0$ for all i, so that no state is instantaneous, and our processes are all assumed conservative, i.e. $\sum_j q_{ij}=0$. We let $P^*=(p^*_{ij})$ denote the transition matrix of the jump chain of the process (X_t), defined by

$$p^*_{ij}=[1-\delta_{ij}]q_{ij}/(-q_{ii})\ . \tag{20}$$

Again we shall consider a second process $(\tilde{X}_t)$, and define all quantities above analogously for $(\tilde{X}_t)$. Our main result is

THEOREM 3. *Suppose the Q-matrices of (X_t) and $(\tilde{X}_t)$ are related by*

$$(1+\varepsilon_i)^{-1}q_{ij}\leqq\tilde{q}_{ij}\leqq(1+\varepsilon_i)q_{ij}\ ,\qquad i\neq j \tag{21}$$

and that

$$(1+\varepsilon)\equiv\prod_i\,(1+\varepsilon_i)<\infty\ .$$

Then if (X_t) is regular and recurrent, or positive recurrent, so is $(\tilde{X}_t)$;

and in the recurrent case their unique invariant measures, (α_j) *and* $(\tilde{\alpha}_j)$, *normalised to have* $\alpha_0=\tilde{\alpha}_0=1$, *are related by*

$$(1+\varepsilon)^{-2}\alpha_j \leqq \tilde{\alpha}_j \leqq (1+\varepsilon)^2\alpha_j \ . \tag{22}$$

In the positive recurrent case the unique invariant probability measures (π_j) *and* $(\tilde{\pi}_j)$ *are related by*

$$(1+\varepsilon)^{-4}\pi_j \leqq \tilde{\pi}_j \leqq (1+\varepsilon)^4\pi_j \ . \tag{23}$$

PROOF. If (X_t) is regular and recurrent, then the jump chain P^* is also obviously recurrent; the converse of this is also true (see Çinlar [2], 3.25, or Tweedie [8], Theorem 1). From (20) and (21), the two jump chain matrices satisfy $(1+\varepsilon_i)^{-2}p^*_{ij} \leqq \tilde{p}^*_{ij} \leqq (1+\varepsilon_i)^2 p^*_{ij}$; so from Theorem 2, $\tilde{P}^*$ is also recurrent, and hence $(\tilde{X}_t)$ is regular and recurrent also.

The unique invariant measure (α_j) for (X_t), normalised to have $\alpha_0=1$, satisfies, from Theorem 4(i) of Tweedie [8],

$$\alpha_j = \lim_{n\to\infty} D_{j0}(n)/D_{00}(n) \ , \tag{24}$$

where now $D_{ji}(n)$ is the cofactor of the (j, i) element of ${}_{(n)}Q$, the $n\times n$ truncation of Q. But from Lemma 1 with $X={}_{(n)}Q$, the $D_{ji}(n)$ can be expanded exactly as in (7), (8) as sums of products with positive coefficients:

$$D_{ji}(n) = \sum_{J_{ji}} (-q_{0k_0})\cdots(-q_{j-1k_{j-1}})(-q_{j+1k_{j+1}})\cdots(-q_{nk_n}) \tag{25}$$

where $j_l \neq l$. Because of the ratio nature of (24), the $(-1)^n$ terms in (25) for $D_{j0}(n)$ and $D_{00}(n)$ cancel, and from (21) we deduce (22). Note that $\sum \alpha_j$ and $\sum \tilde{\alpha}_j$ are finite or not together, whence the preservation of positive recurrence; and the bounds (23) follow trivially from those in (22).

These error bounds in (23) seem cruder than necessary. We conjecture that when the bounds are of the type of (21) on the Q-matrices, the relationship between (π_j) and $(\tilde{\pi}_j)$ should also be of the order of (3). This is lent weight by our final result.

THEOREM 4. *Suppose* (X_t) *is q-bounded, i.e.* $\sup(-q_{ii})<\infty$, *and positive recurrent. Then when* (21) *holds, the invariant probability distributions* (π_j) *and* $(\tilde{\pi}_j)$ *of* (X_t) *and* $(\tilde{X}_t)$ *are related by*

$$(1+\varepsilon)^{-2}\pi_j \leqq \tilde{\pi}_j \leqq (1+\varepsilon)^2\pi_j \ . \tag{26}$$

PROOF. Choose $0<\lambda<\sup_i(-q_{ii}^{-1}, -\tilde{q}_{ii}^{-1})$, and let $P=I+\lambda Q$, $\tilde{P}=I+\lambda\tilde{Q}$, when I is the identity matrix. Now we can use the fact that P and (X_t) are positive recurrent together, and then have the same invariant probability measure; see Jensen and Kendall [3]. Since P and $\tilde{P}$ satisfy (1) for $i \neq j$, from (21), we have the inequality (26) immediately from this fact.

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CSIRO

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