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CHARACTERIZATION OF EQUIREPLICATED VARIANCE-BALANCED BLOCK DESIGNS

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1. Introduction

Consider v treatments arranged in b blocks with the jth block being of size k_j $(j=1, 2, \cdots, b)$ in a block design with incidence matrix $N=\|n_{ij}\|$ such that the ith treatment occurs r_i times $(i=1, 2, \cdots, v)$ and the ith treatment occurs in the jth block n_{ij} times, where n_{ij} can take any of the values, $0, 1, 2, \cdots$, or $n-1$. Such a design is called an n-ary block design. If $n=2$, the design is called binary. When $r_1=r_2=\cdots=r_v$, the design is said to be equireplicated. Let T_i be the total yield for the ith treatment and B_j that for the jth block. On writing $T'=(T_1, T_2, \cdots, T_v)$ and $B'=(B_1, B_2, \cdots, B_b)$ in matrix notation, the adjusted intrablock normal equations for estimating the vector of treatment effects t can be written under the usual assumptions as $Q=C\hat{t}$, where $\hat{t}$ is the estimate of t,

$$Q=T-N\,\mathrm{diag}\,\{k_1^{-1}, k_2^{-1}, \cdots, k_b^{-1}\}B\ ,$$

$$C=\mathrm{diag}\,\{r_1, r_2, \cdots, r_v\}-N\,\mathrm{diag}\,\{k_1^{-1}, k_2^{-1}, \cdots, k_b^{-1}\}N'\ ,$$

and diag stands for a diagonal matrix and A' is the transpose of the matrix A, and further let $\mathrm{diag}\,\{r_1, r_2, \cdots, r_v\}=D_r$ and $\mathrm{diag}\,\{k_1, k_2, \cdots, k_b\}=D_k$. The matrix C is well known as the C-matrix of a block design.

Though the rank of C is at most $v-1$, we consider a case where the rank of C is $v-1$, in which case the design is said to be connected (cf. [3]). We shall deal only with connected designs throughout this paper.

A block design is said to be balanced if every elementary contrast of treatments is estimated with the same variance (cf. [16]). In this sense, this design is also called a variance-balanced block (BB) design. Furthermore, it is known (cf. [6], [8], [9], [10], [11], [13], [16]) that an n-ary BB design with parameters v, b, r_i, k_j $(i=1, 2, \cdots, v;\ j=1, 2, \cdots, b)$ can be given by an incidence matrix N satisfying

$$(C=)\ D_r-ND_k^{-1}N'=\rho\{I_v-(1/v)G_v\}\ ,$$

where $\rho=\left[\sum_{i=1}^{v} r_i-\sum_{j=1}^{b}\left\{(1/k_j)\sum_{i=1}^{v} n_{ij}^2\right\}\right]\Big/(v-1)$, I_v is the unit matrix of order v, $G_v=E_{v\times v}$ and $E_{l\times s}$ is an $l\times s$ matrix with positive unit elements everywhere. Note that for a binary BB design, $\rho=\left(\sum_{i=1}^{v} r_i-b\right)\Big/(v-1)$. For an n-ary BB design, ρ also depends on an incidence structure of the design.

The literature of block designs contains many articles exclusively related to BB designs. The interested reader can refer, for example, to [6], [8], [9], [10], [11], [13], [16] for details. In this paper, some block structure of equireplicated n-ary BB designs is investigated with illustrations. These results include the well known results as special cases.

Finally, since a design uniquely determines its incidence matrix and vice versa, both a design and its incidence matrix are denoted by the same symbol throughout this paper.

2. Characterization

An equireplicated n-ary BB design has the following C-matrix:

$$rI_v-ND_k^{-1}N'=\rho\{I_v-(1/v)G_v\}\ ,$$

i.e.,

$$ND_k^{-1}N'=(r-\rho)I_v+(\rho/v)G_v\ , \tag{2.1}$$

the determinant of the right-hand side of (2.1) is clearly $r(r-\rho)^{v-1}$, where $r=r_1=r_2=\cdots=r_v$. In this case, we first state

THEOREM 2.1. *For an equireplicated n-ary* BB *design with parameters v, b, r and k_j $(j=1, 2,\cdots, b)$ in which $C=\rho\{I_v-(1/v)G_v\}$, the following holds:*
(1) If $v>b$, then $\rho=r$.
(2) If $v=b$, then $k_1k_2\cdots k_b r(r-\rho)^{v-1}$ is a perfect square or zero.
In particular, when the design is binary, $k_1k_2\cdots k_b r\{(b-r)/(v-1)\}^{v-1}$ is a perfect square or zero.

PROOF. If $v>b$, then $|ND_k^{-1}N'|=0$ which from (2.1) yields $\rho=r$. This implies (1). If $v=b$, then N is a square matrix. Hence $|N|^2=k_1k_2\cdots k_b r(r-\rho)^{v-1}$ which must be a perfect square or zero. Especially, when the design is binary, we have $\rho=(vr-b)/(v-1)$ and then $r-\rho=(b-r)/(v-1)$. Thus, the required result (2) is obtained.

Note that the latter case of the result (2) in Theorem 2.1 is previously shown by Atiqullah [1]. In the former of the result (2) of Theorem 2.1, if $k_1=k_2=\cdots=k_b$ $(=k$, say), then $r=k$ and $|N|^2=r^{v+1}(r-$

$\rho)^{v-1}$. Hence we get

COROLLARY 2.1. *For an equireplicated, equiblock-sized n-ary* BB *design with parameters* v, b, r *and* k *in which* $C=\rho\{I_v-(1/v)G_v\}$, *if* v $(=b)$ *is even, then* $r(r-\rho)$ *is a perfect square or zero.*

Note that when $n=2$, Corollary 2.1 yields the well known result that for a symmetrical balanced incomplete block (BIB) design with parameters v, b, r, k and λ, if v is even, then $r-\lambda$ is a perfect square.

Relaxing an assumption of symmetry in (2) of Theorem 2.1, we have the following.

THEOREM 2.2. *For an equireplicated n-ary* BB *design with parameters* v, b, r *and* k_j $(j=1, 2, \cdots, b)$ *in which* $C=\rho\{I_v-(1/v)G_v\}$ *and* $\rho<r$, *if* $v<b$, *then* $r(r-\rho)^{v-1}k_{b-v+1}\cdots k_b|C_{b-v}|$ *is a perfect square or zero, where* $C_{b-v}=I_{b-v}-\{1/(r-\rho)\}\{D_{k_1}^{-1/2}N_1'N_1D_{k_1}^{-1/2}-(\rho/vr)\|\sqrt{k_jk_{j'}}\|\}$ $(j, j'=1, 2, \cdots, b-v)$, $D_{k_1}=\mathrm{diag}\,\{k_1, k_2, \cdots, k_{b-v}\}$, *and* N_1 *is a submatrix consisting of the first* $(b-v)$ *columns of* N.

PROOF. Let the partition of b blocks of N be $N=[N_1 : N_2]$, where N_1 is a $v\times(b-v)$ matrix and N_2 is a square matrix of order v. Further let the partition of sizes of blocks corresponding to the above partition of blocks be

$$D_k=\begin{bmatrix} D_{k_1} & 0 \\ 0 & D_{k_2} \end{bmatrix},$$

where $D_{k_1}=\mathrm{diag}\,\{k_1, k_2, \cdots, k_{b-v}\}$ and $D_{k_2}=\mathrm{diag}\,\{k_{b-v+1}, \cdots, k_b\}$. Then form

$$A=\begin{bmatrix} N_1D_{k_1}^{-1/2} & N_2D_{k_2}^{-1/2} \\ I_{b-v} & 0 \end{bmatrix}, \tag{2.2}$$

from which it follows that

$$AA'=\begin{bmatrix} ND_k^{-1}N' & N_1D_{k_1}^{-1/2} \\ D_{k_1}^{-1/2}N_1' & I_{b-v} \end{bmatrix}.$$

The determinant of AA' can be given by

$$\begin{aligned} |AA'| &= |ND_k^{-1}N'|\,|I_{b-v}-D_{k_1}^{-1/2}N_1'(ND_k^{-1}N')^{-1}N_1D_{k_1}^{-1/2}| \\ &= r(r-\rho)^{v-1}|C_{b-v}|, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} C_{b-v} &= I_{b-v}-D_{k_1}^{-1/2}N_1'(ND_k^{-1}N')^{-1}N_1D_{k_1}^{-1/2} \\ &= I_{b-v}-D_{k_1}^{-1/2}N_1'[(1/vr)G_v+\{1/(r-\rho)\}\{I_v-(1/v)G_v\}]N_1D_{k_1}^{-1/2} \\ &\qquad \text{from (2.1)} \end{aligned}$$

$$=I_{b-v}-\{1/(r-\rho)\}\{D_{k_1}^{-1/2}N_1'N_1D_{k_1}^{-1/2}-(\rho/vr)\|\sqrt{k_jk_{j'}}\|\}$$

for $j, j'=1, 2, \cdots, b-v$. Furthermore, from (2.2)

$$|AA'|=|A|^2=|N_2|^2|D_{k_2}^{-1}|=|N_2|^2(k_{b-v+1}k_{b-v+2}\cdots k_b)^{-1}. \tag{2.4}$$

Therefore, relations (2.3) and (2.4) give the required result.

Remark 2.1. A part of the conditions given in Theorem 2.2, $k_{b-v+1}\cdots k_b|C_{b-v}|$, depends on the partition of blocks of N.

Remark 2.2 If $b=v$, the proof of Theorem 2.2 contains the result (2) of Theorem 2.1.

Remark 2.3. When a binary design is considered, Theorem 2.2 is useful in checking whether for a fixed N_1 a new design $[N_1 : N_2]$ is balanced by juxtaposing an appropriate N_2, because ρ is easily calculated.

It is known (cf. [2]) that for an equireplicated binary BB design, if $b=v$, then the block sizes are constant (and hence the BB design is a symmetrical BIB design). The corresponding result is not generally valid for an equireplicated n-ary BB design with $b=v$. For example, we have a symmetrical n-ary BB design with parameters $v=b=7$, $r=6$, $k_j=3$ or 24, and $C=(14/3)\{I_7-(1/7)G_7\}$, with seven blocks, (1, 2, 2) (1, 3, 3) (1, 4, 4) (1, 5, 5) (1, 6, 6) (1, 7, 7) (2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7) (cf. [7]). A method of constructing a BB design of this type will be mentioned in the last section. As some block structure for an n-ary case, we have

THEOREM 2.3. *For an equireplicated n-ary* BB *design with parameters v, b, r and k_j $(j=1, 2, \cdots, b)$ in which $C=\rho\{I_v-(1/v)G_v\}$, if $\rho<r$ and $v=b$, then both $(r-\rho)k_j+(\rho/vr)k_j^2$ and $\rho k_jk_{j'}/vr$ for all j, j' are integers. Furthermore, regarding a block as a column vector of the incidence matrix,*

(a) *there does not exist a pair of blocks which are identical or proportional;*
(b) *there does not exist a block of type $\alpha E_{v\times 1}$ where α $(<r)$ is any scalar;*
(c) *if the inner product of any two blocks is constant, then $k_1=k_2=\cdots=k_b$;*
(d) *the block sizes of binary blocks are identical;*
(e) *the block intersection numbers of any two binary blocks are constant;*
(f) *if $k_1=k_2=\cdots=k_b$, then $NN'=N'N$ holds.*

PROOF. If $v=b$ and $\rho<r$, then considering the inverse matrix of both the side of (2.1), we have

(2.5) $$D_k=\{1/(r-\rho)\}N'N-\{\rho/vr(r-\rho)\}N'G_vN\ ,$$

or

(2.5)′ $$N'N=(r-\rho)D_k+(\rho/vr)\|k_jk_{j'}\|\ .$$

Comparing the diagonal elements and off-diagonal elements of (2.5)′, respectively, yields that both $(r-\rho)k_j+(\rho/vr)k_j^2$ and $\rho k_jk_{j'}/vr$ are integers for all j, j' $(j\neq j')=1, 2, \cdots, b$. Now, letting $\mu_{jj'}=\sum_{i=1}^{v} n_{ij}n_{ij'}$, we have from (2.5)

(2.6) $$(\rho/vr)k_j^2+(r-\rho)k_j-\mu_{jj}=0\ ,$$

(2.7) $$\mu_{jj'}=(\rho/vr)k_jk_{j'}\qquad (j\neq j')\ .$$

(a) If there exists a pair of blocks which are l $(\geqq 1)$ times proportional, then from (2.7) we have $\mu_{jj}=(\rho/vr)k_j^2$ which yields from (2.6) $\rho=r$ which is a contradiction to $\rho<r$. Hence there does not exist a pair of blocks which are identical or proportional. (b) Similarly, if there exists a block of type $\alpha E_{v\times 1}$, then from (2.6) and (2.7) we have $r=\alpha$ which is a contradiction. (c) If $\mu_{jj'}=\mu$, a constant, for all j, j', then from (2.7) we have $k_jk_{j'}=vr\mu/\rho$ for all j, j' $(j\neq j')=1, 2, \cdots, b$, which yield $k_1=k_2=\cdots=k_b$. (d) If the jth block is binary, then $\mu_{jj}=k_j$. Hence from (2.6) $k_j=vr(1+\rho-r)/\rho$ is a constant for all $j=1, 2, \cdots, b$. (e) Similarly to (d), from (2.7) we have the required result. (f) From (2.5) we have $N'N=(r-\rho)kI_v+(\rho k/v)G_v$. On the other hand, from the definition of the C-matrix we have $NN'=k(r-\rho)I_v+(k\rho/v)G_v$. Hence $NN'=N'N$ holds.

Note that if the design in Theorem 2.3 is binary, then (2.6) yields $k_1=k_2=\cdots=k_b$. This is Bhaskararao's result which was described before Theorem 2.3. However, in an equireplicated binary BB design, if a condition $b=v$ is relaxed, then we obtain

THEOREM 2.4. *For an equireplicated binary* BB *design with parameters v, b, r and k_j $(j=1, 2, \cdots, b)$ in which $C=\rho\{I_v-(1/v)G_v\}$ and $\rho<r$, if $b>v$, then*

$$k_j\geqq \max\left\{2, \frac{vr(v+r-b-1)}{vr-b}\right\}\ ,\qquad j=1, 2, \cdots, b\ .$$

PROOF. Since the design N is equireplicated, from the C-matrix we have

(2.8) $$\begin{aligned} ND_k^{-1}N'&=(r-\rho)I_v+(\rho/v)G_v\\ &=r\{(1/v)G_v\}+(r-\rho)\{I_v-(1/v)G_v\}\end{aligned}$$

which is a spectral decomposition of $ND_k^{-1}N'$. Furthermore, since $ND_k^{-1}N' = ND_k^{-1/2}D_k^{-1/2}N'$, it follows from (2.8) that the spectral decomposition of $D_k^{-1/2}N'ND_k^{-1/2}$ is given by

$$D_k^{-1/2}N'ND_k^{-1/2} = r\left\{\frac{1}{r}D_k^{-1/2}N'\left(\frac{1}{v}G_v\right)ND_k^{-1/2}\right\}$$
$$+(r-\rho)\left\{\frac{1}{r-\rho}D_k^{-1/2}N'\left(I_v - \frac{1}{v}G_v\right)ND_k^{-1/2}\right\} + 0D ,$$

where rank $(D) = b - v$ and D is a projection corresponding to the zero latent root of $D_k^{-1/2}N'ND_k^{-1/2}$, and is expressed as

$$\text{(2.9)} \quad D = I_b - \frac{1}{r}D_k^{-1/2}N'\left(\frac{1}{v}G_v\right)ND_k^{-1/2} - \frac{1}{r-\rho}D_k^{-1/2}N'\left(I_v - \frac{1}{v}G_v\right)ND_k^{-1/2}$$
$$= I_b - \frac{1}{vr}D_k^{-1/2}N'G_vND_k^{-1/2} + \frac{1}{(r-\rho)v}D_k^{-1/2}N'G_vND_k^{-1/2}$$
$$- \frac{1}{r-\rho}D_k^{-1/2}N'ND_k^{-1/2}$$
$$= I_b + \frac{\rho}{vr(r-\rho)}D_k^{-1/2}\|k_jk_{j'}\|D_k^{-1/2} - \frac{1}{r-\rho}D_k^{-1/2}N'ND_k^{-1/2} ,$$

which implies

$$\text{(2.10)} \qquad D_k^{1/2}DD_k^{1/2} = D_k + \frac{\rho}{vr(r-\rho)}\|k_jk_{j'}\| - \frac{1}{r-\rho}N'N .$$

Since diagonal elements of D are nonnegative, comparing the jth diagonal element of (2.10) yields

$$\text{(2.11)} \qquad k_j + \frac{\rho}{vr(r-\rho)}k_j^2 - \frac{1}{r-\rho}\mu_{jj} \geqq 0 , \qquad j = 1, 2, \cdots, b .$$

Now, N is binary. Then we have $\mu_{jj} = k_j$ and $\rho = (vr-b)/(v-1)$. Moreover, from (2.11) we get

$$\frac{vr-b}{vr(b-r)}k_j\left\{k_j - \frac{vr(v+r-b-1)}{vr-b}\right\} \geqq 0 , \qquad j = 1, 2, \cdots, b ,$$

which imply $k_j \geqq vr(v+r-b-1)/(vr-b)$ for $j = 1, 2, \cdots, b$.

Remark 2.4. If $b = v$, then $D_k^{1/2}DD_k^{1/2} = 0$ holds and thus (2.10) and (2.11) yield (2.5) and (2.6), respectively.

We now describe two examples of Theorem 2.4.

Example 2.1. Let M_1 be a symmetrical BIB design with parameters $v^* = b^*$, $r^* = k^*$ and λ^*. Then $N = [E_{v\times 1} : M_1]$ is a BB design with parameters $v=v^*$, $b=v^*+1$, $r=k^*+1$, $k_j=v^*$ or k^* $(j=1, 2, \cdots, b)$. In this case, $vr(v+r-b-1)/(vr-b)=v^*(k^{*2}-1)/(v^*k^*-1)=k^*-(v^*-k^*)/(v^* \cdot k^*-1)$ and $(v^*-k^*)/(v^*k^*-1)<1$. Thus, for this design N, Theorem 2.4 yields $k_j \geqq k^*$ $(j=1, 2, \cdots, b)$ which is attainable. For example, consider a BB design with parameters $v=7$, $b=8$, $r=4$, $k_j=3$ or 7, whose incidence matrix is given by

$$\begin{bmatrix} 1&1&1&1&0&0&0&0 \\ 1&1&0&0&1&1&0&0 \\ 1&0&1&0&1&0&1&0 \\ 1&1&0&0&0&0&1&1 \\ 1&0&0&1&1&0&0&1 \\ 1&0&0&1&0&1&1&0 \\ 1&0&1&0&0&1&0&1 \end{bmatrix} \quad \text{and} \quad C=\frac{10}{3}\left(I_7-\frac{1}{7}G_7\right).$$

In this case, Theorem 2.4 implies $k_j \geqq \max\{2, 14/5\}=14/5$, i.e., $k_j \geqq 3$.

Example 2.2. Let M_2 be a BIB design with parameters v^*, b^* $(=v^*+t-1)$, $r^*=\mu t$, k^* and λ^* for some positive integers μ and t. Then $[E_{v\times\mu}: M_2]$ is a BB design with parameters $v=v^*$, $b=v^*+t+\mu-1$, $r=\mu t+\mu$, $k_j=v^*$ or k^*. In this case, $vr(v+r-b-1)/(vr-b)=v^*\mu(\mu-1)t(t+1)/\{v^*\mu(t+1)-v^*-t-\mu+1\}$ which may be greater than two, provided $\mu \geqq 2$ and $t \geqq 2$. For example, consider a BB design with parameters $v=9$, $b=14$, $r=10$, $k_j=6$ or 9, whose incidence matrix is given by

$$\begin{bmatrix} 1&1&1&0&0&1&1&1&1&0&0&1&1&1 \\ 1&1&0&1&0&0&1&1&1&1&1&0&1&1 \\ 1&1&1&0&1&0&0&1&1&1&1&1&0&1 \\ 1&1&1&1&0&1&0&0&1&1&1&1&1&0 \\ 1&1&1&1&1&0&1&0&0&1&0&1&1&1 \\ 1&1&1&1&1&1&0&1&0&0&1&0&1&1 \\ 1&1&0&1&1&1&1&0&1&0&1&1&0&1 \\ 1&1&0&0&1&1&1&1&0&1&1&1&1&0 \\ 1&1&1&1&1&1&1&1&1&1&0&0&0&0 \end{bmatrix} \quad \text{and} \quad C=\frac{19}{2}\left(I_9-\frac{1}{9}G_9\right).$$

In this case, Theorem 2.4 implies $k_j \geqq \max\{2, 90/19\}=90/19$, i.e., $k_j \geqq 5$.

Remark 2.5. Consider a principal submatrix of order two $(=D^{(2)}$, say) of D in (2.9). For example,

$$D^{(2)}=I_2+\frac{\rho}{vr(r-\rho)}\begin{bmatrix} k_j & \sqrt{k_jk_{j'}} \\ \sqrt{k_jk_{j'}} & k_{j'} \end{bmatrix}-\frac{1}{r-\rho}\begin{bmatrix} 1 & \mu_{jj'}/\sqrt{k_jk_{j'}} \\ \mu_{jj'}/\sqrt{k_jk_{j'}} & 1 \end{bmatrix},$$

where $\mu_{jj'}$ is the number of treatments common to the jth block and the j'th block. Since $\rho=(vr-b)/(v-1)$, we have

$$(2.12)\qquad |D^{(2)}|=-\left(\frac{v-1}{b-r}\right)^2\frac{\mu_{jj'}^2}{k_jk_{j'}}+\frac{2(vr-b)(v-1)}{vr(b-r)^2}\mu_{jj'}$$
$$+\frac{(vr-b)(b-r-v+1)}{vr(b-r)^2}(k_j+k_{j'})+\left(\frac{b-v-r+1}{b-r}\right)^2$$

which must be nonnegative, because D is positive semidefinite. The last fact shows that (2.12) yields a bound on the block intersection number $\mu_{jj'}$. In particular, when the design is a BIB design in which $k_1=k_2=\cdots=k_b$ $(=k$, say),

$$(2.13)\qquad |D^{(2)}|=\frac{1}{(b-r)^2}\left\{\frac{-(v-1)^2}{k^2}\mu_{jj'}^2+\frac{2(k-1)(v-1)}{k}\mu_{jj'}\right.$$
$$\left.+(b-v-r+1)(b-v-r+2k-1)\right\}.$$

Since (2.13) is nonnegative, we have

$$(2.14)\qquad \frac{(v-1)^2}{k^2}\mu_{jj'}^2-\frac{2(k-1)(v-1)}{k}\mu_{jj'}$$
$$-(b-v-r+1)(b-v-r+2k-1)\leqq 0$$

which yields

$$\mu_-\leqq\mu_{jj'}\leqq\mu_+\ ,$$

where

$$\mu_\pm=\frac{k(k-1)}{v-1}\pm\frac{k}{v-1}\sqrt{(k-1)^2+(b-v-r+1)(b-v-r+2k-1)}$$
$$=\frac{k(k-1)}{v-1}\pm\frac{k(b-v-r+k)}{v-1}$$
$$=\frac{k(k-1)}{v-1}\pm\frac{k}{v-1}\left\{k-1+\frac{(v-1)(r-\lambda-k)}{k}\right\}$$
$$=\frac{k(k-1)}{v-1}\pm\left\{\frac{k(k-1)}{v-1}+r-\lambda-k\right\},$$

and so $\mu_+=2k(k-1)/(v-1)+r-\lambda-k=2\lambda k/r+r-\lambda-k$ and $\mu_-=-(r-\lambda-k)$. Thus, it holds that for a BIB design

$$-(r-\lambda-k)\leqq\mu_{jj'}\leqq\frac{2\lambda k}{r}+r-\lambda-k\ ,$$

which corresponds to an important result due to Connor [4]. Furthermore, since it can be shown that $b-v-r+2k-1\geqq 0$ holds in a BIB design, if $\mu_{jj'}=0$, then from (2.14) $b-v-r+1\geqq 0$ holds. This implies that for a BIB design if there exists a pair of disjoint blocks, then Bose's inequality $b\geqq v+r-1$ holds.

Thus, the results described in Theorems 2.1 to 2.4, and Corollary 2.1 can be used as a test for the possible existence of a connected equireplicated BB design having appropriate specified values of parameters.

3. Some constructions

Note that an equiblock-sized binary BB design is equireplicated and hence the design is a BIB design (cf. [9], Theorem 13.1). As a characterization similar to this result for a nonbinary case, we get

THEOREM 3.1. *A nonbinary equiblock-sized* BB *design is a pairwise balanced design of index* $k\rho/v$.

PROOF. For an equiblock-sized BB design $N=\|n_{ij}\|$ with $k_1=k_2=\cdots=k_b$ $(=k$, say), the C-matrix of the design is

$$D_r-\frac{1}{k}NN'=\rho\left(I_v-\frac{1}{v}G_v\right) . \tag{3.1}$$

Comparing the off-diagonal elements of (3.1) yields:

$$\frac{1}{k}\sum_{j=1}^{b} n_{ij}n_{i'j}=\frac{\rho}{v}$$

which implies that N is a pairwise balanced design of index $k\rho/v$.

When a design is equireplicated, the proof of Theorem 3.1 yields

COROLLARY 3.1. *A nonbinary equiblock-sized, equireplicated* BB *design is a balanced n-ary block design of Tocher.*

Here, a balanced n-ary block design of Tocher [19] is an arrangement of V treatments in B blocks, each of size K, such that (i) each treatment occurs in the design R times and (ii) $\sum_{j=1}^{B} n_{ij}n_{i'j}=\Lambda$ (Λ, constant), $i\neq i'$, where n_{ij} is the number of times the ith treatment occurs in the jth block and can take any of the values, $0, 1, 2, \cdots,$ or $n-1$. Corollary 3.1 shows that a balanced n-ary block design of Tocher is a special case of an n-ary BB design. Hence, a number of known methods of constructing balanced n-ary block designs of Tocher (cf. [5], [14], [15], [18], [19]) can be used as construction methods of n-ary BB designs. This fact is very important, because there also exist methods of construction of balanced n-ary block designs using difference sets (cf. [17], [18]). Until now, construction of n-ary BB designs is mainly based on a composition method, and a trial and error method. Nobody ever explicitly presented the method of differences of constructing an n-ary BB design. In this sense, Corollary 3.1 may be very instructive.

Remark 3.1. For a balanced n-ary block design N with parameters V, B, R, K and Λ, we have directly

$$C=RI_V-(1/K)NN'=(\Lambda V/K)\{I_V-(1/V)G_V\}$$

which shows that the design N is an equireplicated n-ary BB design.

Finally, as described before, since some methods of constructing binary BB designs are variously known, we consider construction of nonbinary BB designs. John [8] and Kulshreshtha, Dey and Saha [13] gave some methods for construction of n-ary BB designs. We here present other simple methods of constructing n-ary BB designs by modifying some methods given in [9] and [10].

Method 3.1. If $a=d^2+d$, then the following matrix of order $a+1$ is a symmetrical n-ary BB design with parameters $v=b=a+1$, $r=a$, $k_j=d+1$ or ad^2, and $\rho=d(a+1)/(d+1)$:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 \\ d & & & & d^2 \\ & d & & 0 & d^2 \\ & & \ddots & & \vdots \\ & 0 & & & \vdots \\ & & & d & d^2 \end{bmatrix}.$$

This method can easily be shown by considering the C-matrix of the design. Note that the example before Theorem 2.3 is a special case of Method 3.1 when $a=6$ and $d=2$. We can also give some series of n-ary BB designs by incidence matrices similar to Method 3.1. For example, there are types as

$$\begin{bmatrix} a & a & \cdots & a & 0 \\ b & & & & c \\ & b & & 0 & \vdots \\ & & \ddots & & \vdots \\ & 0 & & & \vdots \\ & & & b & c \end{bmatrix}, \quad \begin{bmatrix} a & a & \cdots & a & 0 & 0 \\ b & b & \cdots & b & 0 & 0 \\ c & & & & d & e \\ & c & & 0 & \vdots & \vdots \\ & & \ddots & & \vdots & \vdots \\ & 0 & & & \vdots & \vdots \\ & & & c & d & e \end{bmatrix}.$$

Method 3.2. In the incidence matrix of a BIB design, by interchanging all 0's and all 1's into appropriate numbers n_1 ($\geqq 0$) and n_2 ($\geqq 0$), respectively, we can obtain n-ary BB designs for some n.

Method 3.3. In the incidence matrix of a binary BB design, by interchanging all 1's only into an appropriate number α ($\geqq 2$), we can get an n-ary BB design.

As mentioned above, we have

Method 3.4. Balanced n-ary block designs in the sense of Tocher are n-ary BB designs.

Remark 3.2. A BB design in the sense of Kiefer [12] is a special case of balanced n-ary block designs of Tocher.

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REFERENCES

[1] Atiqullah, M. (1961). On a property of balanced designs, *Biometrika*, **48**, 215-218.
[2] Bhaskararao, M. (1966). A note on equireplicate balanced designs with $b=v$, *Calcutta Statist. Ass. Bull.*, **15**, 43-44.
[3] Bose, R. C. (1950). *Least Squares Aspects of Analysis of Variance*, Institute of Statistics, Univ. of North Carolina.
[4] Connor, W. S. (1952). On the structure of balanced incomplete block designs, *Ann. Math. Statist.*, **23**, 57-71.
[5] Dey, A. (1970). On construction of balanced n-ary block designs, *Ann. Inst. Statist. Math.*, **22**, 389-393.
[6] Hedayat, A. and Federer, W. T. (1974). Pairwise and variance balanced incomplete block designs, *Ann. Inst. Statist. Math.*, **26**, 331-338.
[7] Ishii, G. (1977). Personal communication to S. Kageyama.
[8] John, P. W. M. (1964). Balanced designs with unequal numbers of replicates, *Ann. Math. Statist.*, **35**, 897-899.
[9] Kageyama, S. (1974). Reduction of associate classes for block designs and related combinatorial arrangements, *Hiroshima Math. J.*, **4**, 527-618.
[10] Kageyama, S. (1976). Constructions of balanced block designs, *Utilitas Math.*, **9**, 209-229.
[11] Kageyama, S. (1977). Note on combinatorial arrangements, *Hiroshima Math. J.*, **7**, 449-458.
[12] Kiefer, J. (1975). Balanced block designs and generalized Youden designs, I. Construction (Patchwork), *Ann. Statist.*, **3**, 109-118.
[13] Kulshreshtha, A. C., Dey, A. and Saha, G. M. (1972). Balanced designs with unequal replications and unequal block sizes, *Ann. Math. Statist.*, **43**, 1342-1345.
[14] Morgan, E. J. (1977). Construction of balanced n-ary designs, *Utilitas Math.*, **11**, 3-31.
[15] Murty, J. S. and Das, M. N. (1968). Balanced n-ary block designs and their uses, *J. Indian Statist. Ass.*, **5**, 73-82.
[16] Rao, V. R. (1958). A note on balanced designs, *Ann. Math. Statist.*, **29**, 290-294.
[17] Saha, G. M. (1975). On construction of balanced ternary designs, *Sankhyā*, B, **37**, 220-227.
[18] Saha, G. M. and Dey, A. (1973). On construction and uses of balanced n-ary designs, *Ann. Inst. Statist. Math.*, **25**, 439-445.
[19] Tocher, K. D. (1952). Design and analysis of block experiments, *J. R. Statist. Soc.*, B, **14**, 45-100.