ON THE CHERNOFF-SAVAGE THEOREM FOR
DEPENDENT SEQUENCES

Ibrahim A. Ahmad* and Pi-Erh Lin

Summary

Given a sequence of ϕ-mixing random variables not necessarily stationary, a Chernoff-Savage theorem for two-sample linear rank statistics is proved using the Pyke-Shorack [5] approach based on weak convergence properties of empirical processes in an extended metric. This result is a generalization of Fears and Mehra [4] in that the stationarity is not required and that the condition imposed on the mixing numbers is substantially relaxed. A similar result is shown to hold for strong mixing sequences under slightly stronger conditions on the mixing numbers.

1. Introduction

Since the appearance of the paper by Chernoff and Savage [2] an ever increasing effort is devoted to study the problem of asymptotic normality of two-sample linear rank statistics. Pyke and Shorack [5], using the concept of weak convergence of certain empirical processes, give an alternative proof of the Chernoff-Savage theorem. Their proof shows that the theorem holds even for a larger class of score functions than initially established. Motivated by interest in the robustness of the two-sample linear rank statistics, Fears and Mehra [4], using the approach of Pyke and Shorack, establish the asymptotic normality of linear rank statistics for strictly stationary ϕ-mixing sequences of random variables satisfying certain regularity conditions.

The purpose of the present investigation is two-fold. For ϕ-mixing sequences we waive the stationarity assumption and considerably

weaken the condition imposed by Fears and Mehra [4] on the mixing numbers. Secondly, we establish, under slightly stronger assumptions on the mixing numbers, analogous results for strong mixing sequences of random variables.

Let $\{X_m\}_{m=1}^{\infty}$ be a sequence of random variables and let $\mathcal{F}_{m,n}$ denote the σ-field generated by $\{X_m, X_{m+1}, \cdots, X_n\}$, $1 \leq m < n \leq \infty$. Further, let $A \in \mathcal{F}_{1,m}$ and $B \in \mathcal{F}_{m+n,\infty}$. Then $\{X_m\}$ is said to be ϕ-mixing or uniformly mixing if

$$\sup_{m \geq 1} \sup_{A \in \mathcal{F}_{1,m}} \sup_{B \in \mathcal{F}_{m+n,\infty}} \frac{1}{P(A)} |P(AB) - P(A) P(B)| = \phi(n)$$

where $\phi(n)$ is nonincreasing such that $\lim_{n \to \infty} \phi(n) = 0$. The sequence $\{X_m\}$ is said to be α-mixing or strongly mixing if

$$\sup_{m \geq 1} \sup_{A \in \mathcal{F}_{1,m}} \sup_{B \in \mathcal{F}_{m+n,\infty}} |P(AB) - P(A) P(B)| = \alpha(n) ,$$

where $\alpha(n)$ is nonincreasing such that $\lim_{n \to \infty} \alpha(n) = 0$. Note that uniform mixing implies strong mixing, but not conversely.

In this paper we consider two independent sequences of random variables $\{X_m\}_{m=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ with common marginals F and G, respectively. Assume that

  (i) $\{X_m\}$ and $\{Y_n\}$ have absolutely continuous finite dimensional distributions with respect to Lebesgue measure, and that either
  (ii) both sequences are uniformly mixing, or
  (ii)′ both sequences are strongly mixing.

For ease of comparison between our result for the case of uniform mixing sequences and that of Fears and Mehra [4] we shall try to be consistent with their presentation and closely follow their notations. Note that the result of Fears and Mehra is obtained assuming strict stationarity for both sequences and $\sum_{n=1}^{\infty} n^2 [\phi(n)]^{1/2} < \infty$ while we waive the stationarity assumption and only assume $\phi(n) = O(n^{-2})$.

Let $\{X_m\}$ and $\{Y_n\}$ be two independent sequences of random variables satisfying Conditions (i) and (ii) above. The two-sample linear rank statistics is defined by

$$(1.1) \qquad\qquad T_N = m^{-1} \sum_{k=1}^{N} c_{Nk} R_{Nk} ,$$

where R_{Nk} denotes the number of X's among $X_1, \cdots, X_m$ which do not exceed the kth order statistic of the combined sample $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ and c_{Nk}, $1 \leq k \leq N$, are a given set of constants with $N = m+n$. As in Pyke and Shorack [5], T_N has another representation that employs certain two-sample empirical processes, viz.,

(1.2)
$$T_N = \int_0^1 F_m H_N^{-1} d\nu_N \,,$$

where $F_m(G_n)$ denotes the empirical distribution function (d.f.) of X_1, $\cdots$, $X_m(Y_1, \cdots, Y_n)$, $H_N = \lambda_N F_m + (1-\lambda_N)G_n$, with $\lambda_N = m/N$ and $H_N^{-1}(t) = \inf\{x : H_N(x) \geqq t\}$, and ν_N is the signed measure which assigns measure c_{Nk} at the point k/N, $k = 1, \cdots, N$. Define $H_\lambda = \lambda F + (1-\lambda)G$ with $\lambda = \lim_{N \to \infty} \lambda_N$ where $F(G)$ is the marginal d.f. of $X(Y)$. Pyke and Shorack [5] prove the asymptotic normality of

(1.3)
$$T_N^* = N^{1/2}(T_N - \mu_N) = \int_0^1 L_N(t) d\nu_N(t) \,,$$

where $\mu_N = \int_0^1 FH^{-1} d\nu_N$ and $L_N(t) = [F_m H_N^{-1}(t) - FH^{-1}(t)]$ with $H = H_{\lambda_N}$, by establishing the weak convergence of $L_N(t)$ to a well defined Gaussian process.

Adopting the approach of Pyke and Shorack, Fears and Mehra [4] prove a corresponding result where $\{X_m\}$ and $\{Y_n\}$ are strictly stationary uniform mixing sequences satisfying the condition $\sum_{n=1}^{\infty} n^2[\phi(n)]^{1/2} < \infty$. The method of proof developed by Pyke and Shorack [5] consists of three main parts. First, the weak convergence is proved for the one-sample process

(1.4)
$$m^{1/2}[F_m F^{-1}(t) - t]/q(t) \,,$$

where $q(t) = K[t(1-t)]^{1/2-\delta}$, $0 \leqq t \leqq 1$ and some δ, $0 < \delta < 1/2$, then the weak convergence of $L_N(t)$ is established and, finally, the asymptotic distribution of T_N^* is obtained. This is the path we follow to present our results, being brief whenever possible.

2.   Two-sample linear rank statistics for uniform mixing processes

The Chernoff-Savage theorem for the two-sample linear rank statistics T_N^* will be established in this section for uniform mixing sequences. The proof proceeds in the following three stages :

(a)   *One sample empirical process.*

Let $U_m(t) = m^{1/2}[F_m F^{-1}(t) - t]$, $0 \leqq t \leqq 1$, and $V_m(t) = U_m(t)/q(t)$. The following main result of this subsection establishes the weak convergence of $V_m(t)$ to a Gaussian process where $\{X_m\}$ is a uniform mixing sequence with $\phi(m) = O(m^{-2})$, not necessarily stationary. Because of the substantial weakening of assumptions on the sequence, a new proof is thus required. Though the proof is slightly long, its important contribution should compensate it.

THEOREM 2.1. *Let $\{X_m\}$ be a uniform mixing sequence with $\phi(m)$ $=O(m^{-2})$. Then $V_m(t)$ converges weakly to a Gaussian process $V(t)=$ $U(t)/q(t)$ where $\mathrm{E}\,U(t)=0$ and, for $0\leqq s\leqq t\leqq1$,*

$$(2.1) \qquad \mathrm{E}\,U(s)U(t)=\lim_{m\to\infty}\mathrm{E}\,U_m(s)U_m(t)=L(s,t)\,, \qquad say$$

provided that the right-hand side of (2.1) exists.

PROOF. First we shall show that $U_m(t)$ converges weakly to $U(t)$ and then use this result to establish that $V_m(t)$ converges weakly to $V(t)$. The proof of the first assertion is an extension of a result of Yoshihara [8] where strict stationarity is assumed; so we shall only mention the necessary changes in his proof. That the finite dimensional distributions are asymptotically normal follows from Corollary 1 (f) of Withers [7] since $\phi(m)=O(m^{-2})$ implies that $\sum_{i=1}^{j} i^2\phi(i)\leqq Kj$, $j=1$, $\cdots, m$. To establish the tightness we need to show that, for all $0\leqq s$ $\leqq t\leqq1$ and some $\gamma>0$,

$$(2.2) \qquad \mathrm{E}\,[U_m(t)-U_m(s)]^4\leqq C_\gamma(t-s)^{1+\gamma}$$

where C_γ is a constant depending on γ. Let

$$(2.3) \qquad \eta_j=I(s<F(X_j)\leqq t)-(t-s)\,, \qquad j=1,\cdots,m\,.$$

Then $U_m(t)-U_m(s)=m^{-1/2}\sum_{j=1}^{m}\eta_j$ and

$$(2.4) \qquad \mathrm{E}\,[U_m(t)-U_m(s)]^4=m^{-2}\,\mathrm{E}\left(\sum_{j=1}^{m}\eta_j\right)^4\,.$$

In what follows all generic constants will be denoted by K. Following Yoshihara [8] we define $l=l_m=[m^\lambda]$ for $0<\lambda<1$ and $p=p_m=[m/2l]$, where $[x]$ denotes the integral part of x. Let $V_i=\sum_{j=1}^{l}\eta_{2il+j}$, $V_i'=$ $\sum_{j=1}^{l}\eta_{(2i+1)l+j}$, $i=0,1,\cdots,p-1$ and $V_p=\sum_{j=(2p-1)l+j}^{m}\eta_j$. Then

$$(2.5) \qquad \mathrm{E}\left(\sum_{j=1}^{m}\eta_j\right)^4=\mathrm{E}\left(\sum_{i=0}^{p}V_i+\sum_{i=0}^{p-1}V_i'\right)^4$$
$$\leqq8\left[\mathrm{E}\left(\sum_{i=0}^{p}V_i\right)^4+\mathrm{E}\left(\sum_{i=0}^{p-1}V_i'\right)^4\right]\,.$$

Note that

$$(2.6) \qquad \mathrm{E}\left(\sum_{i=0}^{p}V_i\right)^4=\mathrm{E}\left(\sum_{i=0}^{p-1}V_i\right)^4+4\,\mathrm{E}\,V_p\left(\sum_{i=0}^{p-1}V_i\right)^3+6\,\mathrm{E}\,V_p^2\left(\sum_{i=0}^{p-1}V_i\right)^2$$
$$+4\,\mathrm{E}\,V_p^3\left(\sum_{i=0}^{p-1}V_i\right)+\mathrm{E}\,V_p^4$$

where

$$(2.7) \quad \mathrm{E}\left(\sum_{i=0}^{p-1} V_i\right)^4 \leqq 4! \sum_{i=0}^{p-1} \sum {}^* \mathrm{E}\,|V_i V_{i+j} V_{i+j+k} V_{i+j+k+u}|$$

$$\leqq 4! \left\{ \sum_{i=0}^{p-1} \mathrm{E}\,|V_i|^4 + \sum_{i=0}^{p-1} \sum_{j=1}^{p-i-1} (|\mathrm{E}\,V_i^3 V_{i+j}| + |\mathrm{E}\,V_i V_{i+j}^3| + |\mathrm{E}\,V_i^2 V_{i+j}^2|) \right.$$

$$+ \sum_{i=0}^{p-1} \sum_{j=1}^{p-i-1} \sum_{k=1}^{p-i-j-1} (|\mathrm{E}\,V_i^2 V_{i+j} V_{i+j+k}| + |\mathrm{E}\,V_i V_{i+j}^2 V_{i+j+k}|$$

$$+ |\mathrm{E}\,V_i V_{i+j} V_{i+j+k}^2|)$$

$$\left. + \sum_{i=0}^{p-1} \sum_{j,k,u \geqq 1} |\mathrm{E}\,V_i V_{i+j} V_{i+j+k} V_{i+j+k+u}| \right\}.$$

The above expression will be further bounded, term by term as follows. Checking the proof of Lemma 1 of Billingsley [1], p. 170, one can show that the same result holds without the assumption of stationarity. Thus it follows from this lemma and the assumption $\phi(m) = O(m^{-2})$ that, for m sufficiently large,

$$(2.8\mathrm{a}) \qquad |\mathrm{E}\,V_i^3 V_{i+j}| \leqq K l^{-3/2} (\mathrm{E}^3\,V_i^4\,\mathrm{E}\,V_{i+j}^4)^{1/4}$$

$$(2.8\mathrm{b}) \qquad |\mathrm{E}\,V_i V_{i+j}^3| \leqq K l^{-1/2} (\mathrm{E}\,V_i^4\,\mathrm{E}^3\,V_{i+j}^4)^{1/4}$$

$$(2.8\mathrm{c}) \qquad |\mathrm{E}\,V_i^2 V_{i+j}^2| \leqq K l^{-1} (\mathrm{E}\,V_i^4\,\mathrm{E}\,V_{i+j}^4)^{1/2} + \mathrm{E}\,V_i^2\,\mathrm{E}\,V_{i+j}^2$$

$$(2.8\mathrm{d}) \qquad |\mathrm{E}\,V_i^2 V_{i+j} V_{i+j+k}| \leqq K l^{-1} (\mathrm{E}^2\,V_i^4\,\mathrm{E}\,V_{i+j}^4\,\mathrm{E}\,V_{i+j+k}^4)^{1/4}$$

$$(2.8\mathrm{e}) \qquad |\mathrm{E}\,V_i V_{i+j}^2 V_{i+j+k}| \leqq K l^{-3/2} (\mathrm{E}\,V_i^4\,\mathrm{E}^2\,V_{i+j}^4\,\mathrm{E}\,V_{i+j+k}^4)^{1/4}$$

$$(2.8\mathrm{f}) \qquad |\mathrm{E}\,V_i V_{i+j} V_{i+j+k}^2| \leqq K l^{-3/2} (\mathrm{E}\,V_i^4\,\mathrm{E}\,V_{i+j}^4\,\mathrm{E}^2\,V_{i+j+k}^4)^{1/4}$$

and

$$(2.8\mathrm{g}) \quad |\mathrm{E}\,V_i V_{i+j} V_{i+j+k} V_{i+j+k+u}| \leqq K l^{-2} (\mathrm{E}\,V_i^4\,\mathrm{E}\,V_{i+j}^4\,\mathrm{E}\,V_{i+j+k}^4\,\mathrm{E}\,V_{i+j+k+u}^4)^{1/4}.$$

The above upper bounds are functions of $\mathrm{E}\,V_i^2$ and $\mathrm{E}\,V_i^4$. We shall obtain upper bounds only for $\mathrm{E}\,V_0^2$ and $\mathrm{E}\,V_0^4$; the same arguments apply to give bounds similar to (2.10) and (2.11) below for $\mathrm{E}\,V_i^2$ and $\mathrm{E}\,V_i^4$, $i = 1, 2, \cdots, p-1$. Now

$$(2.9) \qquad \mathrm{E}\,V_0^4 \leqq 4! \sum_{i=1}^{l} \{\textstyle\sum_{(1)} + \sum_{(2)} + \sum_{(3)}\} |\mathrm{E}\,\eta_i \eta_{i+j} \eta_{i+j+k} \eta_{i+j+k+u}|$$

where $\sum_{(1)}$, $\sum_{(2)}$, and $\sum_{(3)}$ are, respectively, the summations over all indices $i, j, k, u \geqq 1$, $j + k + u \leqq m - i$ such that $j \geqq \max(k, u)$, $k \geqq \max(j, u)$ and $u \geqq \max(j, k)$. Similar to the proof of Lemma 1 of Yoshihara [8], where the assumptions of common marginal F and uniform mixing for $\{X_m\}$ rather than stationarity are crucial, we obtain

$$(2.10) \qquad \mathrm{E}\,V_0^4 \leqq K(t-s) l \left\{ l \left[\sum_{j=1}^{l} \phi^{1/2}(j) \right]^2 + \sum_{j=1}^{l} (j+1)^2 \phi(j) \right\}$$

$$\leqq K(t-s)(l \log l)^2,$$

since, for m sufficiently large, $\sum_{j=1}^{l} \phi^{1/2}(j) = O(\log l)$ and $\sum_{j=1}^{l} (j+1)^2 \phi(j) = O(l)$. Similarly, it can be shown that

$$(2.11) \qquad\qquad \mathrm{E}\, V_0^2 \leq Kl(t-s)\,.$$

Note that, for m sufficiently large, there exists an ε, $0 < \varepsilon < \lambda$, such that $(\log m)^2 \leq m^{\varepsilon}$. Now collecting terms from (2.8) to (2.11), we obtain

$$(2.12) \qquad\qquad \mathrm{E}\left(\sum_{i=0}^{p-1} V_i\right)^4 \leq K[m^2(t-s)^2 + m^{1+\gamma}(t-s)]\,,$$

for some $\gamma > 0$. Next, note that

$$(2.13\mathrm{a}) \qquad \mathrm{E}\, V_p \left(\sum_{i=0}^{p-1} V_i\right)^3 = \sum_{j=2pl+1}^{m} \eta_j \left(\sum_{i=0}^{p-1} V_i\right)^3$$

$$\leq K \sum_{j=2pl+1}^{m} \phi((2p-1)l)\, \mathrm{E}\left|\sum_{i=0}^{p-1} V_i\right|^3$$

$$\leq K\, \mathrm{E}^{3/4}\left(\sum_{i=0}^{p-1} V_i\right)^4,$$

$$(2.13\mathrm{b}) \qquad \mathrm{E}\, V_p^2 \left(\sum_{i=0}^{p-1} V_i\right)^2 \leq K(l^{-1}+1)\, \mathrm{E}^{1/2}\left(\sum_{i=1}^{p-1} V_i\right)^4 \mathrm{E}^{1/2}\, V_p^4$$

$$\leq K\left[\mathrm{E}\left(\sum_{i=0}^{p-1} V_i\right)^4 \mathrm{E}\, V_p^4\right]^{1/2},$$

and

$$(2.13\mathrm{c}) \qquad \mathrm{E}\, V_p^3 \left(\sum_{i=0}^{p-1} V_i\right) \leq K\phi^{1/4}(2p+1)\left[\mathrm{E}\left(\sum_{i=0}^{p-1} V_i\right)^4 \mathrm{E}^3\, V_p^4\right]^{1/4}.$$

Hence, for sufficiently large m, there exists a $\gamma > 0$ such that

$$(2.14\mathrm{a}) \qquad\qquad \mathrm{E}\left(\sum_{i=0}^{p-1} V_i\right)^4 \leq K[m^2(t-s)^2 + m^{1+\gamma}(t-s)]\,.$$

Similarly, it can be shown that, there exists a $\gamma > 0$ such that, for m sufficiently large,

$$(2.14\mathrm{b}) \qquad\qquad \mathrm{E}\left(\sum_{i=0}^{p-1} V_i'\right)^4 \leq K[m^2(t-s)^2 + m^{1+\gamma}(t-s)]\,.$$

Therefore (2.2) follows. The rest of the proof of the tightness coincides with that of Yoshihara [8], and hence, is omitted. To complete the proof of the theorem it remains to show that, given $\varepsilon > 0$, there exist a $\theta \in (0, 1/2)$ and an integer $M = M(\varepsilon, \phi, \delta, K)$ such that for all $m \geq M$

$$(2.15) \qquad\qquad \mathrm{P}\left[\sup_{0 \leq t \leq \theta} |V_m(t)| \geq \varepsilon\right] \leq \varepsilon\,.$$

Let $g_t(X_j) = I(F(X_j) \leq t) - t$, $j = 1, \cdots, m$, $0 \leq t \leq 1$. The balance of the

proof is analogous to Lemma 2.1 of Fears and Mehra [4]. Let $0 < s_1 <$ $s_2 < \cdots < s_R = \theta < 1/2$ be R distinct points with $s_r = r\theta/R$, $r = 1, 2, \cdots, R$. For any pair (j, k) such that $1 < j < k \leq R$ define

(2.16) $\zeta_i^* = [g_{s_k}(X_i)/q(s_{k-1})] - [g_{s_j}(X_i)/q(s_{j-1})]$, $i = 1, \cdots, m$.

Set $\zeta_i = \zeta_i^* / \{2[\sum\limits_{j < r \leq k} q^{-2}(s_{r-1})]^{1/2}\}$, $i = 1, \cdots, m$. Then it follows from (2.7) and (2.8) of Fears and Mehra [4], where the stationarity is not essential, that $\mathrm{E}\, \zeta_i^2 \leq \theta/R$ and $\mathrm{E}\, [g_{s_k}(X_i)/q(s_{k-1})]^2 \leq 2(\theta/R) \sum\limits_{j < r \leq k} q^{-2}(s_{r-1})$, $i = 1, \cdots,$ m. Next, in the proof of the tightness of $U_m(t)$ we have demonstrated that the stationarity is unessential, thus Lemma 2 of Yoshihara [8] becomes: If $\{Z_i\}$ is a sequence of uniform mixing random variables such that $\mathrm{E}\, Z_i = 0$, $\mathrm{E}\, Z_i^2 \leq \tau$, $\mathrm{E}\, |Z_i| \leq C\tau$ for some constant $C > 0$, and $|Z_i| \leq 1$, then there exists a number γ, $0 < \gamma < 1$, such that $\mathrm{E} \left(\sum\limits_{i=1}^{m} Z_i\right)^4 \leq$ $K(m^{2-\gamma}\tau + m^2\tau^2)$ for some constant $K > 0$. We now apply this result to the sequence $\{\eta_i\}$. Note that, for $i = 1, \cdots, m$, $\mathrm{E}\, \zeta_i = 0$, $\mathrm{E}\, \zeta_i^2 \leq \theta/R$, and that, from (2.9) and (2.10) of Fears and Mehra [4], $|\zeta_i| \leq 1$. It remains to show that there exists a positive constant C such that $\mathrm{E}\, |\zeta_i| \leq C\theta/R$ for $i = 1, \cdots, m$. For any pair (j, k), we have

(2.17) $\mathrm{E}\, |\zeta_i| \leq \{2[\sum\limits_{j < r \leq k} q^{-2}(s_{r-1})]^{1/2}\}^{-1} \mathrm{E} \left| \dfrac{g_{s_k}(X_i)}{q(s_{k-1})} - \dfrac{g_{s_j}(X_i)}{q(s_{j-1})} \right|$.

The second factor in the RHS of (2.17) is majorized by

(2.18) $q^{-1}(s_{k-1}) \mathrm{E}\, |g_{s_k}(X_i) - g_{s_j}(X_i)| + \mathrm{E}\, |g_{s_j}(X_i)| [q^{-1}(s_{j-1}) - q^{-1}(s_{k-1})]$

$\leq 2q^{-1}(s_{k-1})(s_k - s_j) + 2s_j[q^{-1}(s_{j-1}) - q^{-1}(s_{k-1})]$

$= 2\left[\dfrac{s_k}{q(s_{k-1})} - \dfrac{s_j}{q(s_{j-1})} \right] + 4s_j[q^{-1}(s_{j-1}) - q^{-1}(s_{k-1})]$.

The above inequality follows from the fact that $\mathrm{E}\, |W - \mathrm{E}\, W| \leq 2\, \mathrm{E}\, |W|$ for any random variable W with $\mathrm{E}\, |W| < \infty$. To obtain a suitable bound for the RHS of (2.18), we recall that $q(t) = K[t(1-t)]^{1/2-\delta}$ for all $t \in [0, 1]$, $\delta \in (0, 1/2)$, and $K > 0$. Thus $0 < q(t) \leq K$ for any $t \in (0, 1)$ and the RHS of (2.18) is then majorized by

(2.19) $\dfrac{2K}{q(s_{k-1})} \left[\dfrac{s_k}{q(s_{k-1})} - \dfrac{s_j}{q(s_{j-1})} \right] + \dfrac{4Ks_j}{q(s_{j-1})} [q^{-1}(s_{j-1}) - q^{-1}(s_{k-1})]$.

Now, from (2.4) and (2.6) of Fears and Mehra [4], and the inequality $|q^2(s_{r-1})/q^2(s_{r-2})| < 2$ for all $2 < r \leq R$, our (2.19) is bounded above by $12K(\theta/R)[\sum\limits_{j < r \leq k} q^{-2}(s_{r-1})]$. Therefore, with $C = 6K[\sum\limits_{j < r \leq k} q^{-2}(s_{r-1})]^{1/2}$, we have

(2.20) $\mathrm{E}\, |\zeta_i| \leq C(\theta/R)$.

Hence, for m sufficiently large, there exists a number γ, $0<\gamma<1$, such that

$$(2.21) \qquad \mathrm{E}\left|\frac{U_m(s_k)}{q(s_{k-1})}-\frac{U_m(s_j)}{q(s_{j-1})}\right|^4 \leq K[\sum_{j<r\leq k} q^{-2}(s_{r-1})]\left[\left(\frac{\theta}{R}\right)^2+\frac{\theta}{R}m^{-\gamma}\right],$$

where K, here and in what follows, denotes generic constants not necessarily the same. If $0<\varepsilon<1$ is a fixed real number such that $(\varepsilon/m)<(\theta/R)$, then the RHS of (2.21) is majorized by

$$(2.22) \qquad K[\sum_{j<r\leq k} q^{-2}(s_{r-1})]^2(1+\varepsilon^{-\gamma})(\theta/R)^{1+\gamma}.$$

Similarly, we also have

$$(2.23) \qquad \mathrm{E}\left|\frac{U_m(s_k)}{q(s_{k-1})}\right|^4 \leq K[\sum_{1<r\leq k} q^{-2}(s_{r-1})](1+\varepsilon^{-\gamma})(\theta/R)^{1+\gamma}.$$

Thus it follows from Theorem 12.2 of Billingsley [1] with $\xi_1=U_m(s_1)/q(s_1)$, $\xi_i=[U_m(s_{j+1})/q(s_j)]-[U_m(s_j)/q(s_{j-1})]$ for $j=2,\cdots,R-1$, that

$$(2.24) \quad \mathrm{P}\left[\max_{1\leq j\leq R}\left|\frac{U_m(s_{j+1})}{q(s_j)}\right|\geq\varepsilon\right]\leq(K/\varepsilon)^4[\sum_{j<r\leq k}q^{-2}(s_{r-1})](1+\varepsilon^{-\gamma})(\theta/R)^{1+\gamma},$$

where $0<\gamma<1$. Given $\varepsilon>0$ and $\theta>0$, if m is sufficiently large it follows that an R can be chosen so that $(\varepsilon/4m)^{4/(1+\gamma)}<(\theta/R)<(\varepsilon/2m)^{2/(1+\gamma)}$ and $(KR^\delta)^{-1}<\varepsilon^{1/2}/4$. Thus for this choice of R we have for some K, $0<K<\infty$, depending on ϕ and q,

$$(2.25) \quad \mathrm{P}\left[\sup_{\theta/R\leq t\leq\theta}\left|\frac{U_m(t)}{q(t)}\right|\geq\frac{\varepsilon}{2}\right]\leq\mathrm{P}\left[2\max_{1\leq j\leq R}\left|\frac{U_m(s_{j+1})}{q(s_j)}\right|\geq\frac{\varepsilon}{8}\right]+\mathrm{P}\left[\left|\frac{U_m(s_1)}{q(s_1)}\right|\geq\frac{\varepsilon}{4}\right]$$

$$\leq(K/\varepsilon^4)(1+\varepsilon^{-\gamma})\left[\int_0^\theta q^{-2}(t)dt\right]^2+K\varepsilon^{-2}s_1^{2\delta},$$

since $\mathrm{E}[U_m(s_1)/q(s_1)]^2\leq Ks_1^{2\delta}$. By choosing θ sufficiently small, the last upper bound of (2.25) can be made less than ε/m for m sufficiently large. But, as in (2.19) of Fears and Mehra [4], we have

$$(2.26) \qquad \mathrm{P}\left[\sup_{0\leq t\leq\theta/R}\left|\frac{U_m(t)}{q(t)}\right|\geq\frac{\varepsilon}{2}\right]\leq\frac{\varepsilon}{2}.$$

Now (2.15) follows from (2.25) and (2.26). The rest of the proof proceeds exactly as in Theorem 2.1 of Fears and Mehra [4] with obvious modifications.

(b) *Weak convergence of $L_N(t)$*

The proof of Theorem 3.1 of Fears and Mehra [4] may be adopted verbatim to our case after replacing their Theorem 2.1 by our Theorem 2.1 and their Lemma 2.1 by our (2.15). Thus Theorem 3.1 of Fears

and Mehra [4] remains valid without the stationarity assumption and with $\phi(m)=O(m^{-2})$ and $\phi(n)=O(n^{-2})$.

(c)   *Chernoff-Savage Theorem*

Let $\mu=\int_0^1 J(t)dFH^{-1}(t)$ where J is a nonconstant function of bounded variation on $(\varepsilon, 1-\varepsilon)$ for all ε, $0<\varepsilon<1/2$, that induces the Lebesgue-Stieltjes measure ν on $(0, 1)$ and satisfies

   (i) $|J(t)|\leq K[t(1-t)]^{1/2-\tau}$ for some $K>0$ and $1/2>\tau>0$,

   (ii) $N^{-1/2}\sum_{j=1}^{N}|C_{Nj}^{*}-j\{\min[j/N, 1-1/N]\}|\leq\delta_N$, with $\delta_N=o(1)$.

Assume that $\nu_N=\nu_0+O(N^{-1/2})$ and that FH^{-1} is differentiable a.e. $|\nu|$ for sufficiently large N.   Let

$$L=(1-\lambda_0)\{\lambda_0^{-1/2}b_0 U(FH_{\lambda_0}^{-1})-(1-\lambda_0)^{-1/2}a_0 V_0(GH_{\lambda_0}^{-1})\}\,,$$

where b_0 (a_0) is the a.c. (wrt Lebesgue measure) derivative of $FH_{\lambda_0}^{-1}$ $(GH_{\lambda_0}^{-1})$.   Hence we arrive at the following result.

THEOREM 2.2.   *Assume that $\{X_m\}$ and $\{Y_n\}$ are two independent and uniform mixing sequences of random variables having absolutely contin-uous finite dimensional distributions with $\phi(m)=O(m^{-2})$ and $\phi(n)=O(n^{-2})$.   Then, under the above conditions, $N^{1/2}(T_N-\mu)$ is asymptotically normally distributed with mean 0 and variance $\sigma^2=\int_0^1\int_0^1 \mathrm{cov}\,(L(u), L(v))\cdot d\nu(u)d\nu(v)$.*

3.   Two-sample linear rank statistics for strong mixing sequences

It is clear from the development of Section 2 that a result analo-gous to Theorem 2.2 may be established for strong mixing (but not necessarily stationary) sequences.   It should be noted here that the arguments of Fears and Mehra [4] can be adopted to show the weak convergence result for a strictly stationary strong mixing sequence with mixing number $\alpha(m)$ satisfying $\sum_{m=1}^{\infty} m^2[\alpha(m)]^{1/2-\tau}<\infty$ for some $\tau\in(0, 1/2)$.   In this section we will again waive the stationarity assump-tion and assume $\alpha(m)=O(m^{-5/2-\delta})$ for some $\delta>0$.   The following theorem obtains the weak convergence of V_m to V for a strong mixing sequence of random variables, from which the asymptotic normality of two-sample linear rank statistics is established.   This is a generalization of a result of Yoshihara [9].

THEOREM 3.1.   *Let $\{X_m\}$ be a strong mixing sequence such that $\alpha(m)=O(m^{-5/2-\delta})$ for some $\delta>0$.   Then V_m converges weakly to a Gaussian process V, as defined in Theorem 2.1.*

PROOF.   As in Theorem 2.1, first we show that $U_m(t)$ converges to $U(t)$, $0 \leqq t \leqq 1$, and then show that, given $\varepsilon > 0$, there exists a $\theta \in (0, 1/2)$ and an integer $M = M(\varepsilon, \alpha, \delta, K)$ such that, for all $m \geqq M$,

$$(3.1) \qquad\qquad P\left(\sup_{0 \leqq t \leqq \theta} |V_m(t)| \geqq \varepsilon\right) \leqq \varepsilon .$$

Now, that $U_m(t)$ is asymptotically normal follows from Corollary 1(f) of Withers [7] since $\sum_{i=1}^{j} i^2 \alpha(i) = \sum_{i=1}^{j} i^2 O(i^{-5/2-\delta}) \leqq \sum_{i=1}^{j} i^2 O(i^{-2}) \leqq Kj$ for some $K > 0$.   To establish the tightness we follow the argument of Yoshihara [9].   Recall the definition of η_i in (2.3) and note that, for all $0 \leqq s \leqq t \leqq 1$,

$$(3.2) \qquad E\,|U_m(t) - U_m(s)|^4 = m^{-2}\, E\left|\sum_{j=1}^{m} \eta_j\right|^4$$

$$\leqq 24 m^{-2} \sum_{i=1}^{m} \sum{}^{*} |E\, \eta_i \eta_{i+j} \eta_{i+j+k} \eta_{i+j+k+u}| ,$$

where $\sum^{*}$ denotes the summation over all combinations of indices such that $1 \leqq i, j, k, u \leqq m$ and $j + k + u \leqq m - i$.   Using Lemma 1 of Davydov [3] without the stationarity assumption we have, for some $r > 1$,

$$(3.3) \qquad |E\, \eta_i(\eta_{i+j}\eta_{i+j+k}\eta_{i+j+k+u})| \leqq |E\, \eta_i \eta_{i+j}|$$

$$\leqq 6\alpha(j)^{1-1/r} (E\,|\eta_i|^r)^{1/r}$$

$$\leqq 6\alpha(j)^{1-1/r} (t-s)^{1/r} .$$

Similarly we have

$$(3.4) \qquad |E\, (\eta_i \eta_{i+j}\eta_{i+j+k})\eta_{j+j+k+u}| \leqq |E\, \eta_{i+j+k}\eta_{i+j+k+u}|$$

$$\leqq 6\alpha(u)^{1-1/r} (t-s)^{1/r} .$$

Finally, with repeated applications of Lemma 1 of Davydov [3], we have

$$(3.5) \qquad |E\, (\eta_i \eta_{i+j})(\eta_{i+j+k}\eta_{i+j+k+u})|$$

$$\leqq |E\, \eta_i \eta_{i+j}||E\, \eta_{i+j+k}\eta_{i+j+k+u}| + 6\alpha(k)^{1-1/r}(E\,|\eta_i \eta_{i+j}|^r)^{1/r}$$

$$\leqq 36\alpha(j)^{2/5}\alpha(u)^{2/5}(t-s)^{6/5} + 6\alpha(k)^{1-1/r}(t-s)^{1/r} .$$

Collecting terms from (3.3), (3.4), and (3.5), it follows that

$$(3.6) \quad E\,|U_m(t) - U_m(s)|^4$$

$$\leqq K m^{-2}\left\{3\sum_{i=1}^{m}\sum{}^{(1)}\alpha(k)^{1-1/r}(t-s)^{1/r} + \sum_{i=1}^{m}\sum{}^{(1)}\alpha(j)^{2/5}\alpha(u)^{2/5}(t-s)^{6/5}\right\} ,$$

where $\sum^{(1)}$ denotes the summation over all indices j, $u \leqq k$, $j + k + u \leqq m - i$, and K, here and hereafter, denotes a generic constant (not necessarily the same).   But for all m sufficiently large,

$$(3.7) \qquad m^{-2} \sum_{i=1}^{m} \sum^{(1)} \alpha(j)^{1-1/r} \leqq m^{-1} \sum_{k=1}^{m} (k+1)^2 \alpha(k)^{1-1/r}$$
$$\leqq K m^{-\rho},$$

where $\rho = (5/2+\delta)(1-1/r)-2$ with $r > (2+\delta)/\delta$ for some $\delta > 0$, as in Lemma 2 of Yoshihara [9]. Also note that, for all m sufficiently large,

$$(3.8) \qquad m^{-2} \sum_{i=1}^{m} \sum_{k=1}^{m-i} \sum_{j,u \leqq k} \alpha(j)^{2/5} \alpha(u)^{2/5} \leqq \left[\sum_{j=1}^{\infty} \alpha(j)^{2/5} \right]^2 \leqq K < \infty,$$

since $\alpha(m) = O(m^{-5/2-\delta})$, for some $\delta > 0$. Therefore, we have

$$(3.9) \qquad E |U_m(t) - U_m(s)|^4 \leqq K [m^{-\rho}(t-s)^{1/r} + (t-s)^{6/5}],$$

which agrees with (16) of Yoshihara [9] with his $\gamma = 1 - 1/r$. Thus the tightness is proved. Next, defining ζ_i^* as in (2.16) and proceeding as in Theorem 2.1 with necessary modifications, it is not difficult to see that (3.1) is satisfied. The balance of the proof remains unaffected. Theorem 3.1 is now proved.

Note that the argument of Fears and Mehra [4] applies to establish the weak convergence of the L_N process. Thus the following Chernoff-Savage theorem is obtained for strong mixing sequences. The detail proof is omitted.

THEOREM 3.2. *Assume that $\{X_m\}$ and $\{Y_n\}$ are independent and strong mixing sequences of random variables having absolutely continuous finite dimensional distributions with $\alpha(m) = O(m^{-5/2-\delta})$ and $\alpha(n) = O(n^{-5/2-\delta})$ for some $\delta > 0$. Then the conclusion of Theorem 2.2 holds.*

MEMPHIS STATE UNIVERSITY
FLORIDA STATE UNIVERSITY

REFERENCES

[1] Billingsley, P. (1968). *Convergence of Probability Measures*, Wiley and Sons, New York.
[2] Chernoff, H. and Savage, I. R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics, *Ann. Math. Statist.*, **29**, 972-994.
[3] Davydov, Y. A. (1968). Convergence of distribution generated by stationary stochastic processes, *Theory Prob. Appl.*, **12**, 691-696.
[4] Fears, T. R. and Mehra, K. L. (1974). Weak convergence of a two-sample empirical process and a Chernoff-Savage theorem for ϕ-mixing sequences, *Ann. Statist.*, **2**, 586-596.
[5] Pyke, R. and Shorack, G. (1968). Weak convergence of two-sample empirical process and a new proof to Chernoff-Savage theorems, *Ann. Math. Statist.*, **39**, 755-771.
[6] Serfling, R. J. (1968). Contributions to central limit theorem for dependent variables, *Ann. Math. Statist.*, **39**, 1158-1175.
[7] Withers, C. S. (1975). Convergence of empirical processes of mixing r.v.'s on [0, 1], *Ann. Statist.*, **3**, 1101-1108.

[8] Yoshihara, K. (1974). Extension of Billingsley's theorems on weak convergence of empirical processes, *Zeit. Wahrscheinlichkeitsth.*, **29**, 87–92.
[9] Yoshihara, K. (1976). Weak convergence of multidimensional empirical processes for strong mixing sequences of stochastic vectors, *Zeit. Wahrscheinlichkeitsth.*, **33**, 133–137.