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ROBUSTNESS OF CONNECTED BALANCED BLOCK DESIGNS

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Summary

It is well known that the connected incomplete block designs with the highest intrablock efficiency factors are balanced. In a connected balanced block (BB) design, every elementary contrast is estimated with the same variance. If a treatment is lost in a connected BB design, then the residual design is not balanced for the most part. In this paper, an upper bound and a lower bound for efficiency of a residual design are derived with some illustrations. Moreover, from these discussions it is conceivable that the loss of efficiency even in the unbalanced case is, in general, small.

1. Introduction

It is known (cf. Raghavarao [5], Roy [6]) that the connected incomplete block designs with the highest intrablock efficiency factors are balanced in the sense that all the elementary treatment contrasts are estimated with the same precision, provided a balanced block (BB) design exists. We consider here the situation in which for some reason one of the treatments in a connected BB design with unequal block sizes and with efficiency $\mathcal{E}$ is missing. In this case the resulting residual design is not, in general, balanced. It is important therefore to ask whether the balanced structure can be so fragile that the loss of a single treatment can produce a residual design so unbalanced that there is a serious loss of efficiency.

In this note, we discuss the general problem of the efficiency E of a residual design and establish a lower bound, $E_{\min}$, and an upper bound, $E_{\max}$, for it. Then $E_{\min} \leqq E \leqq E_{\max}$. The block design attaining $E=E_{\max}$ is called most efficient. Furthermore, $E_{\min}/\mathcal{E} \leqq E/\mathcal{E} \leqq E_{\max}/\mathcal{E}$, which presents the range of a measure of the reduced efficiency. The loss of efficiency due to unbalance in a residual design is less than $(1-E_{\min}/\mathcal{E})$ 100%. From a point of view of efficiency, a design with large value of $E_{\min}/\mathcal{E}$ is called robust concerning one lost of the treatments. A design with large value of $E_{\min}/\mathcal{E}$ for every one lost of the treatments is simply robust. Similar discussions for a balanced incomplete block (BIB) design appear in John [2].

Finally, since a design uniquely determines its incidence matrix and vice versa, both a design and its incidence matrix may be denoted by the same symbol throughout this paper.

2. Bounds for efficiency of residual designs

The efficiency factor of a block design is known to be defined as a ratio, $\bar{V}_R/\bar{V}$, where $\bar{V}$ is the average variance of the estimated intrablock treatment elementary contrasts for the design under consideration and $\bar{V}_R$ that for a randomized block design using the same number of experimental units, $\bar{V}_R$ and $\bar{V}$ being computed on the assumption that the intrablock error variance is the same in both cases.

Let N be a connected binary BB design with parameters v, b, r_i, k_j, $n=\sum_{i=1}^{v} r_i=\sum_{j=1}^{b} k_j$ and $\rho=(n-b)/(v-1)$, in which $C=\rho\{I_v-(1/v)G_v\}$, where I_v is the unit matrix of order v and G_v is a $v\times v$ matrix all of whose elements are unity (cf. Kageyama [3], [4]). The matrix C is well known as the C-matrix of an incomplete block design and is very useful in the theory of block designs. For this BB design N,

$$\bar{V}_R=2\sigma^2/(n/v)$$

and

$$\bar{V}=2\sigma^2\sum_{i=1}^{v-1}\theta_i^{-1}/(v-1)\ ,$$

where $\theta_1, \theta_2, \cdots, \theta_{v-1}$ are the non-zero latent roots of the C-matrix of the design. Furthermore, since $\theta_i=\rho$ for $i=1, 2, \cdots, v-1$, the efficiency of the original design N is given by

$$\mathcal{E}=\frac{v(n-b)}{n(v-1)}\ .$$

Suppose that one of the treatments, the first treatment, say, in N is omitted. The resulting design N^* concerning the lost treatment has the parameters $v^*=v-1$, $b^*=b$, $r_i^*=r_{i+1}$ $(i=1, 2, \cdots, v-1)$, $k_j^*=k_j$ or k_j-1 $(j=1, 2, \cdots, b)$ and $n^*=\sum_{i=1}^{v^*} r_i^*=\sum_{j=1}^{b^*} k_j^*$, which is called a residual design concerning the lost treatment. Note that there may be several different designs depending on which treatment is omitted. It is also supposed that N^* is connected. Efficiency of the residual design N^* is given by

$$E=v^*(v^*-1)\Big/\left(n^* \sum_{i=1}^{v^*-1} \delta_i^{*-1}\right) ,$$

where $\delta_1^*, \delta_2^*, \cdots, \delta_{v-2}^*$ are the non-zero latent roots of C^*-matrix $C^*=\text{diag}\{r_1^*, r_2^*, \cdots, r_{v-1}^*\}-N^*\cdot\text{diag}\{k_1^{*-1}, k_2^{*-1}, \cdots, k_b^{*-1}\}\cdot N^{*\prime}$, where diag $\{a_1, a_2, \cdots, a_m\}$ is an $m\times m$ diagonal matrix with the diagonal elements $a_1, a_2, \cdots, a_m$. We then have

$$E=\frac{v^*(v^*-1)}{n^* \sum_{i=1}^{v^*-1} \delta_i^{*-1}}=\frac{v^*}{n^*}\left(\frac{1}{v^*-1} \sum_{i=1}^{v^*-1} \delta_i^{*-1}\right)^{-1} \tag{1}$$

$$\leqq\frac{v^*}{n^*}\cdot\frac{1}{v^*-1} \sum_{i=1}^{v^*-1} \delta_i^* =\frac{v^*(n^*-b^*)}{n^*(v^*-1)} , \tag{2}$$

since the harmonic mean of a set of positive quantities cannot exceed the arithmetic mean. Thus, the maximum attainable value of E is

$$E_0=v^*(n^*-b^*)/\{n^*(v^*-1)\} .$$

Furthermore, it is clear that the equality sign in (2) holds when and only when $\delta_1^*=\delta_2^*=\cdots=\delta_{v^*-1}^*$, in which case the design N^* is a connected BB design. Therefore, this expression E_0 gives an upper bound for efficiency that is attainable only if N^* is a BB design. In this case

$$E_0/\mathcal{E}=n(v-1)^2(n^*-b)/[n^*(v-2)v(n-b)] ,$$

which presents the maximum of the range of a measure of the reduced efficiency. If *any* residual design N^* concerning every one lost of the treatments is balanced, then the original design N may be exactly robust. However, this case is not interesting to us under the present motivation. As an example of this case, we can produce a trivial BB design $N=[G_v-I_v : I_v]$ with parameters v, $b=2v$, $r=v$, $k_j=1$ or $v-1$, and with $\mathcal{E}=(v-2)/(v-1)$. Any residual design of this design is given by $N^*=[\boldsymbol{j}_{v-1} : G_{v-1}-I_{v-1} : I_{v-1}]$ which is balanced with $E_0=(v^2-3v+1)/\{v(v-2)\}$, after the removal of a block whose all elements are zero, where $\boldsymbol{j}_{v-1}$ is a $(v-1)\times 1$ matrix all of whose elements are unity (cf. Kageyama [3], [4]). In this case $E_0/\mathcal{E}=1-1/\{v(v-2)^2\}$ which tends to unity as v increases indefinitely.

Thus, the main purpose of this paper is to derive bounds for efficiency of a connected residual design N^* which is not balanced (i.e., a relation $\delta_1^*=\delta_2^*=\cdots=\delta_{v^*-1}^*$ does not hold in N^*). From (1), it suffices to determine the smallest and largest stationary values of $\sum_{i=1}^{v^*-1} \delta_i^{*-1}$ for given values of v^*, b^* and n^*. It then follows that

$$\text{trace}\,(C^*)=\sum_{i=1}^{v^*-1} \delta_i^*=n^*-b \qquad (=A,\ \text{say}) ,$$

$$\text{trace}\,(C^{*2})=\sum_{i=1}^{v^*-1}\delta_i^{*2}=b+\sum_{i=2}^{v}r_i-2\sum_{i=2}^{v}\sum_{j=1}^{b}(r_in_{ij})/k_j^*$$
$$+2\sum_{j<j'}p_{jj'}^2/(k_j^*k_{j'}^*)\qquad(=B,\ \text{say})$$

where n_{ij} ($=0$ or 1) is the (i, j)-element of N and $p_{jj'}$ is an intersection number of jth block and j'th block in N^*. It is clear that the value of B depends on block structure of the design. Hence, consider designs with a fixed $\sum_{j<j'} p_{jj'}^2/k_j^*k_{j'}^*$ and hence a fixed B.

Remark. If the original design N is equireplicated (i.e., $r_1=r_2=\cdots=r_v=r$, say), then

$$A=v^*r-b\ ,$$
$$B=v^*r^2-2br+b+2\sum_{j<j'}p_{jj'}^2/(k_j^*k_{j'}^*)\ .$$

Following the approach of Conniffe and Stone [1], stationary values of $\sum_{i=1}^{v^*-1}\delta_i^{*-1}$ subject to the constraints

$$\sum_{i=1}^{v^*-1}\delta_i^*=A\ ,\qquad \sum_{i=1}^{v^*-1}\delta_i^{*2}=B$$

can be shown, by putting $P=[B-A^2/(v^*-1)]^{1/2}$ (>0), as follows.
(i) For designs in which $A^2>B$, $\sum_{i=1}^{v^*-1}\delta_i^{*-1}$ has a minimum given by

$$(v^*-1)\left\{1/[A+(v^*-1)^{1/2}(v^*-2)^{1/2}P]+(v^*-2)\Big/\left[A-\left(\frac{v^*-1}{v^*-2}\right)^{1/2}P\right]\right\}$$
$$(=\Pi_1,\ \text{say})\ .$$

(ii) For designs in which $A^2>(v^*-2)B$, $\sum_{i=1}^{v^*-1}\delta_i^{*-1}$ has a maximum given by

$$(v^*-1)\left\{1/[A-(v^*-1)^{1/2}(v^*-2)^{1/2}P]+(v^*-2)\Big/\left[A+\left(\frac{v^*-1}{v^*-2}\right)^{1/2}P\right]\right\}$$
$$(=\Pi_2,\ \text{say})\ .$$

Hence, from (1), as an upper bound for efficiency of N^*, the smallest stationary value gives for $A^2>B$

$$E_{\max}=v^*\Big/\left\{n^*\left[\frac{1}{A+(v^*-1)^{1/2}(v^*-2)^{1/2}P}+\frac{v^*-2}{A-((v^*-1)/(v^*-2))^{1/2}P}\right]\right\}\ .$$

As a lower bound for efficiency of N^*, the largest stationary value gives for $A^2/B+1>v^*-1>A^2/B$

$$E_{\min}=v^*\Big/\Big\{n^*\Big[\frac{1}{A-(v^*-1)^{1/2}(v^*-2)^{1/2}P}+\frac{v^*-2}{A+((v^*-1)/(v^*-2))^{1/2}P}\Big]\Big\}\ .$$

Furthermore, if B divides A^2, then for $A^2/B+2>v^*-1>A^2/B$ a lower bound for efficiency of N^* can be given by

$$E'_{\min}=v^*\Big/\Big\{n^*\Big[\frac{2}{A-[(v^*-1)(v^*-3)/2]^{1/2}P}+\frac{v^*-3}{A+[2(v^*-1)/(v^*-3)]^{1/2}P}\Big]\Big\}\ .$$

If we can find a reduced design for which $E_{\max}$ is attained, the original design will be the most efficient design. Since it is clear that $d\Pi_1/dP>0$, Π_1 is a minimum for given values of v^*, b^* and n^*, when P is a minimum. These suggest that a residual design with close to maximum efficiency could be obtained by choosing the $\sum_{i<j} p_{ij}^2/(k_i^* k_j^*)$ to minimize P. Similarly, since $d\Pi_2/dP>0$, Π_2 is a maximum when P is a maximum. Note that when $P=0$, the residual design is balanced.

Remark. It is obvious that the discussion here remains valid even if the original design N is a BIB design with parameters v, b, r, k and λ. In this case we have $\mathcal{E}=\lambda v/rk$ and $E_0=[r(v-1)-b]/[r(v-2)]$ which is well known. Moreover, the minimum value $E_{\min}$ of E is given by John [2] for a BIB design. His expression of $E_{\min}$ is very simple and its value is approximately equal to the value of $E_{\min}$ derived here. In this sense, the results here extend John's work on robustness of BIB designs to balanced block designs with possibly unequal block sizes and unequal replication. It is well known that a BIB design has the equal-replications and the equal block-sizes. However, from a practical point of view, it may not be possible to design equi-size blocks accommodating the equi-replications of each treatment in all the blocks. Thus, it may be meaningful to extend or improve upon John's results.

3. Examples

We now consider an application of these discussions. Let N_1 be a symmetrical BIB design with parameters $v=b$, $r=k$, λ and with $v\neq 2k$. Then $N=[N_1: N_1^c]$ is an equireplicated BB design with parameters $v'=v$, $b'=2b$, $r'=b$, having efficiency $\mathcal{E}=1-1/(v-1)$ which tends to unity as v increases indefinitely, where N_1^c is the complementary design of N_1 (cf. Kageyama [3]). Furthermore, a residual design N^* is clearly connected, but it is generally not a BB design. In this case, after elementary calculation, it can be shown that

$$A=v(v-3)$$

and

$$B=v(v^2-5v+2)+2(k-\lambda)^2+2\lambda^2(v-k)[1/(k-1)+(v-k-1)/(2k^2)] \\ +2(k-\lambda)^2[k/(v-k)+(v-k)/k]+k(\lambda-1)^2/(k-1) \\ +k(k-1)(v-2k+\lambda)^2/(v-k)^2+2k(v-k)(k-\lambda-1)^2/[(k-1)(v-k-1)] \\ +[2k(v-2k+\lambda)^2+(v-k)(v-2k+\lambda-1)^2]/(v-k-1)\ ,$$

which should be noted to be dependent only on the parameters of the design. Hence $E_{\max}$ and $E_{\min}$ (or $E'_{\min}$) can easily be calculated.

Example 1. Let N_1 be a symmetrical BIB design with parameters $v=7$, $k=3$ and $\lambda=1$ generated by a difference set $(0, 1, 3) \bmod 7$. $N=[N_1: N_1^c]$ is a BB design with $\mathcal{E}=5/6=0.8\dot{3}$. However, N^* is not a BB design. In this case, $A=28$, $B=156+5/6$ and $P^2=1/30$. Hence $E_{\max}\doteqdot 0.79983$ and $E_{\min}\doteqdot 0.7998261$ which are approximately equal. The actual efficiency is $E=935/1169\doteqdot 0.7998289$. Furthermore, $E_{\min}/\mathcal{E}\doteqdot 0.959784$ which may imply that the original design N is robust concerning any one lost of the treatments. Note that even if N^* is a BB design, its efficiency is $4/5=0.8$ which approximates to the value of $E_{\min}$.

As interesting examples of the other type, we obtain

Example 2. Consider a BB design with parameters $v=5$, $b=15$, $r=9$, $k_j=2$, 3, or 4, $n=45$ and with $\mathcal{E}=5/6$, whose blocks are given by (2, 3, 4, 5) (2, 3, 4, 5) (1, 2, 3) (1, 4) (1, 5) (1, 3, 4, 5) (2, 4) (1, 2, 3) (1, 3, 4, 5) (2, 5) (1, 2, 4, 5) (1, 2, 4, 5) (1, 2, 3) (3, 4) (3, 5), in which $C=(15/2)\{I_5-(1/5)G_5\}$ (cf. Kageyama [3]). The values of bounds for efficiency of residual designs can be computed in two cases: (i) When any one of treatments, 1, 2, or 3, is lost, $E_{\max}=(1321+21\sqrt{3})/27(63+\sqrt{3})\doteqdot 0.77663$ and $E_{\min}=(1321-21\sqrt{3})/27(63-\sqrt{3})\doteqdot 0.77656$. Then $E_{\min}/\mathcal{E}>0.9318$. The actual efficiency is $E=1540/1983\doteqdot 0.77660$. (ii) When any one of treatments, 4 or 5, is lost, $E_{\max}=902/1161\doteqdot 0.77691$ and $E_{\min}=860/1107\doteqdot 0.77687$. Then $E_{\min}/\mathcal{E}>0.9322$. Actually, $E=902/1161=E_{\max}$ which shows that the original design is most efficient concerning one lost of treatments 4 or 5. Note that even if a residual design is balanced, its efficiency is $7/9\doteqdot 0.77777$ which approximates to the values of $E_{\min}$ in the above two cases.

Example 3. Consider a BB design with parameters $v=5$, $b=6$, $r_i=3$ or 4, $k_j=2$ or 4, $n=16$ and $\mathcal{E}=25/32\doteqdot 0.7812$, whose blocks are given by (1, 2)(1, 3)(1, 4)(1, 5)(2, 3, 4, 5)(2, 3, 4, 5), in which $C=(5/2)\{I_5-(1/5)G_5\}$ (cf. Kageyama [3]). When a treatment 1 is lost, the residual design is clearly balanced with efficiency $E_0=2/3$. When any one of treatments 2, 3, 4 or 5 is lost, $E_{\max}=32/45\doteqdot 0.7111$ and $E_{\min}=120/169\doteqdot 0.7100$. Actually, $E=120/169=E_{\min}$. Furthermore, $E_{\min}/\mathcal{E}>0.9088$. Note that even if a residual design in the latter case is balanced, its efficiency is $28/39 \doteqdot 0.7179$ which approximates to the value of E_{min}.

Thus, the values of E_{max}, E_{min}, E_0 and $\mathcal{E}$ can be computed concerning various BB designs which are for example listed by Kageyama [4]. In almost all cases, ratio $E_{min}/\mathcal{E} > 0.90$. Under these circumstances, it seems that the balanced structure on BB designs with unequal block sizes is generally robust concerning any one lost of treatments.

The robustness problem here shall be considered to look at other types of block designs.

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