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FURTHER MODIFIED FORMS OF BINOMIAL AND POISSON DISTRIBUTIONS

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Summary

Some new type of modifications of binomial and Poisson distributions, are discussed. First, we consider Bernoulli trials of length n with success rate p up to time when m times of successes occur, and then, changing the success rate to γp, we continue the remaining trial. The distribution of number of successes is called the modified binomial distribution. The Poisson limit (n tends to infinity and p tends to 0, keeping $np=\lambda$) of the modified binomial is called the modified Poisson distribution. The probability functions of modified binomial and Poisson distributions are given (Section 1).

A new concept of (m, γ)-modification is introduced and fundamental theorem which gives the relations between the factorial moments of any probability function and the factorial moments of its (m, γ)-modification, is presented. Then some lower order moments of the modified binomial and Poisson distributions are given explicitly (Section 2).

The modified Poisson of $m=2$ is fitted to the distribution of number of children for Japanese women in some age group. The fitting procedure is also presented (Section 3). Some historical sketch concerning the modification and generalization of binomial and Poisson distributions is given in Appendix.

1. The modified binomial and Poisson distributions

In an earlier paper [20], we considered the following Bernoulli scheme: First, we try Bernoulli trials with success rate p up to m $(<n)$ times of sucesses occur. Next, changing the success rate p to γp $(0<\gamma<1)$, we continue the remaining trials of the total length n. The distribution $B_m(x)$ of the number of successes was called the *modified binomial distribution*. Throughout of this paper we shall refer this distribution as $MB(n, p; m, \gamma)$.

THEOREM 1 ([20]). *For $MB(n, p; m, \gamma)$, the probability function is given by*

$$(1) \qquad B_m(x)=\begin{cases} \binom{n}{x}p^x(1-p)^{n-x}, & x=0, 1, \cdots, m-1, \\ \sum_{k=m}^{n-x+m}\binom{k-1}{m-1}p^m(1-p)^{k-m}\binom{n-k}{x-m}(\gamma p)^{x-m}(1-\gamma p)^{n-x-k+m}, & \\ & x=m, m+1, \cdots, n. \end{cases}$$

The expression (1) in Theorem 1 is not convenient when one takes a Poisson limit. Then we shall give another expression for (1).

THEOREM 2. *The second expression of* (1) *is equivalent to*

$$(2) \qquad B_m(m+l)=\sum_{k=l}^{n-m} B(m+k; n, p)B(l; k, \gamma),$$

where

$$B(k; n, p)=\binom{n}{k}p^k(1-p)^{n-k}.$$

PROOF. We first note that

$$B(l; n-k, \gamma p)=\sum_{h=l}^{n-k} B(h; n-k, p)B(l; h, \gamma).$$

Then we have

$$\begin{aligned} B_m(m+l) &= \sum_{k=m}^{n-l}\binom{k-1}{m-1}p^m(1-p)^{k-m}B(l; n-k, \gamma p) \\ &= \sum_{k=l}^{n-l}\sum_{h=l}^{n-k}\binom{k-1}{m-1}p^m(1-p)^{k-m}B(h; n-k, p)B(l; h, \gamma) \\ &= \sum_{h=l}^{n-m} B(l; h, \gamma)\sum_{k=m}^{n-h}\binom{k-1}{m-1}p^m(1-p)^{k-m}B(h; n-k, p). \end{aligned}$$

It is easily seen that

$$\sum_{k=m}^{n-h}\binom{k-1}{m-1}p^m(1-p)^{k-m}B(h; n-k, p)=B(m+h; n, p),$$

which completes the proof.

Next, we shall consider the limiting form of $MB(n, p; m, \gamma)$ when

$$(3) \qquad n\to\infty \quad \text{and} \quad p\to 0 \qquad \text{keeping } np=\lambda.$$

We call this the *modified Poisson distribution* and refer to it as $MP(\lambda; m, \gamma)$. It is well known that when the limit (3) is taken

$$B(k; n, p)\to P(k; \lambda),$$

where

$$P(k;\lambda)=e^{-\lambda}\lambda^k/k!\ .$$

Then as the direct consequence of Theorem 2 we obtain

THEOREM 3. *For $MP(\lambda; m, r)$, the probability function is given by*

$$(4)\qquad P_m(x)=\begin{cases} P(x;\lambda)\ , & (0\leqq x\leqq m-1)\ ,\\ \sum_{k=x-m}^{\infty} P(m+k;\lambda)B(x-m;k,\gamma)\ , & (x\geqq m)\ .\end{cases}$$

We note that the following expression is more convenient to calculate the probability function than (4).

$$(5)\qquad P_m(m+l)=P(m+l;\lambda)\gamma^l\frac{(m+l)!}{l!}\sum_{i=0}^{\infty}\frac{(l+i)!}{(m+l+i)!}\frac{[(1-\gamma)\lambda]^i}{i!}\ .$$

We further note that $MP(\lambda; m, \gamma)$ can be regarded as the Poisson point process such that the intensity function is λ by the time of the mth occurence of event and then the intensity is reduced to $\gamma\lambda$. This is a special case of self-exciting point processes, see for example Snyder [19].

For special references, we shall show the first 6 terms in the case of $m=2$ (putting $\alpha=(1-\gamma)\lambda$):

$$P_2(0)=e^{-\lambda}$$

$$P_2(1)=\lambda e^{-\lambda}$$

$$P_2(2)=\frac{e^{-\lambda}}{(1-\gamma)^2}[e^{\alpha}-(1+\alpha)]$$

$$P_2(3)=\frac{\gamma e^{-\lambda}}{(1-\gamma)^3}[(-2+\alpha)e^{\alpha}+(2+\alpha)]$$

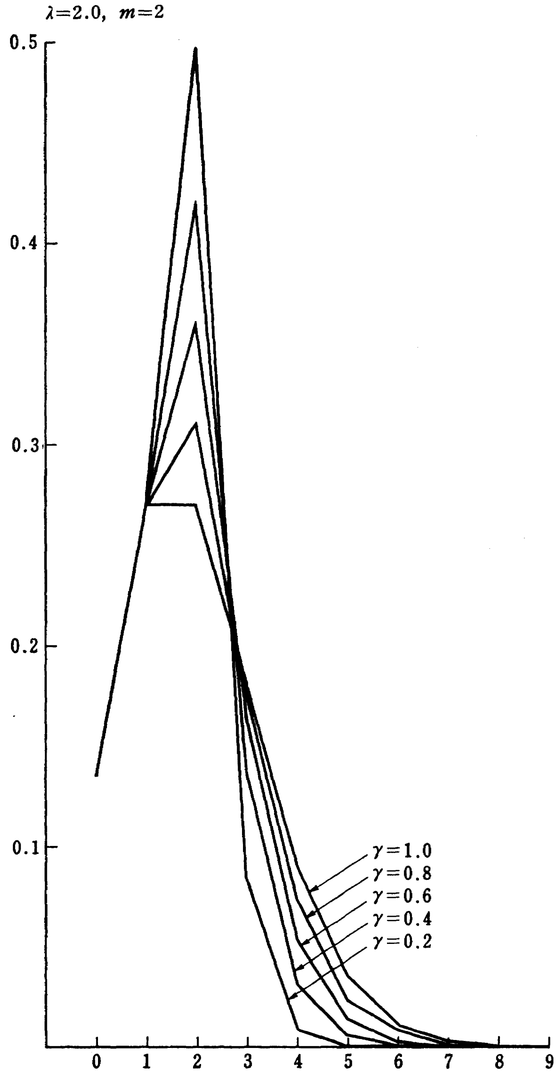
$$P_2(4)=\frac{\gamma^2 e^{-\lambda}}{2(1-\gamma)^4}[(6-4\alpha+\alpha^2)e^{\alpha}-(6+2\alpha)]$$

$$P_2(5)=\frac{\gamma^3 e^{-\lambda}}{6(1-\gamma)^5}[(-24+18\alpha-6\alpha^2+\alpha^3)e^{\alpha}+(24+6\alpha)]\ .$$

Some graphical representation of the probability functions of modified Poisson for some typical parameter values is given in Fig. 1–Fig. 3.

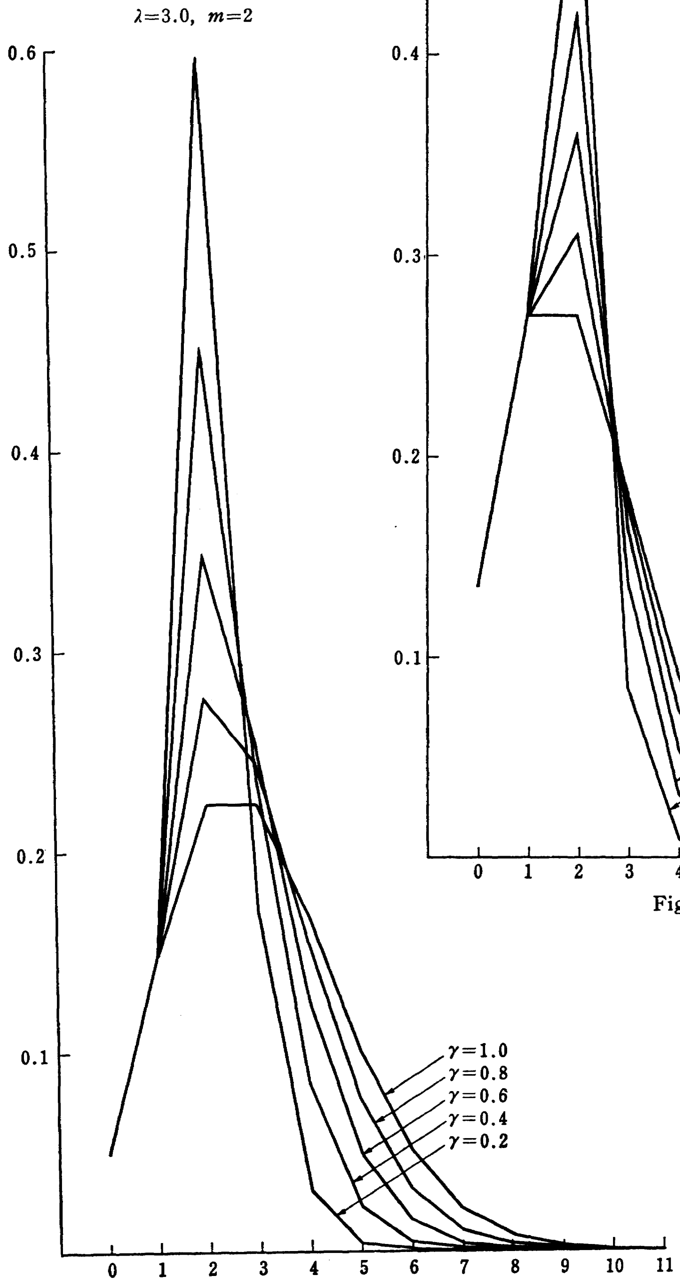
2. The (m, γ)-modification and the relation between moments

One can directly obtain the mean and the variance of modified Binomial or modified Poisson distribution from their probability function. But we generalize our modification and give some formulas on factorial moments for this general case.



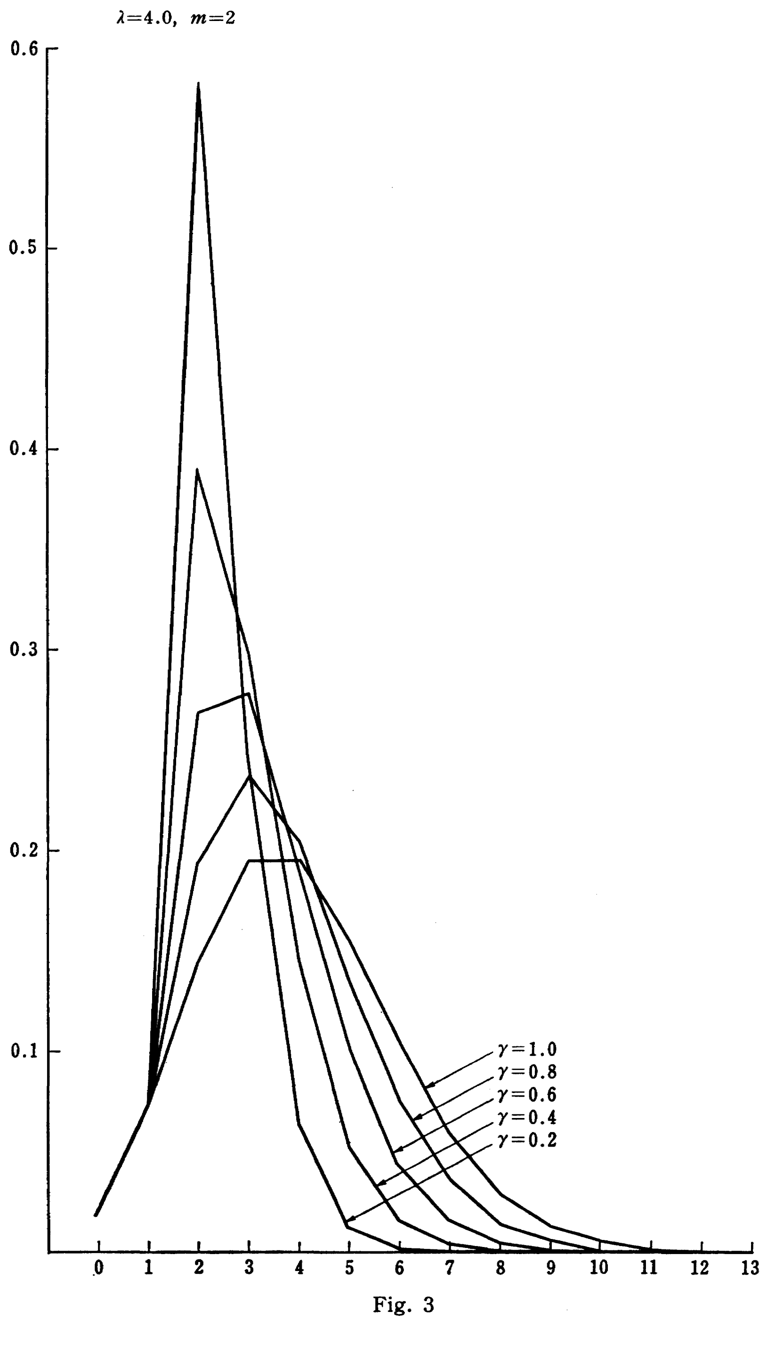
λ=2.0, m=2
0.5
0.4
0.3
0.2
0.1
γ=1.0
γ=0.8
γ=0.6
γ=0.4
γ=0.2
0 1 2 3 4 5 6 7 8 9

Fig. 1



λ=3.0, m=2
0.6
0.5
0.4
0.3
0.2
0.1
γ=1.0
γ=0.8
γ=0.6
γ=0.4
γ=0.2
0 1 2 3 4 5 6 7 8 9 10 11

Fig. 2



λ=4.0, m=2
0.6
0.5
0.4
0.3
0.2
0.1
γ=1.0
γ=0.8
γ=0.6
γ=0.4
γ=0.2
0
1
2
3
4
5
6
7
8
9
10
11
12
13

Fig. 3

Let $H=\{h(x, y)\}$ be a stochastic matrix of finite or denumerable dimension such that

$$\sum_{y\geqq 0} h(x, y)=1 .$$

Then for any probability function $f(x)$ of $x\geqq 0$

$$g(y)=\sum_{x\geqq 0} h(x, y)f(x)$$

becomes also a probability function. g may be called H-modification of f. Especially we put

$$H_{m,\gamma}^{(n)}=\left[\begin{array}{cccc|ccccc}
1 & 0 & \cdots & 0 & & & & & \\
0 & 1 & \cdots & 0 & & & 0 & & \\
\vdots & & \ddots & \vdots & & & & & \\
0 & 0 & \cdots & 1 & & & & & \\
\hline
 & & & & 1 & 1-\gamma & (1-\gamma)^2 & \cdots & (1-\gamma)^{n-m} \\
 & & & & & \gamma & 2\gamma(1-\gamma) & \cdots & \binom{n-m}{1}\gamma(1-\gamma)^{n-m-1} \\
 & & 0 & & & & \gamma^2 & \cdots & \binom{n-m}{2}\gamma^2(1-\gamma)^{n-m-2} \\
 & & & & & & & \ddots & \vdots \\
 & & & & & 0 & & & \gamma^{n-m}
\end{array}\right] .$$

Then it is easily seen that our modified binomial and Poisson distributions are $H_{m,\gamma}^{(n)}$-modification and $H_{m,\gamma}^{(\infty)}$-modification of the ordinary binomial and Poisson distributions respectively. We call this type of modification as *(m, γ)-modification.*

Let $\tilde{f}$ be the (m, γ)-modification of f and we write

$$\mu_{(r)}^{(m)}=\sum_{l\geqq r} l^{(r)}f(m+l) , \qquad \tilde{\mu}_{(r)}^{(m)}=\sum_{l\geqq r} l^{(r)}\tilde{f}(m+l)$$

where

$$l^{(r)}=l(l-1)\cdots(l-r+1) .$$

Then we have

LEMMA 1.

$$\tilde{\mu}_{(r)}^{(m)}=\gamma^r \mu_{(r)}^{(m)} .$$

PROOF. Noting that

$$\sum_{l=r}^{h} l^{(r)}\binom{h}{l}\gamma^l(1-\gamma)^{h-l}=h^{(r)}\gamma^r$$

we have

$$\mu_{(r)}^{(m)}=\sum_{l\geqq r} l^{(r)}\sum_{h\geqq l} f(m+h)\binom{h}{l}\gamma^l(1-\gamma)^{h-l}$$
$$=\sum_{h\geqq r} f(m+h)\sum_{l=r}^{h} l^{(r)}\binom{h}{l}\gamma^l(1-\gamma)^{h-l}$$
$$=\gamma^r\sum_{h\geqq r} h^{(r)}f(m+h)\ .$$

Next, we note that $\mu_{(r)}^{(m)}$ is the rth factorial moment about m. In general, there is the following relation between $\mu_{(r)}^{(m)}$ and the ordinary factorial moment

$$\mu_{(r)}=\sum_{l\geqq r} l^{(r)}f(l)\ .$$

LEMMA 2.

$$\mu_{(r)}^{(m)}=\sum_{i=0}^{r-1}\binom{m-1+i}{m-1}\binom{r}{i}(-1)^i i!\mu_{(r-i)}+(-1)^r r!\pi_r(m)\ ,$$

where

$$\pi_r(m)=\binom{m-1+r}{m-1}-\sum_{j=0}^{m-1}\binom{m-1-j+r}{m-1-j}f(j)\ .$$

PROOF. Noting that

$$\binom{x-m}{r}=(-1)^r\binom{m-x+r-1}{r}\ ,\qquad \text{for } x<m$$

we have

$$\mu_{(r)}^{(m)}=\sum_{x\geqq 0} r!\binom{x-m}{r}f(x)-(-1)^r r!\sum_{x=0}^{m-1}\binom{m-x-r-1}{r}f(x)\ .$$

Furthermore using the relation

$$\sum_{i=0}^{r}(-1)^i\binom{m-1+i}{i}\binom{x}{r-i}=\binom{x-m}{r}$$

we can obtain

$$\mu_{(r)}^{(m)}=\sum_{i=0}^{r}\binom{m-1+i}{i}(-1)^i\sum_{x\geqq 0}\binom{x}{r-i}f(x)-(-1)^r r!\sum_{x=0}^{m-1}\binom{m-x+r-1}{r}f(x)$$
$$=\sum_{i=1}^{r-1}\binom{m-1+i}{i}\binom{r}{i}(-1)^i i!\mu(r-i)$$
$$+(-1)^r r!\left\{\binom{m-1+r}{r}-\sum_{x=0}^{m-1}\binom{m-x+r-1}{r}f(x)\right\}\ .$$

LEMMA 3.

$$\tilde{\mu}_{(r)}=\mu_{(r)}+\sum_{h=0}^{r}\binom{m}{r-h}\binom{r}{h}(r-h)!(\gamma^h-1)\mu_{(h)}^{(m)}\ .$$

PROOF. Noting that

$$\binom{m+l}{r}=\sum_{h=0}^{r}\binom{m}{r-h}\binom{l}{h},$$

we have

$$\begin{aligned}\frac{\tilde{\mu}_{(r)}-\mu_{(r)}}{r!}&=\sum_{l\geq r}\binom{m+l}{r}[\tilde{f}(m+l)-f(m+l)]\\&=\sum_{h=0}^{r}\binom{m}{r-h}\sum_{l\geq h}\binom{l}{h}[\tilde{f}(m+l)-f(m+l)]\\&=\sum_{h=0}^{r}\binom{m}{r-h}\frac{\tilde{\mu}_{(h)}^{(m)}-\mu_{(h)}^{(m)}}{h!}.\end{aligned}$$

Using Lemma 1, the proof is completed.

Combining Lemma 2 and Lemma 3 and changing the order of summations, we obtain

THEOREM 4.

$$\frac{\tilde{\mu}_{(r)}-\mu_{(r)}}{r!}=\sum_{j=1}^{r}\mathrm{C}(r,j)\frac{\mu_{(j)}}{j!}+\sum_{j=1}^{r}(-1)^{j}\binom{m}{r-j}(\gamma^{j}-1)\pi_{j}(m),$$

where

$$\mathrm{C}(r,j)=\sum_{i=1}^{r-j}(-1)^{i}\binom{m}{r-j-i}\binom{m-1+i}{i}(\gamma^{j+i}-1).$$

Especially, we shall show the results of lower order case:

$$\tilde{\mu}_{(1)}=\gamma\mu_{(1)}+(1-\gamma)\pi_{1}(m)$$

$$\tilde{\mu}_{(2)}=2m\gamma(1-\gamma)\mu_{(1)}+\gamma^{2}\mu_{(2)}+2m(1-\gamma)\pi_{1}(m)-2(1-\gamma^{2})\pi_{2}(m).$$

The following theorems are the direct consequence of the above relations.

THEOREM 5. *The mean and the variance of $MB(n, p; m, \gamma)$ are given by*

$$\mu=\gamma np+(1-\gamma)\pi_{1}(m), \tag{6}$$

$$\sigma^{2}=\gamma np(1-\gamma p)+2\gamma(1-\gamma)mnp-(1-\gamma)(2\gamma np-2m-1)\pi_{1}(m)-[(1-\gamma)\pi_{1}(m)]^{2}-2(1-\gamma^{2})\pi_{2}(m). \tag{7}$$

THEOREM 6. *The mean and the variance of $MP(\lambda; m, \gamma)$ are given by*

$$\mu=\gamma\lambda+(1-\gamma)\pi_{1}(m), \tag{8}$$

$$\sigma^{2}=\gamma\lambda+2\gamma(1-\gamma)m\lambda-(1-\gamma)(2\gamma\lambda-2m-1)\pi_{1}(m) \tag{9}$$

$$-[(1-\gamma)\pi_1(m)]^2-2(1-\gamma^2)\pi_2(m) .$$

Note that (8) and (9) are the limits (3) of (6) and (7) respectively.

One may show other statistical properties of (m, γ)-modification. But we shall only show a result on probability generating function.

THEOREM 7. *Let $\tilde{f}$ be the (m, γ)-modification of f and we write*

$$A(t)=\sum_x t^x f(x) , \qquad \tilde{A}(t)=\sum_x t^x \tilde{f}(x) .$$

Then

(i) $\tilde{A}(t)=A(t)-t^m B(t)$,

(ii) $\tilde{A}^{(r)}(t)=A^{(r)}(t)-\sum_{j=0}^{r}\binom{r}{j}\binom{m}{r-j}(r-j)!t^{m-r+j}B^{(j)}(t)$,

where

$$B(t)=\sum_{\nu\geq 1} f(m+\nu)[t^\nu-(1-\gamma+\gamma t)^\nu]$$

and $F^{(r)}(t)$ denotes the rth derivative of $F(t)$ with respect to t.

PROOF. (i)

$$\begin{aligned}\tilde{A}(t)&=\sum_{x\geq 0} t^x \tilde{f}(x)\\&=\sum_{x=0}^{m-1} t^x f(x)+\sum_{x\geq m}\sum_{h\geq x-m}\binom{h}{x-m}\gamma^{x-m}(1-\gamma)^{h-x+m}t^x f(x)\\&=\sum_{x=0}^{m-1} t^x f(x)+t^m\sum_{x\geq m}(1-\gamma+\gamma t)^{x-m}f(x)\\&=A(t)-t^m B(t) .\end{aligned}$$

(ii) It is the direct consequence of (i).

Since it is easily seen that

$$\mu_{(r)}=A^{(r)}(1) , \qquad \tilde{\mu}_{(r)}=\tilde{A}^{(r)}(1) , \qquad B^{(j)}(1)=(1-\gamma^j)\mu_{(j)}^{(m)} ,$$

then we have an alternative proof of Lemma 3.

3. Applications

In Fig. 4, the percentages of Japanese women in the age group less than 50 according to the number of children born to them, are presented (9355 samples are surveyed at 1st June, 1972. Source: Inst. Population Problems [12]). The dotted line is the probability function $MP(2.24; 2, 0.522)$. It will be seen that the fit is satisfactory. In this case the value of $1-\gamma$ is interpreted as the birth-control factor after having two children.

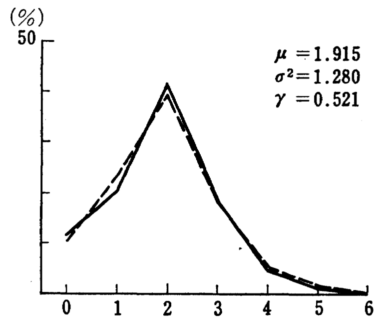


Fig. 4

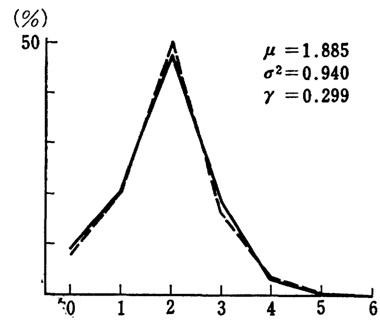


Fig. 5

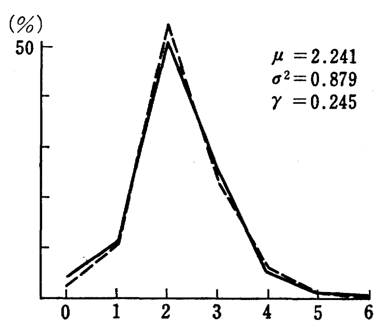


Fig. 6

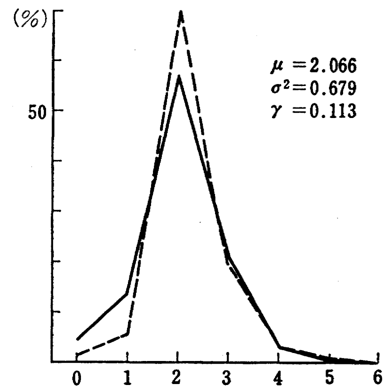


Fig. 7

In the following we shall show the fitting procedure. Omitting λ from the relation (8) and (9), we have

$$A(1-\gamma)^2-2B(1-\gamma)+C=0 ,$$

where

$$A=2\pi_2-4\pi_1+\pi_1^2 , \qquad B=2\pi_2-(2-\mu)\pi_1-2\mu , \qquad C=\mu-\sigma^2 .$$

Solving this equation we have

$$\gamma=D\pm\sqrt{(1-D)^2-E} ,$$

where

$$D=1-\frac{B}{A}=\frac{(\mu-\pi_1)(2-\pi_1)}{(2-\pi_1)^2+2(\pi_2-2)} , \qquad E=\frac{C}{A}=\frac{\mu-\sigma^2}{(2-\pi_1)^2+2(\pi_2-2)} .$$

Furthermore from (8) we have

$$\lambda=(\mu-(1-\gamma)\pi_1)/\gamma .$$

The data of Fig. 4 is the following:

Number of children	0	1	2	3	4	5	6
Percentage	.118	.204	.419	.189	.050	.014	.006
Modified Poisson	.106	.238	.395	.183	.059	.015	.004

Then one can estimate

$$\hat{\mu}=1.915\ , \qquad \hat{\sigma}^2=1.280\ ,$$

$$\hat{\pi}_1=2-2\hat{P}_2(0)-\hat{P}_2(1)=1.560\ ,$$

$$\hat{\pi}_2=3-3\hat{P}_2(0)-\hat{P}_2(1)=2.442\ .$$

Consequently one obtains

$$\hat{\gamma}=0.145+\sqrt{0.142}=0.522\ , \qquad \hat{\lambda}=1.17/0.522=2.24\ .$$

In Fig. 5, we also give the same type data (8598 Japanese women) surveyed at 1st June, 1977 (Source [13]). The percentage of women having two children increased since 5 years ago. This corresponds to the fact that the birth-control factor is strengthened as $\gamma=0.299$.

In Fig. 6, the distribution of 2759 women in the age group 40–49 of the data of 1977. In this case $\gamma=0.245$. Fig. 7 is the distribution of 3411 women in the age group 30–39. The fit is rather poor because there are some couples who intend to have further children.

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APPENDIX

Various modified and generalized forms of binomial and Poisson distributions

We first note that the adjective "modified" and "generalized" have often been used in a restricted kind of modification and generalization. When the original distribution is

$$\Pr\{X=x\}=P(x) \qquad (x=0, 1, 2, \cdots)$$

then the so-called *modified* distribution is given by

$$P'(0)=\theta+(1-\theta)P(0)$$

$$P'(x)=(1-\theta)P(x) \qquad (x=1,2,\cdots)$$

with $0<\theta<1$. We do not list up the works of this truncation type modification, because the book by Johnson and Kotz [15] should serve as a source of references. If $g_1(t)$ and $g_2(t)$ are the probability generating functions of F_1 and F_2 respectively, then the distribution having the probability generating function $g_1(g_2(t))$ is called *generalized* F_1 distribution. And generalizing distribution F_2 is also called "generalizer". Almost all cases, *generalized* distributions are equivalent to the some type of compound distributions (see Gurland [11]).

Woodbury [22] considered a general Bernoulli scheme in which the probability of a success depends on the number of previous successes. Let p_x be the probability of success after x previous success and denote by $P(n, x)$ the probability of x successes in n trials. Then one can formulate the following equation:

$$P(n+1, x+1)=p_xP(n, x)+(1-p_{x+1})P(n, x+1) .$$

When no pair of p_x's are equal one can also obtain

$$P(n, x)=\prod_{i=0}^{x-1} p_i \sum_{j=0}^{i} (1-p_j)^n \Big/ \prod_{i\neq j} (p_i-p_j) .$$

The general form of this distribution is rather complicated.

Rutherford [17] considered the special case where p_x's are determined by two parameters through the relation

$$p_x=p+cx \qquad (c>0) .$$

In this case one has

$$P(n, x)=\frac{\Gamma(p/c+x)}{\Gamma(x+1)\Gamma(p/c+1)} \sum_{j=0}^{x} (-1)^j \binom{x}{j} (1-p-cj)^n .$$

Chaddha [3] treated two different cases

$$p_x=\frac{cx+p}{cx+1} \qquad (c>0) ,$$

$$p_x=\frac{p}{cx+1} \qquad (c>0) ,$$

where c is called the coefficient of contagion. Some graphical representations of these probabilities are presented.

Our modified binomial distribution $MB(n, p; m, \gamma)$ is also the two-parameter special case such as

$$p_x=\begin{cases} p & x\leqq m-1 \\ \gamma p & x\geqq m\,. \end{cases}$$

The Woodbury's model can be further extended to the case in which the probability of a success depends on both the numbers of previous successes and trials, say $p_{n,x}$. Then we also have the recurrence relation:

$$P(n+1,\,x+1)=p_{n,x}P(n,\,x)+(1-p_{n,x+1})P(n,\,x+1)\,.$$

But this general scheme is less interesting because too many parameters are included. We only note the two-parameter special case such as

$$p_{n+1,x}=\frac{a+cx}{a+b+cn}\,.$$

In this case

$$P(n,\,x)=\binom{n}{x}\frac{\prod_{j=1}^{x}(a+(j-1)c)\prod_{j=1}^{n-k}(b+(j-1)c)}{\prod_{j=1}^{n}(a+b+(j-1)c)}\,.$$

This is the so-called Pólya-Eggenberger distribution, which was initially considered by Greenwood and Yule [10] and shortly afterwards was independently rediscovered by Eggenberger and Pólya [7].

Other modification of Binomial scheme was treated by Schelling [8] and Dandekar [6]. They considered a situation in which if a trial actually resulted in a success, then the play is interrupted for m trials. The probability of exactly x successes in n trials is given by

$$P(n,\,x)=Cq^{n-1}\binom{(n-1)-m(x-1)}{x-1}\left(\frac{p}{q^{m+1}}\right)^{x-1},$$
$$x=1,\,2,\cdots,\,\left[\frac{n-1}{m+1}\right]+1\,,$$

where C is the normalizing constant. The moment expressions of this distribution are rather complicated. But for large n the following approximations are available:

$$\mu=\frac{np}{1+mp}\,,\qquad \sigma^2=\frac{npq}{(1+mp)^2}\,.$$

Then using the relation

$$\frac{\mu^2}{\sigma^2}=\frac{np}{1-p}$$

one can estimate the parameter p (or $\lambda=np$). Another parameter m could be estimated from the relation

$$\frac{m}{n}=\frac{1}{\mu}-(1-p)\frac{\sigma^2}{\mu^2} .$$

Consul [4] considered an urn model depending on predetermined strategy and obtained the other type of two-parameter binomial distribution (named by quasi-binomial) such as

$$P(n, x)=\binom{n}{x}p(p+\phi x)^{x-1}(1-p-\phi x)^{n-x} .$$

Especially, putting

$$\phi=\frac{1}{n}, \qquad p=\varepsilon ,$$

then

$$\sum_{x=0}^{[n(1-\varepsilon)]} P(n, x)$$

is reduced to Birnbaum and Tingey's [2] expression for

$$\Pr\{\sup_t \{\hat{F}_n(t)-F(t)\}>\varepsilon\} ,$$

where $\hat{F}_n(t)$ is the empirical distribution function of the theoretical distribution function $F(t)$.

The first and most important modification of Poisson distribution is given by "mixtures" of several Poisson distributions. The resultant distribution is called *compound* distribution and "mixing" distribution is called *compounder*.

The compound Poisson distribution compounded by the gamma distribution is given by

$$PG(x)=\frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty (e^{-\lambda}\lambda^x/x!)\lambda^{\alpha-1}e^{-\lambda/\beta}d\lambda=\frac{\Gamma(\alpha+x)}{x!\Gamma(\alpha)}\left(\frac{\beta}{\beta+1}\right)^x\left(\frac{1}{\beta+1}\right)^\alpha .$$

This (so-called negative binomial distribution) was initially treated by Greenwood and Yule [10].

As the compounder one takes the Poisson distribution the resultant distribution is

$$PP(x)=\begin{cases} (e^{-\lambda}\phi^x/x!)\sum_{j=1}^{\infty}(\lambda e^{-\phi})^j j^x/j! & (x\geqq 1) \\ \exp[-\lambda(1-e^{-\phi})] & (x=0) . \end{cases}$$

This is the Neyman's [16] type A distribution. Since the probability generating function is expressed by

$$\exp[-\lambda(1-e^{-\phi(1-t)})],$$

this is also the *generalized* Poisson with Poisson generalizer, see Feller [8].

Adelson [1] considered the "quasi-compound" Poisson distribution, namely the distribution of the sum of m independent random variables, $X_1, X_2, \cdots, X_m$, with

$$\Pr\{X_j=jk\}=e^{-\lambda_j}\lambda_j^k/k!\ .$$

Since the probability generating function of this distribution is given by

$$\exp\left[-\sum_{j=1}^{m}\lambda_j(1-t^j)\right]=\prod_{j=1}^{\infty}\exp[-\lambda_j(1-t^j)],$$

this is not an usual type mixture. But using the recurrence relation

$$P(x+1)=\frac{1}{x+1}\sum_{i=1}^{x(m)}(i+1)\lambda_{i+1}P(x-i) \qquad (x(m)=\min(x, m))$$

$$P(0)=\exp\left[-\sum_{i=0}^{m}\lambda_i\right]$$

given by Adelson, we can avoid some computational difficulties in the use of this type of distributions.

The two-parameter special case such as

$$\lambda_j=(1-\rho)\rho^{j-1}\lambda\ , \qquad j=1, 2, \cdots$$

was initially considered by Galliher et al. [9], using the term stuttering Poisson. We would like to apply the term "stuttering Poisson" to the class of distributions defined by Adelson.

There is an important distribution which is both the *compound* and is the *generalized* one and is similar to the Neyman Type A distribution. Thomas [21] constructed a model for the distribution of numbers of plants of a given species in randomly placed quadrats. She showed that the probability of x plants in any one quadrat is given by

$$TP(0)=e^{-\lambda},$$

$$TP(x)=\frac{e^{-\lambda}}{x!}\sum_{k=1}^{k}\binom{x}{k}\lambda^k(k\phi)^{x-k}e^{-k\phi} \qquad (x\geqq 1)\ .$$

This distribution was ordinary called the *double Poisson distribution* by Thomas but is now referred as *Thomas distribution*.

Further modified form of two-parameter Poisson was given by Consul and Jain [5] as the limiting version of a generalized negative binomial distribution considered by Jain and Consul [14]. The probability function is given by

$$CP(x)=\lambda(\lambda+x\gamma)^{x-1}e^{-(\lambda+x\gamma)}/x! \qquad x=0, 1, 2, \cdots$$

and the moments are

$$\mu=\frac{\lambda}{1-\gamma}, \qquad \sigma^2=\frac{\lambda}{(1-\gamma)^3}.$$

This is also some limiting form of the urn model's two-parameter quasi-binomial distribution give by Consul [4].

Dandekar [6] obtained another type of modification which is the Poisson limit of Schelling's [18] model. (He did not refer [18].) The cumulative probability function of Dandekar is

$$DF(x)=e^{-(1-\gamma x)\lambda}\sum_{j=0}^{x}\frac{[(1-\gamma x)\lambda]^j}{j!},$$

namely the first $(x+1)$ terms in the Poisson series with the mean $(1-\gamma x)\lambda$. Dandekar gave three examples in which this modified Poisson distribution gives a satisfactory fit to the observed data. He notes that in all three cases, the parameter γ has a negative value. (The value of $\gamma=m/n$ is the interruption rate in Schelling's model.)

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