Ann. Inst. Statist. Math.
32 (1980), Part A, 223-240

NONPARAMETRIC ESTIMATION OF AN AFFINITY MEASURE BETWEEN TWO ABSOLUTELY CONTINUOUS DISTRIBUTIONS WITH HYPOTHESES TESTING APPLICATIONS

IBRAHIM A. AHMAD*

(Received May 10, 1975; revised May 20, 1980)

Abstract

Let F and G denote two distribution functions defined on the same probability space and are absolutely continuous with respect to the Lebesgue measure with probability density functions f and g, respectively. A measure of the closeness between F and G is defined by: $\lambda=\lambda(F, G)=2\int f(x)g(x)dx\Big/\left[\int f^2(x)dx+\int g^2(x)dx\right]$. Based on two independent samples it is proposed to estimate λ by $\hat{\lambda}=\left[\int \hat{f}(x)dG_n(x)+\int \hat{g}(x)dF_n(x)\right]\Big/\left[\int \hat{f}^2(x)dx+\int \hat{g}^2(x)dx\right]$, where $F_n(x)$ and $G_n(x)$ are the empirical distribution functions of $F(x)$ and $G(x)$ respectively and $\hat{f}(x)$ and $\hat{g}(x)$ are taken to be the so-called kernel estimates of $f(x)$ and $g(x)$ respectively, as defined by Parzen [16]. Large sample theory of $\hat{\lambda}$ is presented and a two sample goodness-of-fit test is presented based on $\hat{\lambda}$. Also discussed are estimates of certain modifications of λ which allow us to propose some test statistics for the one sample case, i.e., when $g(x)=f_0(x)$, with $f_0(x)$ completely known and for testing symmetry, i.e., testing $H_0: f(x)=f(-x)$.

1. Introduction

Let F and G be two independent distribution functions (d.f.'s) defined on the same space which are absolutely continuous with respect to the Lebesgue measure with probability density functions (p.d.f.'s) f and g, respectively. Matusita [7] introduced a measure of closeness "affinity" between F and G, defined as follows:

* Present Address: Department of Mathematical Sciences, Memphis State University, Memphis, TN. 38152, U.S.A.

$$\rho=\rho(F, G)=\int [f(x)g(x)]^{1/2}dx\ . \tag{1.1}$$

Assume that f and g are both square integrable. Ahmad and Van Belle [2] introduced another measure of the affinity between f and g, viz.,

$$\lambda=\lambda(F, G)=2\delta/[\Delta(f)+\Delta(g)]\ , \tag{1.2}$$

where $\delta=\int f(x)g(x)dx$, $\Delta(f)=\int f^2(x)dx$, and $\Delta(g)=\int g^2(x)\mathrm{d}x$.

Matusita [7], [11], and [12] discussed various mathematical properties of ρ. When the distributions are discrete, Matusita [7], [8], and Matusita and Akaike [14] used an estimate of ρ to study decision problems. Matusita [9], [10], and [13] studied classification rules for normal populations based on ρ.

Based on two independent samples from F and G a nonparametric estimate of λ is proposed and its large sample theory is studied. Let $X_1,\cdots,X_n$ and $Y_1,\cdots,Y_n$ be two independent samples from F and G respectively. Note that taking equal sample sizes is only for convenience. The results of this paper remain valid if we have samples of sizes m and n such that $m/n\rightarrow\alpha>0$. Let k be a known p.d.f. satisfying the following conditions

$$\sup_u k(u)<\infty \quad \text{and} \quad |u|k(u)\rightarrow 0 \qquad \text{as } |u|\rightarrow\infty\ . \tag{1.3}$$

Furthermore, let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_n\rightarrow 0$ an $n\rightarrow\infty$. The kernel estimates of $f(x)$ and $g(x)$ are given by:

$$\hat{f}(x)=a_n^{-1}\int k[(x-u)/a_n]dF_n(u)=(na_n)^{-1}\sum_{i=1}^{n} k[(x-X_i)/a_n]\ , \tag{1.4}$$

and

$$\hat{g}(x)=a_n^{-1}\int k[(x-u)/a_n]dG_n(u)=(na_n)^{-1}\sum_{i=1}^{n} k[(x-Y_i)/a_n]\ , \tag{1.5}$$

where $F_n(x)$ and $G_n(x)$ denote the empirical d.f. of $F(x)$ and $G(x)$, respectively. Using (1.4) and (1.5) an estimate of λ is given by:

$$\hat{\lambda}=2\hat{\delta}/[\hat{\Delta}(f)+\hat{\Delta}(g)]\ , \tag{1.6}$$

where

$$\hat{\delta}=\frac{1}{2}\left[\int \hat{f}(x)dG_n(x)+\int \hat{g}(x)dF_n(x)\right] \tag{1.7}$$

$$=\frac{1}{2}(n^2a_n)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\{k[(X_i-Y_j)/a_n]+k[(Y_j-X_i)/a_n]\} ,$$

(1.8) $$\hat{\Delta}(f)=\int \hat{f}^2(x)dx , \quad \text{and} \quad \hat{\Delta}(g)=\int \hat{g}^2(x)dx .$$

For convenience, let us assume that $k(u)$ is a symmetric p.d.f., in which case

(1.9) $$\hat{\delta}=\int \hat{f}(x)dG_n(x)=\int \hat{g}(x)dF_n(x)=(n^2a_n)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}k[(X_i-Y_j)/a_n] .$$

Throughout this paper the estimate (1.9) will be used for estimating λ. Note also that in all of the above and in what follows whenever no limits are given, integrals are taken over the entire real line.

The functional $\Delta(f)$ is of interest of its own right because it appears as the main term in the asymptotic efficacy of several rank statistics. The estimate $\hat{\Delta}(f)$ was shown to be consistent in the mean by Bhattacharayya and Roussas [3], if $na_n\to\infty$, as $n\to\infty$. Conditions under which $\hat{\Delta}(f)$ is strongly consistent are given in Section 2. An equivalent estimate to $\Delta(f)$ may be defined by:

(1.10) $$\tilde{\Delta}(f)=\int \hat{f}(x)dF_n(x)=(n^2a_n)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}k[(X_i-X_j)/a_n] .$$

It can be shown that $\tilde{\Delta}(f)$ is also consistent in the mean and strongly consistent estimate of $\Delta(f)$ under the same conditions as those of the estimate $\hat{\Delta}(f)$. In this paper, however, we shall use $\hat{\Delta}(f)$.

In Section 2, asymptotic properties of $\hat{\lambda}$ are discussed. Conditions under which $\hat{\lambda}$ is consistent, strongly consistent, and asymptotically normal are given. A two-sample goodness-of-fit test statistic based on $\hat{\lambda}$ is proposed for testing $H_0: f=g$. In Section 3, estimation of two special forms of λ is considered. First we discuss estimating λ when $g=f_0$ is known and based on this estimate a one-sample test statistic is proposed for $H_0: f=f_0$ (known). Second, an estimate is given for

(1.11) $$\lambda^*=\delta^*/\Delta(f) ,$$

where $\Delta(f)$ is a given in (1.2) and $\delta^*=\int f(x)f(-x)dx$. Based on this estimate a test statistic is proposed for testing $H_0: f$ is symmetric about zero, i.e., $f(x)=f(-x)$ for all real x.

Throughout this paper it is assumed that the set of discontinuity points of f and g are, respectively, null sets.

2. Asymptotic properties of $\hat{\lambda}$

The following two lemmas summarize some properties of $\hat{\delta}$ and $\hat{\Delta}(f)(\hat{\Delta}(g))$ that will be used in the main theorem.

LEMMA 2.1.

(i) *If* $na_n \to \infty$ *as* $n \to \infty$ *and if* $\int f^2(x)g(x)dx < \infty$ *or* $\int g^2(x)f(x)dx < \infty$, *then*

$$\mathrm{E}\,|\hat{\delta}-\delta| \to 0 \quad \text{as } n \to \infty . \tag{2.1}$$

(ii) *If k is a continuous function of bounded variation, and if for any* $\gamma > 0$, $\sum_{n=1}^{\infty} \exp(-\gamma n a_n^2) < \infty$, *then*

$$\hat{\delta} \to \delta \text{ with probability one as } n \to \infty . \tag{2.2}$$

(iii) *If f and g have bounded second derivatives, if* $\int f^3(x)g(x)dx < \infty$ *and* $\int g^3(x)f(x)dx < \infty$, *if* $\int u^2 k(u)du < \infty$, *and if* $na_n \to \infty$ *and* $na_n^4 \to 0$ *as* $n \to \infty$, *then* $n^{1/2}(\hat{\delta}-\delta)$ *is asymptotically normal with mean* 0 *and variance* σ^2 *given by*

$$\sigma^2 = \int f(x)[g(x)-\delta]^2 dx + \int g(x)[f(x)-\delta]^2 dx . \tag{2.3}$$

PROOF.

(i) Recall that $\hat{\delta} = \int \hat{f}(x)dG_n(x)$. Then

$$\begin{aligned} |\hat{\delta}-\delta| &\leqq \int |\hat{f}(x)-f(x)|dG_n(x) + \left|\int \hat{f}(x)dG_n(x) - \int f(x)dG(x)\right| \\ &= I_{1n} + I_{2n} , \qquad \text{say} . \end{aligned} \tag{2.4}$$

$$\begin{aligned} \mathrm{E}\, I_{1n} &= n^{-1}\sum_{i=1}^{n} \mathrm{E}\,|\hat{f}(Y_i)-f(Y_i)| = \mathrm{E}\,|\hat{f}(Y_1)-f(Y_1)| \\ &= \int \mathrm{E}\,|\hat{f}(x)-f(x)|dG(x) , \end{aligned} \tag{2.5}$$

since by Theorems 1A and 2A of Parzen [16], $\mathrm{E}\,|\hat{f}(x)-f(x)| \to 0$ as $n \to \infty$, for each continuity point x of f, and $\mathrm{E}\,|\hat{f}(x)-f(x)| \leqq \mathrm{E}\,\hat{f}(x)+f(x) = h_n(x)$, say, which is integrable and converges to the integrable function $2f(x)$ at each continuity point x of f, hence by the extended Lebesgue dominated convergence theorem (ELDCT), see Royden [20], p. 89, we have, since $\int \mathrm{E}\,\hat{f}(x)dx = 1 = \int \hat{f}(x)dx$ for all n so that $\int h_n(x)dx$

$=2$, that

$$\mathrm{E}\int |\hat{f}(x)-f(x)|g(x)dx \to 0 \quad \text{as } n\to\infty . \tag{2.6}$$

Next, note that

$$\mathrm{E}\, I_{2n} = \mathrm{E}\left| n^{-1}\sum_{i=1}^{n} f(Y_i)-\delta\right| \to 0 \quad \text{as } n\to\infty , \tag{2.7}$$

since $\mathrm{E}\, f(Y_1)=\delta$, and $\mathrm{Var}\, f(Y_1)=\int f^2(x)g(x)dx-\delta^2$, thus $\mathrm{Var}\left(n^{-1}\sum_{i=1}^{n} f(Y_i)\right)$ $\to 0$ as $n\to\infty$.

(ii) First note that

$$|\mathrm{E}\,\hat{\delta}-\delta| = \left|\int \mathrm{E}\,\hat{f}(x)dG(x)-\int f(x)dG(x)\right| \leqq \int |\mathrm{E}\,\hat{f}(x)-f(x)|dG(x) , \tag{2.8}$$

which converges to 0 as $n\to\infty$, since for all continuity points x of f, $|\mathrm{E}\,\hat{f}(x)-f(x)|\to 0$ as $n\to\infty$ by Theorem 1A of Parzen [16] and $|\mathrm{E}\,\hat{f}(x)-f(x)|\leqq \mathrm{E}\,\hat{f}(x)+f(x)=h_n(x)$, say which is integrable and converges to the integrable function $2f(x)$ for all continuity points x of f, and $\int h_n(x)dx$ $=2$, hence the ELDCT applies. Thus, it suffices to show that $|\hat{\delta}-\mathrm{E}\,\hat{\delta}|$ $\to 0$ with probability one as $n\to\infty$. To this end, note that

$$\begin{aligned} |\hat{\delta}-\mathrm{E}\,\hat{\delta}| &= \left|\int \hat{f}(x)dG_n(x)-\int \mathrm{E}\,\hat{f}(x)dG(x)\right| \\ &\leqq \int |\hat{f}(x)-\mathrm{E}\,\hat{f}(x)|dG_n(x)+\int |\hat{g}(x)-\mathrm{E}\,\hat{g}(x)|dF(x) \\ &\leqq \sup_x |\hat{f}(x)-\mathrm{E}\,\hat{f}(x)|+\sup_x |\hat{g}(x)-\mathrm{E}\,\hat{g}(x)| . \end{aligned} \tag{2.9}$$

where the first inequality follows since $\int \mathrm{E}\,\hat{f}(x)dG_n(x)=a_n^{-1}\int\int k[(x-y)/a_n]dF(y)dG_n(x)=a_n^{-1}\int\int [k(y-x)/a_n]dG_n(x)dF(y)=\int \hat{g}(x)dF(x)$ and by adding and subtracting $\int \mathrm{E}\,\hat{f}(x)dG_n(x)$ and since $\int \mathrm{E}\,\hat{f}(x)dG(x)=\int \mathrm{E}\,\hat{g}(x)dF(x)$, while the second inequality follows since $\int dG_n(x)=1$. Since k is a continuous function of bounded variation, then by integration by parts with μ, the total variation of k, we have (see Nadaraya [15]) that

$$\sup_x |\hat{f}(x)-\mathrm{E}\,\hat{f}(x)| \leqq (\mu/a_n) \sup_x |F_n(x)-F(x)| . \tag{2.10}$$

Hence using a result of Dvoretzky, Kiefer, and Wolfowitz [6], we have for any $\varepsilon>0$

$$(2.11) \qquad \mathrm{P}\,[\sup_x |\hat{f}(x)-\mathrm{E}\,\hat{f}(x)| \geqq \varepsilon] \leqq \mathrm{P}\,[\sup_x |F_n(x)-F(x)| \geqq (\varepsilon a_n/\mu)] \leqq C \exp(-\varepsilon^2 n a_n^2/\mu^2)\,,$$

where $C>0$ is some constant.

Thus in view of the Borel-Cantelli lemma and the assumption that $\sum_{n=1}^{\infty} \exp(-\gamma n a_n^2) < \infty$ for all $\gamma > 0$ we conclude that $\sup |\hat{f}(x)-\mathrm{E}\,\hat{f}(x)| \to 0$ with probability one as $n \to \infty$. Similarly we can show that $\sup |\hat{g}(x)-\mathrm{E}\,\hat{g}(x)| \to 0$ with probability one as $n \to \infty$. Thus we conclude that $|\hat{\delta}-\mathrm{E}\,\hat{\delta}| \to 0$ with probability one as $n \to \infty$.

(iii) Let $p_1(x/a_n) = a_n^{-1}\,\mathrm{E}\,\{k[(X-Y)/a_n]\,|\,X=x\}$ and $p_2(y/a_n) = a_n^{-1}\,\mathrm{E}\,\{k[(X-Y)/a_n]\,|\,Y=y\}$. Then $\mathrm{E}\,p_1(X_1/a_n) = \mathrm{E}\,p_2(Y_1/a_n) = \mathrm{E}\,\hat{\delta}$. Let

$$(2.12) \qquad V_n = n^{-1}\sum_{i=1}^{n} p_1(X_i/a_n) - \mathrm{E}\,\hat{\delta} \quad \text{and} \quad W_n = n^{-1}\sum_{i=1}^{n} p_2(Y_i/a_n) - \mathrm{E}\,\hat{\delta}\,.$$

Then we have that

$$(2.13\mathrm{a}) \qquad \mathrm{Var}\,p_1(X_1/a_n) \to \int f(x)[g(x)-\delta]^2 dx \qquad \text{as } n \to \infty\,,$$

$$(2.13\mathrm{b}) \qquad \mathrm{E}\,|p_1(X_1/a_n)-\mathrm{E}\,\hat{\delta}|^3 \to \int f(x)\,|g(x)-\delta|^3 dx \qquad \text{as } n \to \infty\,,$$

$$(2.13\mathrm{c}) \qquad \mathrm{Var}\,p_2(Y_1/a_n) \to \int g(x)[f(x)-\delta]^2 dx\,, \qquad \text{as } n \to \infty, \text{ and}$$

$$(2.13\mathrm{d}) \qquad \mathrm{E}\,|p_2(Y_1/a_n)-\mathrm{E}\,\hat{\delta}|^3 \to \int g(x)\,|f(x)-\delta|^3 dx\,, \qquad \text{as } n \to \infty\,.$$

We will sketch the proof of (2.13a) and (2.13b), (2.13c) and (2.13d) are shown similarly. Note that $\mathrm{E}\,p_1(X_1/a_n) = \mathrm{E}\,\hat{\delta} \to \delta$ as $n \to \infty$. Next,

$$\begin{aligned}(2.14) \qquad \mathrm{E}\,p_1^2(X_1/a_n) &= \int\!\!\int\!\!\int k(z_1)k(z_2)f(x)g(x-a_nz_1)g(x-a_nz_2)dz_1dz_2dx \\ &\leqq \int\!\!\int k(z_1)k(z_2)\left[\int g^2(x-a_nz_1)f(x)dx\right]^{1/2} \\ &\quad \cdot \left[\int g^2(x-a_nz_2)f(x)dx\right]^{1/2} dz_1dz_2 \\ &= \left\{\int k(z)\left[\int g^2(x-a_nz)f(x)dx\right]^{1/2} dz\right\}^2 .\end{aligned}$$

But by Theorem 1A of Parzen [16] the integrand $\int g^2(x-a_n3)f(x)dx$ converges as $n \to \infty$ to $\int g^2(x)f(x)dx$, since by assumption $\int g^3(x)f(x)dx < \infty$. Hence

$$\limsup_n \mathrm{E}\, p_1^2(X_1/a_n) \leqq \int g^2(x)f(x)dx \ .$$

Since $p_1(x/a_n) \to g(x)$ as $n \to \infty$ at every continuity point x of g, by Fatou Lemma (Royden [20], p. 83) we have

$$\liminf_n \mathrm{E}\, p_1^2(X_1/a_n) \geqq \int g^2(x)f(x)dx \ . \tag{2.15}$$

Hence from (2.14) and (2.15) it follows that $\mathrm{E}\, p_1^2(X_1/a_n) \to \int g^2(x)f(x)dx$ as $n \to \infty$, thus

$$\mathrm{Var}\, p_1(X_1/a_n) \to \int g^2(x)f(x)dx - \delta^2 = \int f(x)[g(x)-\delta]^2 dx \ .$$

Now, since $|p_1(x/a_n) - \mathrm{E}\,\hat{\delta}|^3 \to |g(x)-\delta|^3$ as $n \to \infty$ for each continuity point x of g, by Fatou's lemma, we have

$$\liminf_n \mathrm{E}\, |p_1(X_1/a_n) - \mathrm{E}\,\hat{\delta}|^3 \geqq \int f(x)|g(x)-\delta|^3 dx \ . \tag{2.16}$$

On the other hand

$$\begin{aligned}
&\mathrm{E}\,|p_1(X_1/a_n) - \mathrm{E}\,\hat{\delta}|^3 \\
&= \int f(x) \left| \int k(z) g(x - a_n z) dz - \mathrm{E}\,\hat{\delta} \right|^3 dx \\
&\leqq \int f(x) \int\int\int k(z_1)k(z_2)k(z_3) |g(x-a_n z_1) - \mathrm{E}\,\hat{\delta}| |g(x-a_n z_2) - \mathrm{E}\,\hat{\delta}| \\
&\qquad \cdot |g(x-a_n z_3) - \mathrm{E}\,\hat{\delta}| dx dz_1 dz_2 dz_3 \\
&= \int\int\int\int k(z_1)k(z_2)k(z_3) f(x) |g(x-a_n z_1) - \mathrm{E}\,\delta| |g(x-a_n z_2) - \mathrm{E}\,\delta| \\
&\qquad \cdot |g(x-a_n z_3) - \mathrm{E}\,\delta| dx dz_1 dz_2 dz_3 \\
&\leqq \int\int\int k(z_1)k(z_2)k(z_3) \left\{ \int f(x) |g(x-a_n z_1) - \mathrm{E}\,\hat{\delta}|^3 dx \right\}^{1/3} \\
&\qquad \cdot \left\{ \int f(x) |g(x-a_n z_2) - \mathrm{E}\,\hat{\delta}|^{3/2} |g(x-a_n z_3) - \mathrm{E}\,\hat{\delta}|^{3/2} dx \right\}^{2/3} dz_1 dz_2 dz_3 \\
&\leqq \left\{ \int k(z) \left\{ \int f(x) |g(x-a_n z) - \mathrm{E}\,\hat{\delta}|^3 dx \right\}^{1/3} dz \right\}^3 .
\end{aligned} \tag{2.17}$$

But by Theorem 1A of Parzen [16], the integrand $\int f(x)|g(x-a_n z) - \mathrm{E}\,\hat{\delta}|^3 dx$ converges to $\int f(x)|g(x)-\delta|^3 dx$ as $n \to \infty$ since $\int f(x)|g(x-a_n z) - \mathrm{E}\,\hat{\delta}|^3 dx < \infty$ for all $n \geqq 1$, and since $\int f(x)|g(x-a_n z) - \mathrm{E}\,\hat{\delta}|^3 dx \leqq C\left\{\int f(x) \cdot |g(x-a_n z) - \delta|^3 dx + \int f(x)|\mathrm{E}\,\hat{\delta} - \delta|^3 dx\right\}$ with C a constant and the second term converges to 0. Hence, (2.18) $$\limsup_n \mathrm{E}\,|p_1(X_1/a_n)-\mathrm{E}\,\hat{\delta}|^3 \leqq \int f(x)|g(x)-\delta|^3 dx \ .$$

Thus from (2.16) and (2.18) it follows that $\mathrm{E}\,|p_1(X_1/a_n)-\mathrm{E}\,\delta|^3 \to \int f(x)|g(x)-\delta|^3 dx$ as $n\to\infty$. Hence it follows that $\mathrm{E}\,|p_1(X_1/a_n)-\mathrm{E}\,\hat{\delta}|^3/[\{\mathrm{Var}\, p_1(X_1/a_n)]^{3/2}\sqrt{n}\}\to 0$ as $n\to\infty$, thus Liapounv's condition of the central limit theorem is satisfied, hence $n^{1/2}V_n$ is asymptotically normal with mean 0 and variance $\int f(x)[g(x)-\delta]^2 dx$. Similarly we can show that $\mathrm{E}\,|p_2(Y_1/a_n)-\mathrm{E}\,\delta|^3/\{[\mathrm{Var}\, p_2(Y_1/a_n)]^{3/2}\sqrt{n}\}\to 0$ and thus $n^{1/2}W_n$ is asymptotically normal with mean 0 and variance $\int g(x)[f(x)-\delta]^2 dx$. Hence $n^{1/2}(V_n+W_n)$ is asymptotically normal with mean 0 and variance σ^2 as given in (2.3), provided it is positive. Next, we need to show that

(2.19) $$n^{1/2}(\hat{\delta}-V_n-W_n-\mathrm{E}\,\hat{\delta})\to 0 \quad \text{in probability as } n\to\infty \text{ and}$$

(2.20) $$n^{1/2}(\mathrm{E}\,\hat{\delta}-\delta)\to 0 \quad \text{as } n\to\infty \ .$$

Let us prove (2.19) first. Let $d[I(u\leqq x)]=I(u\leqq x+dx)-I(u\leqq x)$ and set $\xi_{ij}=d[I(X_i\leqq x_j)]$ and $\eta_{ij}=d[I(Y_i\leqq y_j)]$ so that $\mathrm{P}\,[\xi_{ij}=1]=dF(x_j)=p_j$, and $\mathrm{P}\,[\eta_{ij}\geqq 1]=dG(y_j)=q_j$, say $i=1,\cdots,n$, $j=1,2$. Hence

(2.21) $$\begin{aligned}&\mathrm{E}\,(\hat{\delta}-V_n-W_n-\mathrm{E}\,\hat{\delta})^2\\ &\quad=a_n^{-2}\int\!\!\int\!\!\int\!\!\int k[(x_1-y_1)/a_n]k[(x_2-y_2)/a_n]\,\mathrm{E}\,\{d[F_n(x_1)-F(x_1)]\\ &\qquad\cdot d[F_n(x_2)-F(x_2)]\}\cdot\mathrm{E}\,\{d[G_n(y_1)-G(y_1)]d[G_n(y_2)-G(y_2)]\ .\end{aligned}$$

But note that

(2.22) $$\begin{aligned}&n^2|\mathrm{E}\,d[F_n(x_1)-F(x_1)]d[F_n(x_2)-F(x_2)]|\\ &\quad=\left|\mathrm{E}\prod_{j=1}^{2}\left\{\sum_{i=1}^{n} d[I(X_i\leqq x_j)-F(x_j)]\right\}\right|\\ &\quad\leqq\sum_{i=1}^{n}\sum_{l=1}^{n}|\mathrm{E}\,(\xi_{i1}-p_1)(\xi_{l2}-p_2)|\\ &\quad=\sum_{i=1}^{n}\mathrm{E}\,|(\xi_{i1}-p_1)(\xi_{i2}-p_2)|\\ &\quad\leqq 4np_1p_2=4ndF(x_1)dF(x_2)\ .\end{aligned}$$

Similarly we can show that

$$n^2|\mathrm{E}\,d[G_n(y_1)-G(y_1)]d[G_n(y_2)-G(y_2)]|\leqq 4ndG(y_1)dG(y_2)\ .$$

Hence

(2.23) $$\mathrm{E}\,(\hat{\delta}-V_n-W_n-\mathrm{E}\,\hat{\delta})^2\leqq n^{-1}(\mathrm{E}\,\hat{\delta})^2\ ,$$

which converges to 0 as $n\to\infty$ provided $na_n\to\infty$ proving (2.19).

$$(2.24)\qquad n^{1/2}|\mathrm{E}\,\hat{\delta}-\delta|=n^{1/2}\left|\int \mathrm{E}\,\hat{f}(x)dG(x)-\int f(x)dG(x)\right|$$
$$\leqq n^{1/2}\sup_x|\mathrm{E}\,\hat{f}(x)-f(x)|\leqq C_1(na_n^4)^{1/2}\sup_x|f''(x)|$$

or $n^{1/2}|\mathrm{E}\,\hat{\delta}-\delta|\leqq C_2(na_n^4)\sup_x|g''(x)|\to 0$ as $n\to\infty$, where C_1 and C_2 are positive constants, proving (2.20) and part (iii).

LEMMA. 2.2.
(i) *If* $na_n\to\infty$ *as* $n\to\infty$, *then*

$$(2.25)\qquad \mathrm{E}\,|\hat{\Delta}(f)-\Delta(f)|\to 0 \qquad \text{as } n\to\infty\,.$$

(ii) *If* $na_n^2\to\infty$ *as* $n\to\infty$, *then*

$$(2.26)\qquad \mathrm{Var}\,\hat{\Delta}(f)\to 0 \qquad \text{as } n\to\infty\,.$$

(iii) *If* k *is a continuous function of bounded variation and if for any* $\gamma\geqq 0$, $\sum_{n=1}^{\infty}\exp(-\gamma na_n^2)<\infty$, *then*

$$(2.27)\qquad \hat{\Delta}(f)\to\Delta(f) \qquad \text{with probability one as } n\to\infty\,.$$

PROOF. The proof of (i) and (ii) are given by Bhattacharayya and Roussas [3], Theorems 2.2 and 2.3 respectively. To prove (iii) note that $|\mathrm{E}\,\hat{\Delta}(f)-\Delta(f)|\to 0$ as $n\to\infty$, since $\sum_{n=1}^{\infty}\exp\{-\gamma na_n^2\}<\infty$ implies that $na_n\to\infty$, thus it suffices to show that $|\hat{\Delta}(f)-\mathrm{E}\,\hat{\Delta}(f)|\to 0$ with probability one as $n\to\infty$.

$$(2.28)\qquad |\hat{\Delta}(f)-\mathrm{E}\,\hat{\Delta}(f)|=\left|\int \hat{f}^2(x)dx-\int \mathrm{E}\,\hat{f}^2(x)dx\right|$$
$$\leqq\left|\int \hat{f}^2(x)dx-\int (\mathrm{E}\,\hat{f}(x))^2dx\right|$$
$$+\left|\mathrm{E}\,\hat{f}^2(x)dx-\int(\mathrm{E}\,\hat{f}(x))^2dx\right|\,.$$

Now, it follows from the proof of Theorem 2.2 of Bhattacharayya and Roussas [3] that

$$\left|\int \mathrm{E}\,\hat{f}^2(x)dx-\int(\mathrm{E}\,\hat{f}(x))^2dx\right|\to 0 \qquad \text{as } n\to\infty\,.$$

Also note that

$$(2.29)\qquad \left|\int \hat{f}^2(x)dx-\int(\mathrm{E}\,\hat{f}(x))^2dx\right|$$
$$\leqq\sup_x|\hat{f}(x)-\mathrm{E}\,\hat{f}(x)|\left\{\int \hat{f}(x)dx+\int \mathrm{E}\,\hat{f}(x)dx\right\}$$

$$=2\sup_x|\hat{f}(x)-\mathrm{E}\,\hat{f}(x)|\,.$$

since $\int\hat{f}(x)dx=\int\mathrm{E}\,\hat{f}(x)dx=1$. Thus the upper bound of (2.28) converges to 0 with probability one as $n\to\infty$ as shown in (2.10).

With the aid of the above two lemmas we now state and prove the main result of this section.

THEOREM 2.1.

(i) *If* $na_n\to\infty$, *as* $n\to\infty$ *and if* $\int f^2(x)g(x)dx<\infty$ *or* $\int f(x)g^2(x)dx<\infty$ *then,*

(2.30) $$\hat{\lambda}\to\lambda \quad \textit{in probability as } n\to\infty\,.$$

(ii) *If* k *is a continuous function of bounded variation and if for any* $\gamma>0$, $\sum_{n=1}^{\infty}\exp(-\gamma na_n^2)<\infty$ *then*

(2.31) $$\hat{\lambda}\to\lambda \quad \textit{with probability one as } n\to\infty\,.$$

(iii) *If* $na_n^2\to\infty$ *and* $na_n^4\to 0$ *as* $n\to\infty$, *if* f *and* g *have bounded second derivative, if* $\int f^3(x)g(x)dx<\infty$, *and* $\int f(x)g^3(x)dx<\infty$, *and if* $\int u^2k(u)du<\infty$, *then* $n^{1/2}(\hat{\lambda}-\lambda)$ *is asymptotically normal with mean* 0 *and variance* σ^2 *given by*

(2.32) $$\sigma^2=4(\xi_{10}+\xi_{01})/[\Delta(f)+\Delta(g)]^2\,,$$

where $\xi_{10}=\int f(x)[g(x)-\delta]^2dx$ *and* $\xi_{01}=\int g(x)[f(x)-\delta]^2dx$.

PROOF. (i) follows immediately from (i) of Lemma 2.1 and (i) of Lemma 2.2. (ii) again follows from (ii) of Lemma 2.1 and (iii) of Lemma 2.2. It remains to prove (iii). Note that

(2.33) $$\begin{aligned}\hat{\lambda}-\lambda&=2(\hat{\delta}-\delta)/(\hat{\Delta}(f)+\hat{\Delta}(g))-2\delta[(\hat{\Delta}(f)-\Delta(f))+(\hat{\Delta}(g)-\Delta(g))]\\&\quad/(\hat{\Delta}(f)+\hat{\Delta}(g))(\Delta(f)+\Delta(g))\\&=U_{1n}-U_{2n}\,,\qquad\text{say}\,.\end{aligned}$$

First, it will be established that $n^{1/2}U_{1n}$ is asymptotically normal with mean 0 and variance σ^2 given by (2.32). But this follows directly from Doob's Theorem in conjunction with parts (iii) of Lemma 2.1 and (i) of Lemma 2.2. Next, let us show that $n^{1/2}U_{2n}\to 0$ in probability as $n\to\infty$. To this end it suffices to show that $n^{1/2}(\hat{\Delta}(f)-\Delta(f))\to 0$ in probability and that $n^{1/2}(\hat{\Delta}(g)-\Delta(g))\to 0$ in probability, as $n\to\infty$. Note that it follows from the proof of Theorem 2.3 of Bhattacharayya and Roussas [3] and Lemma 2.2 above that

$$(2.34)\qquad (na_n)^2 \operatorname{Var} \hat{\Delta}(f) \to \left(\int k^2(u)du\right)^2 \quad \text{as } n\to\infty .$$

Thus $n \operatorname{Var} \hat{\Delta}(f) \to 0$ since $na_n^2 \to \infty$, $n\to\infty$. Next, note that,

$$\begin{aligned}(2.35)\quad \mathrm{E}\,\hat{\Delta}(f) &= \int \mathrm{E}\,\hat{f}^2(x)dx \\ &= (na_n)^{-2}\int \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{E}\, k[(x-X_i)/a_n]k[(x-X_j)/a_n]dx \\ &= (na_n^2)^{-1}\int \mathrm{E}\, k^2[(x-X_1)/a_n]dx \\ &\quad + a_n^{-2}((n-1)/n)\int \{\mathrm{E}\, k[(x-X_1)/a_n]\}^2 dx \\ &= (na_n)^{-1}\int k^2(u)du + ((n-1)/n)\int\left[\int k(u)f(x-a_nu)du\right]^2 dx\end{aligned}$$

where the last equality is obtained by changing variables $(x-y)/a_n = u$ and observing that $\int f(x-a_nu)dx = 1$. Thus,

$$\begin{aligned}(2.36)\quad &|\mathrm{E}\,\hat{\Delta}(f) - \Delta(f)| \\ &= \left|(na_n)^{-1}\int k^2(u)du + [(n-1)/n]\int\left[\int k(u)f(x-a_nu)du\right]^2 dx \right. \\ &\qquad \left. - \int f^2(x)dx\right| \\ &\leqq (na_n)^{-1}\int k^2(u)du + [(n-1)/n]\left|\int\left[\int k(u)f(x-a_nu)du\right]^2 dx \right. \\ &\qquad \left. - \int f^2(x)dx\right| + n^{-1}\int f^2(x)dx \\ &\leqq (na_n)^{-1}\int k^2(u)du + [(n-1)/n]\int\left|\left[\int k(u)f(x-a_nu)du\right]^2 \right. \\ &\qquad \left. - f^2(x)\right| dx + \Delta(f)/n ,\end{aligned}$$

where the integral in the middle term in the above last upper bound is less than or equal to

$$\begin{aligned}&\int\left|\int k(u)f(x-a_nu)du - f(x)\right|\left\{\int k(u)f(x-a_nu)du + f(x)\right\}dx \\ &\leqq \sup_x \left|\int [f(x-a_nu)-f(x)]k(u)du\right| \\ &\quad \cdot \left\{\int\int k(u)f(x-a_nu)dxdu + \int f(x)dx\right\}\end{aligned}$$

$$\leq a_n^2 \sup_x |f''(x)| \int u^2 k(u) du .$$

Hence we have

$$n^{1/2} |\mathrm{E}\, \hat{\Delta}(f) - \Delta(f)| \leq (na_n^2)^{-1/2} \int k^2(u) du + (na_n^4)^{1/2} [(n-1)/n] \sup_x |f''(x)| \cdot \int u^2 k(u) du + n^{-1/2} \Delta(f) ,$$

which converges to 0 as $n \to \infty$ since $na_n^2 \to \infty$, $na_n^4 \to 0$, and $a_n \to 0$, as $n \to \infty$. Similarly it can be shown that $n^{1/2} |\mathrm{E}\, \Delta(G) - \hat{\Delta}(G)| \to 0$ as $n \to \infty$. This completes the proof of (iii) and the theorem.

Consider the hypothesis testing problem $H_0: F = G$. If F and G admit square integrable p.d.f.'s f and g, then a nonparametric test ϕ_n based on $\hat{\lambda}$ may be proposed by rejecting H_0 for small values of $\hat{\lambda}$. Thus if the conditions of (iii) of Theorem 2.3 are satisfied, then $(na_n)^{1/2} \cdot (\hat{\lambda} - \lambda)/\sigma$ is asymptotically standard normal and hence we can perform approximate testing using the normal variate. It is interesting to compare this test against other nonparametric tests such as the Chernoff-Savage statistics, see Chernoff and Savage [5]. The exact null and nonnull distribution of $\hat{\lambda}$, though perhaps difficult to find, is an interesting open question. Note that

$$\hat{\sigma}^2 = 4 \left\{ \int \hat{f}(x) [\hat{g}(x) - \hat{\delta}]^2 dx + \int \hat{g}(x) [\hat{f}(x) - \hat{\delta}]^2 dx \right\} \Big/ (\hat{\Delta}(f) + \hat{\Delta}(g))$$

is a consistent estimate for σ, thus an approximate $(1-\alpha)\,100\%$ confidence interval of λ is given by $\hat{\lambda} \pm z_{\alpha/2}\, \hat{\sigma}/(na_n)^{1/2}$, with z_α denoting the standard normal variate. Robustness of the ϕ_n-test against dependence in or between samples need to be explored. Ahmad and Lin [1] studied the large sample theory of the kernel estimate of p.d.f.'s when the observations are strong mixing in the sense of Rosenblatt [19], or uniformly mixing, see Billingsley [4] p. 168, or mixing in the sense of Philipp [17]. The results of the paper by Ahmad and Lin [1] should prove helpful in establishing large sample properties of $\hat{\lambda}$ when each sample is taken from a stationary mixing processes. It is also interesting to study the properties of $\hat{\lambda}$ when $\{(X_n, Y_n)\}$ is a random sample from a bivariate d.f. H having marginals F and G, respectively.

3. Certain one-sample cases

In this section we shall discuss the case when g is either completely known, i.e. $g(x) = f_0(x)$, for all x where f_0 is a known p.d.f., or is such that $g(x)=f(-x)$ for all x. The former case is applied to the one-sample hypothesis testing problem $H_0: F=F_0$ a known absolutely continuous d.f. while the latter is applied to testing $H_0^*: F$ is symmetric about 0. Let $X_1, \cdots, X_n$ denote a random sample from F.

(A) *One-sample hypothesis testing.* In this case an estimate of λ may be given by:

$$\hat{\lambda}_0=2n^{-1}\sum_{i=1}^{n} f_0(X_i)/(\hat{\Delta}(f)+C_0) , \tag{3.1}$$

where $\hat{\Delta}(f)$ is as given by (1.8), and $C_0=\int f_0^2(x)dx$ a known constant. Using arguments similar to those of Theorem 2.1 we can prove the following theorem.

THEOREM 3.1.

(i) *If* $na_n \to \infty$, *as* $n \to \infty$, *and if* $\int f_0^2(x)f(x)dx<\infty$, *then*

$$\hat{\lambda}_0 \to \lambda_0 = 2\int f(x)f_0(x)dx/(\Delta(f)+C_0) \qquad \textit{in probability as } n\to\infty . \tag{3.2}$$

(ii) *If k is a right continuous function of bounded variation and if for any* $\gamma>0$, $\sum_{n=1}^{n} \exp(-\gamma na_n^2)<\infty$, *then*

$$\hat{\lambda}_0 \to \lambda_0 \qquad \textit{with probability one as } n\to\infty . \tag{3.3}$$

(iii) *If* $\int f_0^3(x)f(x)dx<\infty$ *and if* $na_n^2\to\infty$, *then* $n^{1/2}(\hat{\lambda}_0-\lambda_0)$ *is asymptotically normal with mean* 0 *and variances* σ_0^2 *given by*

$$\sigma_0^2=4\int f(x)[f_0(x)-\delta_0]^2dx/(\Delta(f)+C_0)^2 , \tag{3.4}$$

where $\delta_0=\int f(x)f_0(x)dx$.

(B) *Testing for symmetry.* In this case λ may be estimated by

$$\hat{\lambda}^*=\hat{\delta}^*/\hat{\Delta}(f) , \tag{3.5}$$

where $\hat{\Delta}(f)$ is as given in (1.8) and

$$\hat{\delta}^*=(n^2a_n)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} k[(X_i+X_j)/a_n] .$$

The following lemma is needed in the proof of Theorem 3.2.

LEMMA 3.1.

(i) *If* $na_n \to \infty$, *as* $n \to \infty$ *and if* $\int f^2(x)f(-x)dx < \infty$, *then*

$$\mathrm{E}\,|\hat{\delta}^* - \delta^*| \to 0 \qquad \text{as } n \to \infty\,, \tag{3.6}$$

where $\delta^* = \int f(x)f(-x)dx$.

(ii) *If* k *is a continuous function of bounded variation and if for any* $\gamma > 0$, $\sum_{n=1}^{\infty} \exp(-\gamma n a_n^2) < \infty$, *then*

$$\hat{\delta}^* \to \delta^* \qquad \text{with probability one as } n \to \infty\,. \tag{3.7}$$

(iii) *If* $na_n \to \infty$ *and* $na_n^4 \to 0$ *as* $n \to \infty$, *if* f *has a bounded second derivative, if* $\int f^3(x)f(-x)dx < \infty$, *and if* $\int u^2 k(u)du < \infty$, *then* $n^{1/2}(\hat{\delta}^* - \delta^*)$ *is asymptotically normal with mean* 0 *and variance* $\sigma^{*2} = \int f(x)[f(-x) - \delta^*]^2 dx / \Delta^2(f)$.

PROOF. (i) First we show that $\mathrm{E}\,\hat{\delta}^* \to \delta^*$ as $n \to \infty$. But

$$\begin{aligned}
\mathrm{E}\,\delta^* &= (na_n)^{-1}\,\mathrm{E}\,k(2X_1/a_n) + [(n-1)/n]a_n^{-1}\,\mathrm{E}\,k[(X_1+X_2)/a_n] \\
&= (na_n)^{-1}\int k(2u/a_n)f(u)du \\
&\quad + [(n-1)/n]a_n^{-1}\int\int k[(u+v)/a_n]f(u)f(v)dudv \\
&= (na_n)^{-1}\int k(v)f(a_n v/2)dv \\
&\quad + [(n-1)/n]\int\int k(w)f(wa_n - v)f(v)dwdv
\end{aligned} \tag{3.8}$$

and, as

$$(na_n)^{-1}\int k(v)f(a_n v/2)dv \to 0 \qquad \text{as } n \to \infty\,,$$

it suffices to show that

$$\int\int k(w)f(wa_n - v)f(v)dwdv \to \delta^* = \int f(-v)f(v)dv\,, \qquad \text{as } n \to \infty\,.$$

Setting $\theta_n(v) = \int k(w)f(wa_n - v)dw$ we obtain from Parzen [16], Theorem 1A that, $\theta_n(v) \to f(-v)$, as $n \to \infty$. Thus, to apply the extended Lebesgue dominated convergence theorem to conclude that $\int \theta_n(v)f(v)dv \to \int f(-v) \cdot f(v)dv$, as $n \to \infty$ one need only to prove that there exists a function

$g(v)$ such that $|\theta_n(v)f(v)| \leqq g(v)$ and $\int g(v)dv < \infty$. However, as $|\theta_n(v)| = \int k(w)f(wa_n - v)dw \leqq \sup_x f(x)$, we have only to set $g(v) = \sup_x f(x)f(v)$.

Next we show that $\mathrm{E}\,\hat{\delta}^{*2} \to \delta^{*2}$ as $n \to \infty$. Note that

$$\mathrm{E}\,\hat{\delta}^{*2} = (n^4 a_n^2)^{-1} \sum_i \sum_j \sum_p \sum_q \mathrm{E}\, k[(X_i + X_j)/a_n] k[(X_p + X_q)/a_n] \,. \tag{3.9}$$

In the above right-hand side the contribution from terms with i, j, p, and q all different is in the order of

$$(n^4 a_n^2)^{-1} n(n-1)(n-2)(n-3)\{\mathrm{E}\, k[(X_1 + X_2)/a_n]\}^2 \,, \tag{3.10}$$

which converges to $\left\{\int f(x)f(-x)dx\right\}^2 = \delta^{*2}$. Next, the total contribution from all the remaining terms is at most $B(n^4 a_n)^{-1}[n^4 - n(n-1)(n-2)(n-3)]$, where $B > 0$ is a constant independent of n, which converges to 0 as $n \to \infty$. In order to see this note that since $k(u)$ is bounded, then $\int k^2(u)du < \infty$, and $\int f^2(x)dx < \infty$, by assumption,

$$\mathrm{E}\, k[(X_1 + X_2)/a_n] k[(X_1 + X_3)/a_n] \leqq B_1\, \mathrm{E}\, k[(X_1 + X_2)/a_n] \,,$$

and

$$\begin{aligned} a_n^{-1}\, \mathrm{E}\, k[(X_1 + X_2)/a_n] &= \int\int k(w)f(a_n w - v)f(v)dwdv \\ &\leqq \left(\int f^2(x)dx\right)^{1/2} \left(\int k^2(u)du\right)^{1/2} \leqq B_1 \,, \end{aligned} \tag{3.11}$$

where B_1 is a positive constant. Thus (i) is proved.

(ii) It follows from (i) that $|\mathrm{E}\,\hat{\delta}^* - \delta^*| \to 0$ as $n \to \infty$, thus it suffices to show that $|\hat{\delta}^* - \mathrm{E}\,\hat{\delta}^*| \to 0$ with probabilty one as $n \to \infty$. But

$$\begin{aligned} |\hat{\delta}^* - \mathrm{E}\,\hat{\delta}^*| &= \left| \int \hat{f}(-x)dF_n(x) - \int \mathrm{E}\,\hat{f}(-x)dF(x) \right| \\ &\leqq \sup_x |\hat{f}(x) - \mathrm{E}\,\hat{f}(x)| + \left| \int\int a_n^{-1} k[(x-y)/a_n] f(y) dF_n(x)dy \right. \\ &\qquad \left. - \int\int a_n^{-1} k[(x-y)/a_n] f(y) dF(x)dy \right| \\ &= J_{1n} + J_{2n} \,, \qquad \text{say} \,. \end{aligned} \tag{3.12}$$

Now, $J_{1n} \to 0$ with probability one as $n \to \infty$ as in (2.11), while upon integration by parts, $J_{2n} \leqq (\mu a_n^{-1}) \sup |F_n(x) - F(x)|$, where μ is the total variation of $k(u)$. Thus for any $\varepsilon > 0$

$$\mathrm{P}\,[J_{2n} \geqq \varepsilon] \leqq \mathrm{P}\,[\sup_x |F_n(x) - F(x)| \geqq \varepsilon an/\mu] \leqq 2\exp(-\varepsilon^2 n a_n^2/\mu^2) \,. \tag{3.13}$$

Thus by a result of Dvoretzky, Kiefer, and Wolfowitz [6], since for any $\gamma>0$, $\sum_{n=1}^{\infty} \exp(-\gamma n a_n^2)<\infty$, we have for any $\varepsilon>0$, $\sum_{n=1}^{\infty} \mathrm{P}[J_{2n}\geqq\varepsilon]<\infty$, thus by the Borel-Cantelli lemma we have $J_{2n}\to 0$ with probability one as $n\to\infty$. Hence $|\hat{\delta}^*-\mathrm{E}\,\hat{\delta}^*|\to 0$ with probability one as $n\to\infty$ and (ii) is proved.

(iii) Let $l_1(x/a_n)=a_n^{-1}\,\mathrm{E}\,\{k[(X_1+X_2)/a_n]\,|\,X_1=x\}$. Then $\mathrm{E}\,l_1(X_1/a_n)=\mathrm{E}\,\hat{\delta}^*$ which converges to δ^* as $n\to\infty$. Using an argument similar to that of (i) it can be shown that

$$\mathrm{Var}\,l_1(X_1/a_n)\to\int f(x)[f(-x)-\delta^*]^2dx\,, \qquad \text{as } n\to\infty\,, \tag{3.14}$$

and

$$\mathrm{E}\,|l_1(X_1/a_n)-\mathrm{E}\,\delta^*|^3\to\int f(x)|f(-x)-\delta^*|^3dx \qquad \text{as } n\to\infty\,, \tag{3.15}$$

since the right-hand sides of (3.14) and (3.15) are finite by assumption. It follows that the Layaponouff's conditions of the central limit theorem are satisfied, and hence $n^{1/2}\left[n^{-1}\sum_{i=1}^{n} l_1(X_i/a_n)-\mathrm{E}\,\delta^*\right]$ is asymptotically normal with mean 0 and variance σ^{*2}. As in the proof of Lemma 2.1 it can be shown that $n^{1/2}\left[\delta^*-n^{-1}\sum_{i=1}^{n} l_1(X_i/a_n)\right]$ converges to zero in probability as $n\to\infty$, and that if f has a bounded second derivative we have $n^{1/2}(\mathrm{E}\,\hat{\delta}^*-\delta^*)\to 0$ as $n\to\infty$ provided that $na_n^4\to 0$ as $n\to\infty$. This completes the proof of (iii) and the lemma.

With the aid of Lemma 3.1 in conjunction with Lemma 2.2 we arrive at the following theorem.

THEOREM 3.2.

(i) *If* $na_n\to\infty$, *as* $n\to\infty$ *and if* $\int f^2(x)f(-x)dx<\infty$, *then*

$$\hat{\lambda}^*\to\lambda^* \qquad \textit{in probability as } n\to\infty\,. \tag{3.16}$$

(ii) *If* k *is right continuous function of bounded variation, and if for any* $\gamma>0$, $\sum_{n=1}^{\infty}\exp(-\gamma n a_n^2)<\infty$, *then*

$$\hat{\lambda}^*\to\lambda^* \qquad \textit{with probability one as } n\to\infty\,. \tag{3.17}$$

(iii) *If* $na_n^2\to\infty$, *and* $na_n^4\to 0$ *as* $n\to\infty$, *if* f *has a bounded second derivative, if* $\int f^3(x)f(-x)dx<\infty$, *and if* $\int u^2k(u)du<\infty$, *then* $n^{1/2}(\hat{\lambda}^*-\lambda^*)$

is asymptotically normal with mean 0 *and variance* σ^{*2} *given by*

$$\sigma^{*2}=\int f(x)[f(-x)-\delta^*]^2dx/\Delta^2(f) . \tag{3.18}$$

Acknowledgement

The author is deeply indebted to the referee for his careful reading of the manuscript and detailed remarks which resulted in a much improved paper, to Mr. S. Aki for providing a proof of the first part of Lemma 3.1 (i), and to the editor, Professor K. Matusita for his constant help. This research was supported in part by the National Research Council of Canada while the author was with McMaster University and by the Research Board of Memphis State University.

McMaster University

References

[1] Ahmad, I. A. and Lin, P. E. (1977). Non parametric density estimation for dependent variables with application, under revision.

[2] Ahmad, I. A. and Van Belle, G. (1974). Measuring affinity of distributions. *Reliability and Biometry, Statistical Analysis of Life Testing,* (eds. Proschan and R. J. Serfling), SIAM, Philadelphia, 651-668.

[3] Bhattacharayya, G. K. and Roussas, G. (1969). Estimation of certain functional of probability density function, *Skand. Aktuarietidskr.*, **52**, 203-206.

[4] Billingsley, P. (1968). *Convergence of Probability Measures,* John Wiley and Sons, New York.

[5] Chernoff, H. and Savage, I. R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics, *Ann. Math. Statist.*, **29**, 972-994.

[6] Dvoretzky, A., Kiefer, J. and Wolfowitz, J. (1956). Asymptotic minimax character of the sample distribution function and of the classical multinomial estimator, *Ann. Math. Statist.*, **27**, 642-669.

[7] Matusita, K. (1955). Decision rules based on the distance for the problems of fit, two samples, and estimation, *Ann. Math. Statist.*, **26**, 631-640.

[8] Matusita, K. (1964). Distance and decision rules, *Ann. Inst. Statist. Math.*, **16**, 305-315.

[9] Matusita, K. (1966). A distance and related statistics in multivariate analysis, *Multivariate Analysis* I, (ed. P. R. Krishnaiah), Academic Press, New York, 187-200.

[10] Matusita, K. (1967a). Classification Based on Distance in Multivariate Gaussian Cases, *Proc. Fifth Berkeley Symp. Math. Statist. Prob.*, Vol. I, 299-304.

[11] Matusita, K. (1967b). On the notion of affinity of several distributions and some of its applications, *Ann. Inst. Statist. Math.*, **19**, 181-192.

[12] Matusita, K. (1971). Some properties of affinity and applications, *Ann. Inst. Statist. Math.*, **23**, 137-155.

[13] Matusita, K. (1973). Correlation and affinity in Gaussian cases, *Multivariate Analysis* III, (ed. P. R. Krishnaiah), Academic Press, New York, 345-349.

[14] Matusita, K. and Akaike, H. (1956). Decision rules based on the distance for the problem of independence, invariance, and two samples, *Ann. Inst. Statist. Math.*, **7**, 67-80.

[15] Nadaraya, E. A. (1965). On nonparametric estimation of density function and regression curve, *Theory Prob. Appl.*, **10**, 186-190.

[16] Parzen, E. (1962). On the estimation of a probability density function and mode, *Ann. Math. Statist.*, **33**, 1065-1076.

[17] Philipp, W. (1969). The central limit theorem for mixing sequences of random variables, *Z. Wahrscheinlickeitsth.*, **12**, 155-171.

[18] Resenblatt, M. (1956a). Remarks on some nonparametric estimates of a density function, *Ann. Math. Statist.*, **27**, 832-837.

[19] Rosenblatt, M. (1956b). A central limit theorem and a strong mixing condition, *Proc. Nat. Acad. Sci. USA*, **42**, 43-47.

[20] Royden, H. L. (1968). *Real Analysis* (Second Edition), Macmillan, New York.