Ann. Inst. Statist. Math.
32 (1980), Part A, 107-110

CHARACTERIZATION OF CERTAIN BALANCED n-ARY BLOCK DESIGNS

SANPEI KAGEYAMA

(Received Aug. 31, 1979)

Summary

A balanced n-ary block design with $RK=\Lambda V$ of Tocher [4] is completely characterized. It is noted that $B\geqq V$ holds for a *non-trivial* balanced n-ary block design, where B is the number of blocks and V, the number of treatments.

1. Introduction

Following Tocher [4], a balanced n-ary block design N $(=\|n_{ij}\|$ in its $V\times B$ incidence matrix) with parameters, V, B, R, K and Λ is an arrangement of V treatments in B blocks being of size K $\left(=\sum_{i=1}^{V} n_{ij}\text{, all } j\right)$ such that (i) the ith treatment occurs in the jth block n_{ij} times, (ii) each treatment occurs R $\left(=\sum_{j=1}^{B} n_{ij}\text{, all } i\right)$ times, and (iii) $\sum_{j=1}^{B} n_{ij}n_{i'j}=\Lambda$ for all i, i' $(i\neq i')=1, 2, \cdots, V$, where n_{ij} can take any of the values, $0, 1, \cdots$, or $n-1$. When $n=2$, the design is called binary. Otherwise, the design is said to be non-binary.

Fisher [1] proved that the inequality $b\geqq v$ holds for a balanced incomplete block (BIB) design, where b is the number of blocks and v, the number of treatments. It is well known that this inequality is still valid for various block designs. The purpose of this note is to characterize a balanced n-ary block design with $RK=\Lambda V$ and to note that the inequality $B\geqq V$ holds for any "non-trivial" balanced n-ary block design.

2. Characterization

Let N $(=\|n_{ij}\|)$ be a $V\times B$ incidence matrix of a balanced n-ary block design with parameters V, B, R, K and Λ. In this case, we get

$$(1)\qquad NN'=\begin{bmatrix} r_1 & & & \Lambda \\ & r_2 & & \\ & & \ddots & \\ \Lambda & & & r_V \end{bmatrix},$$

where $r_i=\sum_{j=1}^{B} n_{ij}^2$ for $i=1, 2,\cdots, V$. Since

$$\sum_{j=1}^{B} n_{ij}^2+\Lambda(V-1)=RK \qquad \text{for all } i\,,$$

holds in a balanced n-ary block design, the relation (1) becomes

$$(2)\qquad NN'=\begin{bmatrix} RK-\Lambda V+\Lambda & & \Lambda \\ & \ddots & \\ \Lambda & & RK-\Lambda V+\Lambda \end{bmatrix},$$

the eigenvalues of which can easily be shown to be RK and $RK-\Lambda V$ with multiplicities 1 and $V-1$, respectively. Since NN' is a positive semi-definite matrix, it is obvious that $RK-\Lambda V\geqq 0$. If $RK-\Lambda V>0$, then $V=\text{rank}\,(NN')=\text{rank}\,(N)\leqq B$, i.e., $B\geqq V$ which was derived first by Fisher [1] for a BIB design and is then called Fisher's inequality. The remaining case $RK-\Lambda V=0$ is now investigated for the general validity of Fisher's inequality in a balanced n-ary block design. A characterization of this case may be important as a property of a balanced n-ary block design and is the main purpose of this note.

When $RK-\Lambda V=0$, we have, from (2),

$$(3)\qquad NN'=\Lambda J_V\,,$$

where $J_V=J_{V\times V}$ and $J_{s\times t}$ is an $s\times t$ matrix all of whose elements are unity. Relation (3) yields that

$$(4)\qquad \sum_{j=1}^{B} n_{ij}^2=\Lambda\,, \qquad i=1, 2,\cdots, V;$$

$$(5)\qquad \sum_{j=1}^{B} n_{ij}n_{i'j}=\Lambda\,, \qquad i, i'\,(i\neq i')=1, 2,\cdots, V\,.$$

Case I: $n=2$ (binary). In this case, $n_{ij}=0$ or 1 which, from (4), implies that $R=\Lambda$ and then, from $RK=\Lambda V$, we have $V=K$ which, from (3), yields that $N=J_{V\times B}$ which is an incidence matrix of a complete block design.

Case II: $n\geqq 3$ (non-binary). Relations (4) and (5) yield, by Cauchy-Schwarz's inequality, the following

$$\Lambda^2=\Big(\sum_{j=1}^{B} n_{ij}n_{i'j}\Big)^2\leqq\Big(\sum_{j=1}^{B} n_{ij}^2\Big)\Big(\sum_{j=1}^{B} n_{i'j}^2\Big)=\Lambda^2\,.$$

Since the equality holds in Cauchy-Schwarz's inequality, we have, for some positive integer m,

(6) $\quad mn_{ij}=n_{i'j}\,, \qquad i\neq i',\ i, i'=1, 2,\cdots, V;\ j=1, 2,\cdots, B\,.$

Furthermore, (4), (5) and (6) yield $m^2\Lambda=\Lambda$ and so $m=1$ which implies that

$$n_{ij}=n_{i'j}\,, \qquad i, i'\,(i\neq i')=1, 2,\cdots, V;\ j=1, 2,\cdots, B\,.$$

Hence, we can write N as

$$N=\begin{bmatrix} \alpha_1 & \alpha_2 & & \alpha_B \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_1 & \alpha_2 & & \alpha_B \end{bmatrix} \tag{7}$$

for some integers $\alpha_1, \alpha_2,\cdots, \alpha_B \in \{0, 1,\cdots, n-1\}$. Furthermore, since the block size is constant (i.e., $\sum_{i=1}^{V} n_{ij}=K$ for all $j=1, 2,\cdots, B$) in a balanced n-ary block design, a relation (7) yields $\alpha_1=\alpha_2=\cdots=\alpha_B=K/V$ $(=\Lambda/R)$.

Therefore, we have established

THEOREM. *A balanced n-ary block design with parameters V, B, R, K and Λ such that $RK-\Lambda V=0$ is only of the following type*

$$N=(K/V)\begin{bmatrix} 1 & 1\cdots 1 \\ 1 & 1\cdots 1 \\ \vdots & \vdots \quad\ \vdots \\ 1 & 1\cdots 1 \end{bmatrix}.$$

In particular, if $n=2$, then $V=K$, i.e., the design is a complete block design.

Remark. It is known (cf. [3]) that the efficiency factor of a balanced n-ary block design with parameters V, B, R, K and Λ is given by $\Lambda V/(RK)$. In this sense, the efficiency factor of the design described in the theorem is 1, and hence the design is the best possible, but has obvious structure.

The form given in the theorem is called trivial for a balanced n-ary block design. Otherwise, a design is said to be non-trivial, in which case $RK-\Lambda V>0$ and the efficiency factor is less than 1. Thus, we can also assert that for a non-trivial balanced n-ary block design Fisher's inequality holds.

Recently, Dey [2] has shown that Fisher's inequality holds for equireplicated n-ary balanced block designs, barring designs with incidence matrices having identical rows. This "balanced" property means that every elementary contrast among the treatment effects is estimated with the same variance. In this sense, it is obvious that a balanced n-ary block design of Tocher is "balanced". Upon the whole, the results of this note may be similar to Dey's results. However, there is an important point different from Dey in the sense that this note consists in the *complete* classification of a balanced n-ary block design with $RK=\Lambda V$ among the parameters, which is new and cannot be derived from Dey's results. As mentioned before, this characterization plays an important role in a balanced n-ary block design of Tocher.

HIROSHIMA UNIVERSITY

REFERENCES

[1] Fisher, R. A. (1940). An examination of the different possible solutions of a problem in incomplete blocks, *Ann. Eugen.*, **10**, 52-75.

[2] Dey, A. (1975). A note on balanced designs, *Sankhyā*, B, **37**, 461-462.

[3] Saha, G. M. (1975). On construction of balanced ternary designs, *Sankhyā*, B, **37**, 220-227.

[4] Tocher, K. D. (1952). Design and analysis of block experiments, *J. R. Statist. Soc.*, B, **14**, 45-100.