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ON SEQUENTIAL POINT ESTIMATION OF THE MEAN OF A NORMAL DISTRIBUTION

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1. Introduction

Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent random variables from a normal distribution with mean μ and variance σ^2. For each n, define $\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i$ and $S_n = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$. Then the loss incurred in estimating μ by $\bar{X}_n$, when fixed sample size is n, is

$$L_n = a(\bar{X}_n - \mu)^2 + c^* n \tag{1.1}$$

where $a>0$ and cost $c^*>0$. The risk for (1.1) is given by

$$\nu_n(c) = \frac{a\sigma^2}{n} + c^* n \ . \tag{1.2}$$

If σ^2 is known, it turns out the integer value n^* which minimizes (1.2) is given by

$$n^* = \inf\left\{n \geqq 1 \,|\, n^2 \geqq \frac{\sigma^2}{c}\right\} , \tag{1.3}$$

where $c = c^*/a$. If σ^2 is unknown, we estimate σ^2 by estimator S_n, based on $X_1, \cdots, X_n$, that is,

$$N_1 = \inf\left\{n \geqq m \,|\, n^2 \geqq \frac{S_n}{c}\right\} , \tag{1.4}$$

where m is a positive constant integer. Then if $N_1 = n$, we estimate μ by $\bar{X}_n$. The sequential procedure given by (1.4) is due to Robbins [5]. But we note that Ray [4] and Starr [6] believed the expected value of N_1 defined by (1.4) becomes smaller than n^* in application. So Starr [6] modified (1.4) as follows:

$$N_2 = \inf\left\{n \geqq m \,|\, n^2 \geqq \frac{k_n^2 S_n}{c}\right\} , \tag{1.5}$$

where k_n is decreasing sequence and converges to one. Furthermore he investigated in some detail for the risk efficiency from asymptotic viewpoints as $c \to 0$. However he does not give how to choose k_n. After a time, Starr and Woodroofe [7], in fact, proved that $\mathrm{E}\, S_{N_1} < \sigma^2$. So we propose to use the unbiased estimator of the standard deviation σ instead of one of σ^2. That is, in (1.5) we define

$$k_n = \left\{\sqrt{\frac{n-1}{2}}\right\} \Gamma\left(\frac{n-1}{2}\right) \Big/ \Gamma\left(\frac{n}{2}\right) . \tag{1.6}$$

This sequence k_n will be shown to be decreasing and be one as $n \to \infty$, later. On the other hand, Simons [8] considered the reverse stopping variable M_1 to evaluate $\mathrm{E}\, N_1$, that is,

$$\begin{aligned} M_1 &= \sup \left\{ n \geqq m \mid n^2 < \frac{1}{c} S_n \right\} \\ &= m-1 \qquad \text{if } n^2 \geqq \frac{1}{c} S_n \text{ for all } n \geqq m . \end{aligned} \tag{1.7}$$

Then by the general theory in martingale, he obtained $\mathrm{E}\, S_{M_1} = \sigma^2$. Unfortunately we can not use the reverse stopping time M_1 as the usual stopping time. But we have $\mathrm{E}\, S_{N_1} < \sigma^2 = \mathrm{E}\, S_{M_1}$. This fact shows us that "middle point" between N_1 and M_1 will be near to n^* and the risk (1.2) will be small. Thus we propose the following stopping variable:

$$N = \inf \left\{ n \geqq m \mid (n-1)^2 \geqq \frac{S_{(n-1)}}{c},\ n^2 \geqq \frac{S_n}{c} \right\} . \tag{1.8}$$

Then when $N=n$, we estimate μ by $\bar{X}_n$. At first we remark $N_1 \leqq N$. Furthermore corresponding to (1.5) we propose the following rule:

$$N_3 = \inf \left\{ n \geqq m \mid (n-1)^2 \geqq \frac{k_{(n-1)}^2 S_{(n-1)}}{c},\ n^2 \geqq \frac{k_n^2 S_n}{c} \right\} . \tag{1.9}$$

To avoid the complexity of the sequence k_n in the procedure N_3, the properties of N are investigated and we give numerical comparison for the above four procedures.

2. Properties when c is fixed

At first we have the following:

THEOREM 2.1. *For N defined by* (1.8), *we have*

$$\mathrm{P}\,(N > n) \leqq 2\rho^n \tag{2.1}$$

where $0 < \rho < 1$.

PROOF. Since we have

$$\text{(2.2)}\qquad \mathrm{P}(N>n)\leqq \mathrm{P}\left((n-1)^2<\frac{1}{c}S_{n-1} \text{ or } n^2<\frac{1}{c}S_n\right)$$
$$\leqq 2\max\left\{\mathrm{P}\left((n-1)^2<\frac{1}{c}S_{n-1}\right),\ \mathrm{P}\left(n^2<\frac{1}{c}S_n\right)\right\}.$$

As we can express S_n as $\sigma^2\sum_{i=1}^{n-1}Z_i/(n-1)$, where Z_i $(i=1,2,\cdots,n-1)$ are independent random variables distributed according to χ^2 distribution with one degree of freedom, we have for $0<t<1/2$

$$\text{(2.3)}\qquad \mathrm{E}\left[\exp\left(t\sum_{i=1}^{n-1}\left\{Z_i-\frac{n^2c}{\sigma^2}\right\}\right)\right]=\left\{\exp\left(-\frac{n^2ct}{\sigma^2}\right)(1-2t)^{-1/2}\right\}^{(n-1)}$$
$$\geqq \mathrm{P}\left(n^2<\frac{1}{c}S_n\right).$$

Then for large n_0, we have $\exp[-n_0^2ct/\sigma^2](1-2t)^{-1/2}=\rho<1$ for all $n\geqq n_0$. By the same consideration we obtain the desired conclusion.

We remark that $\mathrm{E}\,N^l$ exists for all $l>0$ from the above theorem.

THEOREM 2.2. *We have*

$$\text{(2.4)}\qquad \mathrm{E}\,\bar{X}_N=\mu\,,\qquad \mathrm{Var}\,(\bar{X}_N)=\sigma^2\,\mathrm{E}\left(\frac{1}{N}\right)$$

and $(\sqrt{N})(\bar{X}_N-\mu)/\sigma$ *and* N *stochastically independent and the former is distributed according to a standard normal distribution.*

PROOF. This proof is based on that $\bar{X}_n$ and S_k (for all $k\leqq n$) are stochastically independent. Refer to Ray [4] and Robbins [5] in detail.

Next we evaluate $\mathrm{E}\,N^l$. We define the following reverse stopping variable corresponding to M_1 in (1.7)

$$\text{(2.5)}\qquad K=\sup\left\{k\geqq m-1\,|\,k^2<\frac{1}{c}S_k\right\}$$
$$=m-2\qquad \text{if } k^2\geqq\frac{1}{c}S_k \text{ for all } k\geqq m-1\,.$$

Then $K=(m-2)I_A+KI_{\bar{A}}$, where $A=\bigcap_{k=m-1}^{\infty}\{k^2\geqq(1/c)S_k\}$ and I_A stands for an indicator function of A. Since $N\leqq K+2$, we have

$$\text{(2.6)}\qquad \mathrm{E}\,N\leqq \mathrm{E}\,K+2\leqq(m-2)+\frac{1}{\sqrt{c}}(\mathrm{E}\sqrt{S_K})+2$$
$$\leqq m+\frac{1}{\sqrt{c}}\{\mathrm{E}\,S_K\}^{1/2}=m+\frac{\sigma}{\sqrt{c}}\,.$$

On the other hand, since $\mathrm{E}\,N_1 \leqq m+\sigma/\sqrt{c}$, nevertheless $N_1 \leqq N$, we remark the expectation of N does not increase so much. For $\mathrm{E}\,N^l$, by considering submartingale as in Nagao [3], we have the following theorem:

THEOREM 2.3. *For N defined by* (1.8), *we have*

$$\mathrm{E}\,N \leqq m+\frac{\sigma}{\sqrt{c}}, \tag{2.7}$$

$$\mathrm{E}\,N^l \leqq m^l+\sum_{i=1}^{l}\sigma^i c^{-i/2}\binom{l}{i}2^{l-i}\left[\left\{1+\frac{2(i-1)}{m-3}\right\}\cdots\left\{1+\frac{2}{m-3}\right\}\right]^{1/2}. \tag{2.8}$$

3. Properties when $c\to 0$

Taking c sufficient small, we have from (1.8)

$$\max\left(\frac{1+\sqrt{S_{N-1}/c}}{n^*}, \frac{\sqrt{S_N/c}}{n^*}\right) \leqq \frac{N}{n^*} < \frac{2+\sqrt{S_{N-2}/c}}{n^*}. \tag{3.1}$$

Since $S_N \to \sigma^2$ a.s. as $c\to 0$, we have $\lim_{c\to 0} N/n^*=1$. Since $\mathrm{E}\,[\sup_{n\geqq 2} S_n]<\infty$ (see, for example, Zacks [9], p. 561), $\lim_{c\to 0} \mathrm{E}\,N/n^*=1$. Thus we have

THEOREM 3.1. *For N defined by* (1.8), *as $c\to 0$ we have*

$$\lim_{c\to 0}\frac{N}{n^*}=1 \quad\text{and}\quad \lim_{c\to 0}\frac{\mathrm{E}\,N}{n^*}=1. \tag{3.2}$$

To obtain the limiting distribution of N, we show the following lemma.

LEMMA 3.1. *Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent random variables distributed according to chi-square distribution with one degree of freedom and let $N(c)$ be a stopping variable such that $N(c)$ is monotone increasing tending to infinity as $c\to 0$ and $n(c)$ be an increasing sequence tending to infinity as $c\to 0$, where $c>0$. If $N(c)/n(c)\to 1$ in probability, then the random variable $X_{N(c)}/\sqrt{N(c)}$ converges to zero in probability.*

PROOF. Since $N(c)/n(c)\to 1$ in probability, for any $0<\tau$, $\eta<1$ there exists $c_0>0$ such that for all $c<c_0$

$$\mathrm{P}\left(\left|\frac{N(c)}{n(c)}-1\right|>\eta\right)\leqq\tau. \tag{3.3}$$

Then we have

$$\mathrm{P}\left(\frac{X_{N(c)}}{\sqrt{N(c)}}\geqq\varepsilon\right)\leqq\mathrm{P}\left(\frac{X_{N(c)}}{\sqrt{N(c)}}\geqq\varepsilon,\ \left|\frac{N(c)}{n(c)}-1\right|\leqq\eta\right) \tag{3.4}$$

$$+\mathrm{P}\left(\frac{X_{N(c)}}{\sqrt{N(c)}}\geqq\varepsilon,\ \left|\frac{N(c)}{n(c)}-1\right|>\eta\right)$$
$$\leqq\mathrm{P}\left(\frac{X_{N(c)}}{\sqrt{N(c)}}\geqq\varepsilon,\ \left|\frac{N(c)}{n(c)}-1\right|\leqq\eta\right)+\tau$$
$$<\sum_{k=[n(c)(1-\eta)]}^{[n(c)(1+\eta)]}\mathrm{P}\left(\frac{X_k}{\sqrt{k}}\geqq\varepsilon\right)+\tau\,.$$

As the distribution of X_k is the same as that of X_1, we have

$$\mathrm{P}\left(\frac{X_{N(c)}}{\sqrt{N(c)}}\geqq\varepsilon\right)\leqq\sum_{k=[n(c)(1-\eta)]}^{[n(c)(1+\eta)]}\mathrm{P}\left(\frac{X_1}{\sqrt{k}}\geqq\varepsilon\right)+\tau\,. \tag{3.5}$$

But

$$\mathrm{P}\,(X_1\geqq\varepsilon\sqrt{k}\,)=\left(\sqrt{\frac{2}{\pi}}\right)\int_{\varepsilon^{1/2}k^{1/4}}^{\infty}\exp\left(-\frac{x^2}{2}\right)dx \tag{3.6}$$
$$\leqq\left(\sqrt{\frac{2}{\pi}}\right)\frac{1}{\varepsilon^{1/2}k^{1/4}}\exp\left(-\frac{\varepsilon\sqrt{k}}{2}\right)\,.$$

Thus as $c\to0$ we have $\lim_{c\to0}\mathrm{P}\,((X_{N(c)}/\sqrt{N(c)})\geqq\varepsilon)\leqq\tau$. Therefore we obtain the desired conclusion.

THEOREM 3.2. *As $c\to0$, the limiting distribution of $\{\sqrt{2n^*}\}(N/n^*-1)$ is a standard normal distribution.*

PROOF. By (3.1) we have

$$\max\left\{\sqrt{2n^*}\left[\frac{\sqrt{c}}{\sigma}+\left(\frac{\sqrt{S_{N-1}}}{\sigma}-1\right)\right],\ \sqrt{2n^*}\left(\frac{\sqrt{S_N}}{\sigma}-1\right)\right\} \tag{3.7}$$
$$\leqq\sqrt{2n^*}\left(\frac{N}{n^*}-1\right)<\sqrt{2n^*}\left\{\frac{2\sqrt{c}}{\sigma}+\left(\frac{\sqrt{S_{N-2}}}{\sigma}-1\right)\right\}\,.$$

Then the limiting distribution of the R.H.S. is distributed according to a standard normal distribution by Theorem 3.1 and Anscombe [1]. Thus we must show that the L.H.S. is so. Then S_n can be expressed as $(1/(n-1))\sum_{i=1}^{n-1}z_i$ $(n=2,3,\cdots)$ where $\{Z_n\}$ are independent random variables with the same distribution as σ^2 times chi-square distribution with one degree of freedom, we have

$$\sqrt{S_{N-1}}=\sqrt{\frac{N-1}{N-2}S_N}\sqrt{1-\frac{1}{N-1}\frac{Z_{N-1}}{S_N}}\,. \tag{3.8}$$

Since $(N-2)^{-1}=o_p(n^{*-1/2})$ and $1-(N-1)^{-1}Z_{N-1}/S_N=1+o_p(n^{*-1/2})$ by Lemma 3.1 we have

$$\sqrt{2n^*}\left[\frac{\sqrt{c}}{\sigma}+\left(\frac{\sqrt{S_{N-1}}}{\sigma}-1\right)\right]=\sqrt{2n^*}\left\{\frac{\sqrt{S_N}}{\sigma}-1\right\}+o_p(1)\ . \tag{3.9}$$

Therefore the theorem is proved.

As possible measures of the usefulness of this procedure, we consider the risk efficiency and the regret. By Theorem 2.2, since

$$\bar{\nu}(c)=\mathrm{E}\,L_N=\sigma^2\,\mathrm{E}\left(\frac{1}{N}\right)+c\,\mathrm{E}\,N\ , \tag{3.10}$$

the risk efficiency is given by

$$\varepsilon(c)=\frac{\bar{\nu}(c)}{\nu_{n^*}(c)}=\frac{1}{2}\left\{\mathrm{E}\left(\frac{N}{n^*}\right)+\mathrm{E}\left(\frac{n^*}{N}\right)\right\}\ . \tag{3.11}$$

Then we shall show $\lim_{c\to 0}\mathrm{E}\,(n^*/N)=1$. Since $N/n^*\to 1$ in probability as $c\to 0$, for any $\varepsilon, \eta>0$, the exists $c_0>0$ such that for all $c<c_0$

$$\mathrm{P}\left(\left|\frac{N}{n^*}-1\right|\leqq\varepsilon\right)\geqq 1-\eta\ . \tag{3.12}$$

Then we have

$$\begin{aligned}\mathrm{E}\frac{n^*}{N}&=\int_{|N/n^*-1|\leqq\varepsilon}\frac{n^*}{N}dP+\int_{|N/n^*-1|>\varepsilon}\frac{n^*}{N}dP\leqq\frac{1}{1-\varepsilon}\\&\quad+\frac{n^*}{m}\mathrm{P}\left(\frac{N}{n^*}\leqq 1-\varepsilon\right)+\frac{\eta}{1+\varepsilon}\\&\leqq\frac{1}{1-\varepsilon}+\frac{n^*}{m}\mathrm{P}\left(\frac{N_1}{n^*}\leqq 1-\varepsilon\right)+\frac{\eta}{1+\varepsilon}\ .\end{aligned} \tag{3.13}$$

Since $\mathrm{P}\,(N_1/n^*\leqq 1-\varepsilon)=o(c^{(m-1)/2})$ by a similar calculation as Simons [8], we have $\overline{\lim}\,\mathrm{E}\,n^*/N\leqq 1$ if $m\geqq 3$. On the other hand, by Fatou's lemma $\underline{\lim}\,\mathrm{E}\,n^*/N\geqq 1$. Therefore we have the following:

THEOREM 3.3. *For the stopping variable N defined by* (1.8), *the risk efficiency $\varepsilon(c)$ is asymptotically one as $c\to 0$ if $m\geqq 3$.*

Next we consider the regret $\omega(c)$ which is given by

$$\omega(c)=\mathrm{E}\,L_N-\mathrm{E}\,L_{n^*}=\sigma^2\,\mathrm{E}\,\frac{n^*-N}{n^*}+c\,\mathrm{E}\,(N-n^*)=c\,\mathrm{E}\,\frac{(N-n^*)^2}{N}\ . \tag{3.14}$$

We shall prove $\omega(c)\to 0$ as $c\to 0$. By Theorem 2.3,

$$\mathrm{E}\,\frac{(N-n^*)^2}{N}=\mathrm{E}\left\{(N-n^*)+\frac{n^*(n^*-N)}{N}\right\} \tag{3.15}$$

$$\leqq m + n^* \mathrm{E}\left|\frac{n^*}{N} - 1\right| .$$

Thus we shall show $\mathrm{E}|n^*/N-1| \to 0$ as $c \to 0$. Since $n^*/N \to 1$ a.s., $\mathrm{E}(n^*/N) < \infty$ and $\mathrm{E}(n^*/N) \to 1$ when $c \to 0$ as we show it in the proof of Theorem 3.3, n^*/N is uniformly integrable. (See, for example, Chow, Robbins and Siegmund [2], p. 4). Thus $\mathrm{E}|(n^*/N)-1| \to 0$ as $c \to 0$. Therefore we have the following:

THEOREM 3.4. *For the regret* $\omega(c)$ *defined by* (3.14) *we have*

$$\lim_{c \to 0} \omega(c) = 0 . \tag{3.16}$$

4. Numerical example

Before we present the results of Monte-Carlo experiments for comparison of four procedures, first of all we show the monotonicity of K_n defined by (1.6). Let $x=(n-1)/2$, then k_n can be expressed as $f(x) = \Gamma[x+1]/\{\sqrt{x}\,\Gamma[x+1/2]\}$. Thus we show that this function $f(x)$ is monotone decreasing for all $x>0$. Taking a logarithm for $f(x)$, we have

$$g(x) = \log f(x) = -\frac{1}{2}\log x + \log \Gamma[x+1] - \log \Gamma\left[x+\frac{1}{2}\right] . \tag{4.1}$$

Then we have

$$\begin{aligned} \frac{dg(x)}{dx} &= -\frac{1}{2x} + \sum_{n=0}^{\infty}\left(\frac{1}{x+1/2+n} - \frac{1}{x+1+n}\right) \\ &= -\frac{1}{2x} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(x+1+n)(x+1/2+n)} \\ &< -\frac{1}{2x} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(x+1+n)(x+n)} = 0 . \end{aligned} \tag{4.2}$$

Thus the function $f(x)$ is monotone decreasing. Also k_n converges to one as $n \to \infty$ by Stirling's formula.

Next we fix $\sigma^2 = 5^2$ and we obtain the following results by repeating the experiment 20,000 times in TOSBAC 5600. We remark that the expected values of S_N's for the procedures N_2 and N_3 in examples below stand for $\mathrm{E}\,k_N^2 S_N$.

Example 4.1. This example shows the change of the expectations of N's and other for two different initial values of pseudo normal random number in case $n^*=10$ and $m=4$.

Procedure	Expected value of N's	Variance of N's	Efficiency $\varepsilon(c)$	Regret $\omega(c)$	Expected value of S_N's
N	10.66 10.65	7.721 7.665	1.0503 1.0497	0.25157 0.24857	21.855 21.863
N_1	9.40 9.40	7.958 7.942	1.0703 1.0699	0.35132 0.34969	21.079 21.126
N_2	9.75 9.71	7.666 7.770	1.0604 1.0618	0.30192 0.30877	22.586 22.389
N_3	11.02 10.97	7.290 7.426	1.0457 1.0468	0.22838 0.23410	23.256 23.065

From this table it turns out that each value does not depend on an initial value of pseudo normal random number by taking suitable order so much.

Example 4.2. This example shows the change of the expectation of N's and other for one of the initial sample size m in case $n^*=20$.

Value of m	Expected value of N	Variance of N	Efficiency $\varepsilon(c)$	Regret $\omega(c)$	Expected value of S_N's
			Procedure N		
5	20.82	13.6	1.023	0.058	23.6
10	20.88	11.9	1.017	0.041	23.7
20	21.87	4.6	1.008	0.020	24.2
25	25.15	0.34	1.027	0.066	24.9
			Procedure N_1		
5	19.55	14.9	1.032	0.080	23.3
10	19.66	12.7	1.020	0.051	23.4
20	21.28	3.3	1.005	0.013	24.4
25	25.06	0.11	1.026	0.064	25.0
			Procedure N_2		
5	19.78	15.0	1.031	0.078	23.8
10	19.97	12.5	1.019	0.047	24.1
20	21.38	3.5	1.006	0.014	24.9
25	25.07	0.15	1.026	0.064	25.4
			Procedure N_3		
5	21.12	12.9	1.021	0.053	24.2
10	21.17	11.9	1.016	0.041	24.3
20	22.08	5.0	1.009	0.023	24.9
25	25.18	0.41	1.027	0.067	25.4

In this example, if m is small, the goodness of procedures shows to be invariant from viewpoint of efficiency and regret.

Example 4.3. The following example gives the comparison of four procedures.

True value	Procedure	Expected value of N's	Variance of N's	Efficiency $\varepsilon(c)$	Regret $\omega(c)$	Expected value of S_N's
	N	10.71	7.6	1.049	0.246	22.0
$n^*=10$	N_1	9.41	7.9	1.070	0.346	21.1
$m=4$	N_2	9.73	7.6	1.060	0.298	22.5
	N_3	11.00	7.4	1.046	0.229	23.2
	N	15.67	11.8	1.041	0.135	22.9
$n^*=15$	N_1	14.36	13.2	1.061	0.203	21.1
$m=4$	N_2	14.65	12.8	1.054	0.180	23.2
	N_3	15.98	11.3	1.038	0.124	23.7
	N	20.86	12.9	1.021	0.0512	23.7
$n^*=20$	N_1	19.64	14.0	1.026	0.0658	23.5
$m=6$	N_2	19.94	13.7	1.024	0.0608	24.1
	N_3	21.12	12.6	1.019	0.0484	24.2
	N	25.85	15.7	1.015	0.0307	23.9
$n^*=25$	N_1	24.67	16.7	1.020	0.0390	23.8
$m=6$	N_2	24.93	16.1	1.017	0.0346	24.2
	N_3	26.15	15.0	1.014	0.0275	24.4
	N	30.89	17.7	1.012	0.0194	24.1
$n^*=30$	N_1	29.73	18.7	1.014	0.0235	24.0
$m=6$	N_2	29.99	18.1	1.013	0.0217	24.4
	N_3	31.17	17.6	1.011	0.0187	24.5
	N	40.93	22.2	1.0076	0.0095	24.4
$n^*=40$	N_1	39.74	23.5	1.0090	0.0112	24.3
$m=6$	N_2	40.08	22.1	1.0079	0.0100	24.7
	N_3	41.16	21.9	1.0076	0.0095	24.6
	N	50.93	27.4	1.0061	0.0060	24.5
$n^*=50$	N_1	49.85	27.7	1.0063	0.0062	24.5
$m=6$	N_2	50.11	26.8	1.0061	0.0060	24.8
	N_3	51.21	26.6	1.0056	0.0056	24.7
	N	70.98	36.1	1.004	0.0027	24.6
$n^*=70$	N_1	69.86	37.2	1.004	0.0030	24.6
$m=6$	N_2	70.11	36.4	1.004	0.0028	24.8
	N_3	71.25	36.4	1.004	0.0027	24.8
	N	101.00	52.1	1.003	0.0013	24.8
$n^*=100$	N_1	99.87	52.3	1.003	0.0014	24.8
$m=6$	N_2	100.10	51.6	1.003	0.0013	24.9
	N_3	101.32	50.7	1.003	0.0013	24.9

From this table it first turns out that N_2 procedure defined by (1.5) with (1.6) is best and N_3 procedure defined by (1.9) with (1.6) is worst at the point of expected values of N's. Also we can say the procedures N_3 and N are better than other procedures from the points of efficiency and regret.

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