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ON A STOCHASTIC GAME WITH ONE-CHANCE RECOVERY

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Summary

Bernoulli trials with success rate p are considered. Peter, who is a gambler of success rate p, gets 1 unit if the first trial results in success and loses the same unit otherwise. For the kth trial ($k \geqq 2$), he gets or loses 1 according as success or failure unless his previous gain S_{k-1} is negative. When S_{k-1} is minus, Peter gets or loses $-S_{k-1}$. Then Peter's gain S_n in n trials is the sum of "dependent" random variables. Therefore, Peter has always the chance p of recovering his minus gain instantaneously.

The probability function of S_n is given and the expected gain is compared with the ordinary (non-symmetric) random walk situation. It will be concluded that Peter should not play the game with one-chance recovery because when p is less than 1/2, he must be afraid of suffering a bigger risk than the usual case.

1. Statement of the problem

The usual type of random walk is found in Feller [1]. Put $\omega_k^0=1$ if the kth trial results in success with the probability p, $\omega_k^0=-1$ otherwise. Then $S_n^0=\omega_1^0+\omega_2^0+\cdots+\omega_n^0$ is the gain (the accumulated excess of successes over failures) in n trials and the probability function of S_n^0 can be expressed by the binomial distribution. Some topics of a general random walk which is the sum of independent random variables with an arbitrary common distribution, are discussed in Feller [2].

We consider Bernoulli trials with the probability of success p and put $\omega_1=1$ if the first trial results in success and $\omega_1=-1$ otherwise. Next we shall define for the succeeding trials as follows. When

$$S_{k-1}=\omega_1+\omega_2+\cdots+\omega_{k-1}$$

is not smaller than 0, put

$$\omega_k=\begin{cases} 1 & \text{(success)}\,, \\ -1 & \text{(failure)}\,, \end{cases}$$

and when $S_{k-1}<0$, put

$$\omega_k=\begin{cases} -S_{k-1} & \text{(success)}, \\ S_{k-1} & \text{(failure)}, \end{cases}$$

S_k corresponds to the so-called "Peter's accumulated net gain" (see for example Feller [1]). It is the significant difference from the ordinary random walk problem that S_k is not the sum of independent random variables and that Peter has always the probability p of recovering his minus gain instantaneously. We shall call the problem of this type STOCHASTIC GAME WITH ONE-CHANCE RECOVERY. We are now interested in some stochastic properties of gain S_n in a sequence of n tossing.

2. Preliminaries

For any positive integer n, the arrangement $\{s_1, s_2, \cdots, s_n\}$ of integers satisfying

$$s_0=0\,, \qquad s_i-s_{i-1}=\varepsilon_i=\pm 1 \ (i=1, 2, \cdots, n)$$

is called "path". For $i \leqq n/2$, let $\mathrm{C}(n, i)$ be the number of paths which satisfy

$$s_1 \geqq -1,\ s_2 \geqq -1, \cdots, s_{2i} \geqq -1 \quad \text{and} \quad s_n=n-2i\,.$$

LEMMA.

(i) $\mathrm{C}(n, i)=\binom{n}{i}-\binom{n}{i-2}$

(ii) $\sum_{l=2j}^{n-1} \mathrm{C}(l, j)=\mathrm{C}(n, j+1)$

(iii) $\sum_{i=0}^{j} \mathrm{C}(m+2i, i)\,\mathrm{C}(n-2i, j-i)=\mathrm{C}(m+n+2, j)$

(iv) $\sum_{i=0}^{n-2} C_i C_{n-2-i}+2C_{n-1}=C_n$,

where

$$C_i=\mathrm{C}(2i, i)\,.$$

PROOF. When $i \leqq 2$, the relation (i) is trivial. Next, from the definition of $\mathrm{C}(n, i)$ it is easily verified that

$$\mathrm{C}(2i+2, i+1)=\mathrm{C}(2i+1, i)+\mathrm{C}(2i, i)\,,$$

$$\mathrm{C}(n, i+1)=\mathrm{C}(n-1, i)+\mathrm{C}(n-1, i+1) \qquad (n \geqq 2i+3)\,.$$

Furthermore noting that

$$C(2i, i)=\left[\binom{2i}{i}+\binom{2i}{i-1}\right]-\left[\binom{2i}{i-1}+\binom{2i}{i-2}\right]$$
$$=\binom{2i+1}{i}-\binom{2i+1}{i-1}=\binom{2i+1}{i+1}-\binom{2i+1}{i-1},$$

we have

$$C(2i+2, i+1)=\binom{2i+2}{i+1}-\binom{2i+2}{i-1}.$$

Similarly

$$C(n, i+1)=\binom{n}{i+1}-\binom{n}{i-1}.$$

Therefore, it is shown that (i) is true for $i \geqq 3$ using mathematical induction.

For the proof of (ii) we first note that

$$\binom{n}{j+1}-\binom{2j}{j+1}=\sum_{i=1}^{n-2j}\left\{\binom{n+1-i}{j+1}-\binom{n-i}{j+1}\right\}=\sum_{i=1}^{n-2j}\binom{n-i}{j}.$$

Therefore, from (i) we have

$$\sum_{l=2j}^{n-1} C(l, j)=\sum_{i=1}^{n-2j}\left\{\binom{n-i}{j}-\binom{n-i}{j-2}\right\}$$
$$=\binom{n}{j+1}-\binom{2j}{j+1}-\binom{n}{j-1}+\binom{2j}{j-1}$$
$$=\binom{n}{j+1}-\binom{n}{j-1}=C(n, j+1).$$

Next, we can also see that the relation (iii) is true for $j=1$. Then we suppose (iii) for some j. We further note that

$$C(n-2i, j+1-i)=C(n-1-2i, j-i)+C(n-3-2(i-1), j-(i-1)).$$

Then using the assumption and the relation (ii)

$$\sum_{i=1}^{j} C(m+2i, i)\, C(n-2i, j+1-i)$$
$$=\sum_{i=1}^{j} C(m+2i, i)\, C(n-1-2i, j-i)$$
$$+\sum_{i=0}^{j-1}\sum_{l=1}^{m+1} C(l+2i, i)\, C(n-3-2i, j-i)$$
$$=C(m+n+1, j)-C(n-1, j)+\sum_{l=0}^{m+1}\{C(l+n-1, j)-C(l+2j, j)\}$$
$$=C(m+n+1, j)-C(n-1, j)+C(m+n+1, j+1)$$
$$-C(n-1, j+1)-C(m+2j+2, j+1)$$
$$=C(m+n+2, j+1)-C(n, j+1)-C(m+2j+2, j+1).$$

From the definition of $\mathrm{C}(n, i)$ we have

$$\mathrm{C}(2n, n)=\mathrm{C}(2n-1, n-1)+\mathrm{C}(2n-2, n-1)$$
$$=\mathrm{C}(2n-2, n-2)+2\,\mathrm{C}(n-2, n-1)\ .$$

Using the relation (iii) for the term $\mathrm{C}(2n-2, n-2)$, we have the relation (iv).

3. Probability function of S_n

We shall find the probability function of S_n. Put

$$P_n(s)=\Pr\{S_n=s\}\ ,$$

then it is easily seen from Figure 1 that the range of P_n is

$$\{n, n-2, n-3, \cdots, 0, -1, -2, -4, \cdots, -2^{n-3}, -2^{n-1}\}$$

and

$$P_n(n)=p^n\ ,\quad P_n(n-2)=np^{n-1}q\ ,\quad P_n(n-3)=p^{n-2}q^2\ ,\quad P_n(-2^{n-1})=q^n\ .$$

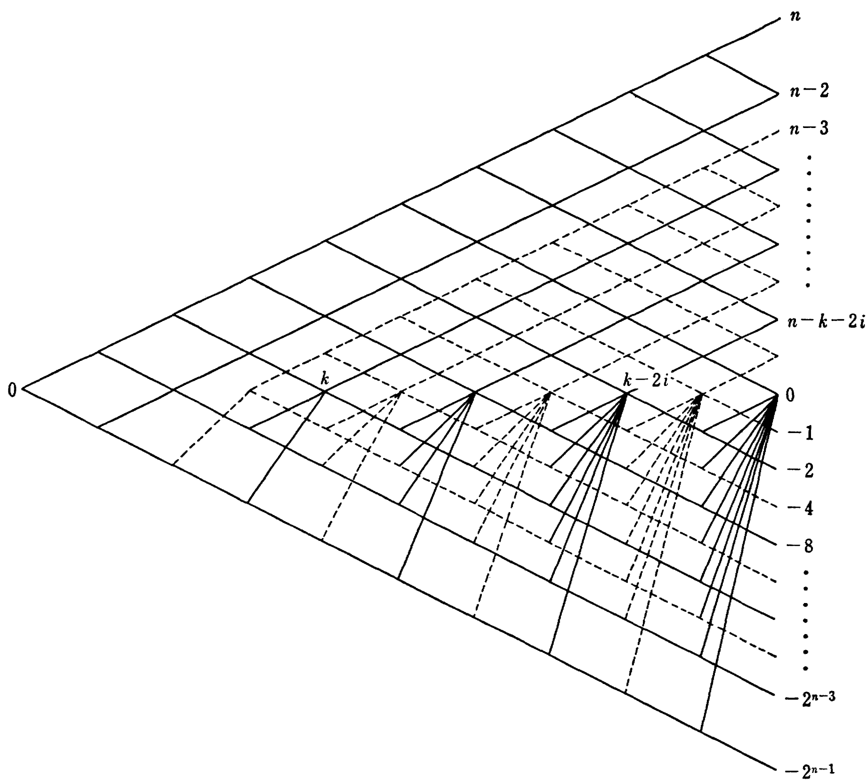

Fig. 1. Game with one-chance recovery. (The vertical scale below -2 is taken logarithic.)

For other cases we first obtain

PROPOSITION 1.

$$Q(k, n-k-2i)=\Pr\{S_{k+2j}>0\ (1\leqq j\leqq i),\ S_n=n-k-2i\,|\,S_k=0\}$$
$$=\mathrm{C}(n-k-2, i)p^{n-k-i}q^i ,$$

where

$$\mathrm{C}(n, i)=\binom{n}{i}-\binom{n}{i-2} .$$

PROPOSITION 2.

$$P_n(n-2b)=\sum_{a=0}^{b} P_{2a}(0)Q(2a, n-2b) \qquad \left(b\leqq\left[\frac{n-2}{2}\right]\right) ,$$

$$P_n(n-2b-1)=\sum_{a=1}^{b} P_{2a+1}(0)Q(2a+1, n-2b-1) \qquad \left(b\leqq\left[\frac{n-3}{2}\right]\right) ,$$

$$P_n\left(n-2\left[\frac{n}{2}\right]\right)=P_{2[n/2]}(0)p^{n-2[n/2]} ,$$

$$P_n\left(n-2\left[\frac{n-1}{2}\right]-1\right)=P_{2[(n-1)/2]+1}(0)p^{n-2[(n-1)/2]-1} ,$$

$$P_n(-2^{-l})=P_{n-1-l}(0)q^{l+1} \qquad (l=0, 1,\cdots, n-3, n-1) .$$

Proposition 1 is trivial from the definition of $\mathrm{C}(n, i)$. Proposition 2 is the direct consequence of Proposition 1 and the definition of our game (see Figure 1). From Proposition 2 it is sufficient to have the expressions for $P_j(0)$.

PROPOSITION 3. For $k\geqq 2$,

$$(1) \qquad P_{2k}(0)=\sum_{i=1}^{2k-3} pq^iP_{2k-1-i}(0)+pq^{2k-1}+\sum_{i=1}^{k} C_{i-2}P_{2k-2i}(0)p^iq^i ,$$

$$(2) \qquad P_{2k+1}(0)=\sum_{i=1}^{2k-2} pq^iP_{2k-i}(0)+pq^{2k}+\sum_{i=1}^{k-1} C_{i-2}P_{2k-2i+1}(0)p^iq^i ,$$

where

$$C_j=\begin{cases} \mathrm{C}(2j, j) & j\geqq 0 , \\ 1 & j=-1 . \end{cases}$$

PROOF. We first note the following relations hold:

$$(3) \qquad P_{2k-1}(1)=\sum_{i=1}^{k} C_{i-2}P_{2k-2i}(0)p^iq^{i-1} ,$$

$$(4) \qquad P_{2k}(1)=\sum_{i=1}^{k-1} C_{i-2}P_{2k+1-2i}(0)p^i q^{i-1} .$$

For, we devide the event $[S_{2k-1}=1]$ as follows:

$$[S_{2k-1}=1]=\bigcup_{i=1}^{k} [S_{2k-1}=1,\ S_{2k-2i}=0,\cdots,\ S_{2k-2i+2}>0,\cdots,\ S_{2k-2}>0] .$$

Noting that

$$\Pr\{S_{2k-2i+2}>0,\cdots,\ S_{2k-2}>0,\ S_{2k-1}=1\,|\,S_{2k-2i}=0\}=C_{i-2}p^i q^{i-1}$$

we have the relation (3). In the same way we can also prove the relation (4).

Next we note that

$$\begin{aligned}\Pr\{S_{2k-1}<0\}&=\sum_{l=0}^{2k-4} P_{2k-1}(-2^l)+P_{2k-1}(-2^{2k-2})\\ &=\sum_{l=0}^{2k-4} P_{2k-2-l}(0)q^{l+1}+q^{2k-1} .\end{aligned}$$

Inserting these expressions into

$$P_{2k}(0)=p\Pr\{S_{2k-1}<0\}+qP_{2k-1}(1) ,$$

we obtain the recurrence relation (1). Similarly, we can show the relation (2).

PROPOSITION 4.

$$(5) \qquad P_l(0)=\sum_{i=1}^{[l/2]} \mathrm{C}(l-1,\ i-1)p^i q^{l-i}+C(l) ,$$

where

$$C(l)=\begin{cases} C_{k-1}p^k q^k , & l=2k \text{ (even)}\\ 0 , & \text{otherwise} .\end{cases}$$

PROOF. Suppose that the relation (5) holds for $l<2k$. From Proposition 3 we have

$$\begin{aligned}P_{2k}(0)&=\sum_{h=2}^{2k-2} pq^{2k-1-h}P_h(0)+pq^{2k-1}+\sum_{j=0}^{k-1} C_{k-2-j}P_{2j}(0)(pq)^{k-j}\\ &\equiv A+pq^{2k-1}+B .\end{aligned}$$

Using the assumption and Lemma (ii)

$$\begin{aligned}A&=\sum_{h=2}^{2k-2} pq^{2k-1-h}\sum_{j=1}^{[h/2]} \mathrm{C}(h-1,\ j-1)p^j q^{h-j}+\sum_{h=1}^{k-1} pq^{2k-1-h}C_{h-1}p^h q^h\\ &=\sum_{j=1}^{k-1} p^{j+1}q^{2k-j-1}\sum_{h=2j}^{2k-2} \mathrm{C}(h-1,\ j-1)+\sum_{h=1}^{k-1} p^{h+1}q^{2k-h-1}C_{h-1}\end{aligned}$$

$$=\sum_{j=1}^{k-1} p^{j+1}q^{2k-j-1}\,\mathrm{C}\,(2k-2,\, j)\ .$$

In the similar way

$$\begin{aligned}B&=\sum_{j=1}^{k-1} C_{k-2-j}\sum_{h=1}^{j}\mathrm{C}\,(2j-1,\, h-1)p^{k-j+h}q^{k+j-h}\\&\quad+\sum_{j=1}^{k-1} C_{k-2-j}C_{j-1}p^kq^k+C_{k-2}p^kq^k\\&=\sum_{i=2}^{k} p^iq^{2k-i}\sum_{l=1}^{i-1} C_{l-2}\,\mathrm{C}\,(2k-1-2l,\, i-1-l)+\sum_{j=0}^{k-1} C_{k-2-j}C_{j-1}p^kq^k\\&=\sum_{i=2}^{k}\mathrm{C}\,(2k-2,\, i-2)p^iq^{2k-i}+C_{k-1}p^kq^k\ .\end{aligned}$$

Consequently, we have

$$\begin{aligned}P_{2k}(0)&=\sum_{i=1}^{k} p^iq^{2k-i}[\mathrm{C}\,(2k-2,\, i-1)+\mathrm{C}\,(2k-2,\, i-2)]+C_{k-1}p^kq^k\\&=\sum_{i=1}^{k}\mathrm{C}\,(2k-1,\, i-1)p^iq^{2k-i}+C(2k)\ .\end{aligned}$$

This shows that the relation (5) holds for $l=2k$. Along the same line, we can also prove that (5) holds for $l=2k+1$ assuming the truth of (5) for $l\leqq 2k$.

We can now state the fundamental results of the probability function of S_n in the range $\{n, n-2, n-3, \cdots, 2\}$.

THEOREM 1.

$$P_n(n-2k)=\mathrm{C}\,(n,\, k)p^{n-k}q^k+\sum_{i=1}^{k-1}\mathrm{C}\,(n-1,\, k-1-i)p^{n-k-i}q^{k+i}\qquad\left(k\leqq\left[\frac{n-2}{2}\right]\right)\tag{6}$$

$$P_n(n-2k-1)=\sum_{i=1}^{k}\mathrm{C}\,(n-1,\, k-i)p^{n-k-i}q^{k+i}\qquad\left(k\leqq\left[\frac{n-3}{2}\right]\right)\ .\tag{7}$$

PROOF. We shall prove only the relation (6). We first note that

$$\begin{aligned}P_n(n-2k)&=\sum_{i=0}^{k} P_{2k-2i}(0)\,\mathrm{C}\,(n-2k-2+2i,\, i)p^{n-2k+i}q^i\\&=I+\mathrm{C}\,(n-2,\, k)p^{n-k}q^k\ ,\end{aligned}$$

where

$$I=\sum_{l=1}^{k} P_{2l}(0)\,\mathrm{C}\,(n-2-2l,\, k-l)p^{n-k-l}q^{k-l}\ .$$

Using Proposition 4 and Lemma (iii), (iv)

$$I=\sum_{l=1}^{k}\sum_{i=1}^{l} \mathrm{C}(2l-1,\, i-1)\,\mathrm{C}(n-2-2l,\, k-l)p^{n-k-l+i}q^{k+l-i}$$
$$+\sum_{l=1}^{k} C_{l-1}\,\mathrm{C}(n-2-2l,\, k-l)p^{n-k}q^{k}$$
$$=\sum_{h=0}^{k-1} p^{n-k-h}q^{k+h}\sum_{l=h+1}^{k} \mathrm{C}(2l-1,\, l-1-h)\,\mathrm{C}(n-2-2l,\, k-l)$$
$$+\mathrm{C}(n-2,\, k-1)p^{n-k}q^{k}$$
$$=\sum_{h=0}^{k-1} \mathrm{C}(n-1,\, k-h-1)p^{n-k-h}q^{k+h}+\mathrm{C}(n-2,\, k-1)p^{n-k}q^{k}$$
$$=\sum_{h=1}^{k-1} \mathrm{C}(n-1,\, k-1-h)p^{n-k-h}q^{k+h}+\{\mathrm{C}(n-1,\, k-1)$$
$$+\mathrm{C}(n-2,\, k-1)\}\,p^{n-k}q^{k}\,.$$

Noting that

$$\mathrm{C}(n-1,\, k-1)+\mathrm{C}(n-2,\, k-1)+\mathrm{C}(n-2,\, k)=\mathrm{C}(n,\, k)\ ,$$

we have the relation (6).

4. Moments of S_n

We write the factorial moments of S_n as

$$\mu_1(n)=\sum sP_n(s)\ , \qquad \mu_2(n)=\sum s(s-1)P_n(s)\ .$$

Then it is easily seen that

$$\mu_1(1)=p-q\ , \qquad \mu_2(1)=2q$$
$$\mu_1(2)=2(p-q)\ , \qquad \mu_2(2)=2p^2+6q^2\ .$$

Therefore

$$\sigma^2(1)=4pq\ , \qquad \sigma^2(2)=8pq\ .$$

In general, for $n\geqq 3$ we have

THEOREM 2.

(8) $\quad \mu_1(n)=\mu_1(n-1)+(p-q)(1+\pi_1(n-2)-\pi_0(n-2))$

(9) $\quad \mu_2(n)=\mu_2(n-1)+2(p-q)\mu_1(n-1)-(p-q)(\pi_2(n-2)-\pi_1(n-2))$
$\qquad +2q\pi_0(n-2)$

where

$$\pi_k(l)=\sum_{j=0}^{l} P_{l-j}(0)2^{kj}q^{j+1}\ , \qquad k=0,\, 1\ .$$

PROOF. We first note that

$$P_n(n)=pP_{n-1}(n-1) ,$$

$$P_n(s)=pP_{n-1}(s-1)+qP_{n-1}(s+1) \qquad (1\leqq s\leqq n-2) ,$$

$$P_n(-1)=qP_{n-1}(0) ,$$

$$p_n(-2^l)=qP_{n-1}(-2^{l-1}) , \qquad (l=1, 2,\cdots, n-3, n-1) .$$

Therefore

$$\begin{aligned}\mu_1(n)&=\sum sP_n(s)\\&=npP_{n-1}(n-1)+\sum_{s=1}^{n-2} s[pP_{n-1}(s-1)+qP_{n-1}(s+1)]\\&\quad -qP_{n-1}(0)+\sum_{l=1}^{n-3}(-2^l)qP_{n-1}(-2^{l-1})+(-2^{n-1})qP_{n-1}(-2^{n-2}) .\end{aligned}$$

Denoting the summation of $j=0, 1,\cdots, n-4, n-2$ by $\sum'$, we have

$$\begin{aligned}\mu_1(n)&=p\sum_{j=0}^{n-1}(j+1)P_{n-1}(j)+q\sum_{j=0}^{n-1}(j-1)P_n(j)+2q\sum{}'(-2^j)P_{n-1}(-2^j)\\&=\sum_{j=0}^{n-1} jP_{n-1}(j)+(p-q)\sum_{j=0}^{n-1}P_{n-1}(j)+[1-(p-q)]\sum{}'(-2^j)P_{n-1}(-2^j)\\&=\mu_1(n-1)+(p-q)(1+\delta_{n-1}) ,\end{aligned}$$

where

$$\begin{aligned}\delta_{n-1}&=\sum{}'(2^j-1)P_{n-1}(-2^j)\\&=\sum_{j=0}^{n-2}(2^j-1)P_{n-2-j}(0)q^{j+1}=\pi_1(n-2)-\pi_0(n-2) .\end{aligned}$$

This completes the proof of (8).

In the similar way,

$$\begin{aligned}\mu_2(n)&=\sum s(s-1)P_n(s)\\&=n(n-1)pP_{n-1}(n-1)+\sum_{s=1}^{n-2} s(s-1)[pP_{n-1}(s-1)+qP_{n-1}(s+1)]\\&\quad+2qP_{n-1}(0)+2q\left\{\sum_{j=0}^{n-4}(-2^j)(-2^j-1)P_{n-1}(-2^j)\right.\\&\qquad\left.+(-2^{n-2})(-2^{n-2}-1)P_{n-1}(-2^{n-2})\right\}\\&=p\sum_{j=0}^{n-1} j(j+1)P_{n-1}(j)+q\sum_{j=0}^{n-1}(j-1)(j-2)P_{n-1}(j)\\&\quad+[1+2(p-q)-3(p-q)]\sum{}'(-2^j)(-2^j-1)P_{n-1}(-2^j)\\&=\sum_{j=0}^{n-1} j(j-1)P_{n-1}(j)+2(p-q)\sum_{j=0}^{n-1} jP_{n-1}(j)+2q\sum_{j=0}^{n-1}P_{n-1}(j)\\&\quad+\sum{}'(-2^j)(-2^j-1)P_{n-1}(-2^j)+2(p-q)\sum{}'(-2^j)P_{n-1}(-2^j)\end{aligned}$$

$$+\sum{}' [2(p-q)2^j(2^j+2)-3(p-q)2^j(2^j+1)+2q]P_{n-1}(-2^j)$$
$$=\mu_2(n-1)+2(p-q)\mu_1(n-1)-(p-q)\sum{}' 2^j(2^j-1)P_{n-1}(-2^j)$$
$$+2q[1-\sum{}' P_{n-1}(-2^j)]$$
$$=\mu_2(n-1)+2(p-q)\mu_1(n-1)-(p-q)\sum_{j=0}^{n-2} 2^j(2^j-1)P_{n-2-j}(0)q^{j+1}$$
$$+2q\sum_{j=0}^{n-2} P_{n-2-j}(0)q^{j+1} .$$

This completes the proof of (9).

From Theorem 2 we obtain

$$\begin{aligned}\mu_1(n)&=\mu_1(n-1)+(p-q)(1+\delta_{n-1})\\ &=\mu_1(n-2)+(p-q)(1+\delta_{n-2})+(p-q)(1+\delta_{n-1})\\ &=\mu_1(n-2)+(p-q)(2+\delta_{n-2}+\delta_{n-1})\\ &=\mu_1(2)+(p-q)\left(n-2+\sum_{l=2}^{n-1}\delta_l\right) .\end{aligned}$$

Since

$$\mu_1(2)=2p^2-2q^2=2(p-q) ,$$

then we have

$$\mu_1(n)=\mu_1^0(n)+(p-q)\Delta(n) ,$$

where $\mu_1^0(0)=(p-q)n$ is the expected value of gain in ordinary random walk situation and

$$\begin{aligned}\Delta(n)&=\sum_{l=2}^{n-1}\delta_l=\sum_{l=2}^{n-1}\sum_{j=1}^{l-1}(2^j-1)P_{l-1-j}(0)q^{j+1}\\ &=\sum_{j=1}^{n-2}(2^j-1)\left[\sum_{i=0}^{n-2-j}P_i(0)\right]q^{j+1} .\end{aligned}$$

For special references, we shall show the first 5 terms of $\Delta(n)$.

$$\Delta(3)=\delta_2=q^2 ,$$

$$\Delta(4)=[P_0(0)+P_1(0)]q^2+3P_0(0)q^3=(1+3q)q^2 ,$$

$$\begin{aligned}\Delta(5)&=[P_0(0)+P_1(0)+P_2(0)]q^2+3[P_0(0)+P_1(0)]q^3+7[P_0(0)]q^4\\ &=(1+2pq)q^2+3q^3+7q^4=(1+5q+5q^2)q^2 ,\end{aligned}$$

$$\Delta(6)=(1+5q+12q^2+8q^3)q^2 ,$$

$$\Delta(7)=(1+6q+20q^2+q^3+33q^4)q^2 .$$

When $p<q$, player with one-chance recovery strategy must incur the additional risk $(q-p)\Delta(n)$ ($\Delta(n)$ is always positive!!). From the expression for $\Delta(n)$ it is clear that

$$\Delta(n) > \Delta(n-1)\ .$$

However, the inequality

$$\Delta(n)/n > \Delta(n-1)/(n-1) \tag{10}$$

does not always hold.

PROPOSITION 5.

$$\pi_0(l) = [P_l(0) + \pi_0(l-1)]q \tag{11}$$

$$\pi_1(l) = [P_l(0) + 2\pi_1(l-1)]q \tag{12}$$

PROOF. We shall show only the proof of (12).

$$\begin{aligned}\pi_1(l) - \pi_1(l-1) &= \sum_{j=0}^{l} P_{l-j}(0) 2^j q^{j+1} - (p+q) \sum_{j=0}^{l-1} P_{l-1-j}(0) 2^j q^{j+1} \\ &= q\left\{ \sum_{j=0}^{l} P_{l-j}(0) 2^j q^j - \sum_{j=1}^{l} P_{l-j}(0) 2^{j-1} q^j \right\} \\ &\quad - p \sum_{j=1}^{l} P_{l-j}(0) 2^{j-1} q^j \\ &= P_l(0) q + (q-p) \sum_{j=1}^{l} P_{l-j}(0) 2^{j-1} q^j \\ &= P_l(0) q + (q-p) \pi_1(l-1)\ .\end{aligned}$$

THEOREM 3. *When* $p \leqq q$ *the inequality* (10) *always holds.*

PROOF. We first note that

$$\begin{aligned}&(n-1)\Delta(n) - n\Delta(n-1) \\ &\quad = (n-1)[\Delta(n) - \Delta(n-1)] - \Delta(n-1) \\ &\quad\quad > (n-3)[\Delta(n) - \Delta(n-1)] - \Delta(n-1) \\ &\quad = (n-3)[\pi_1(n-2) - \pi_0(n-2)] - \sum_{l=1}^{n-3} [\pi_1(l) - \pi_0(l)] \\ &\quad = \sum_{l=1}^{n-3} \{[\pi_1(n-2) - \pi_0(n-2)] - [\pi_1(l) - \pi_0(l)]\}\ .\end{aligned}$$

Consequently, it is sufficient to show that

$$[\pi_1(l) - \pi_0(l)] - [\pi_1(l-1) - \pi_0(l-1)] > 0\ .$$

Using Proposition 5 we have

$$\begin{aligned}&\pi_1(l) - \pi_1(l-1) - [\pi_0(l) - \pi_0(l-1)] \\ &\quad = P_l(0)q + (q-p)\pi_1(l-1) - [P_l(0)q - \pi_0(l-1)p] \\ &\quad = (q-p)\pi_1(l-1) + \pi_0(l-1)p\ .\end{aligned}$$

The last expression is positive for $p \leqq q$.

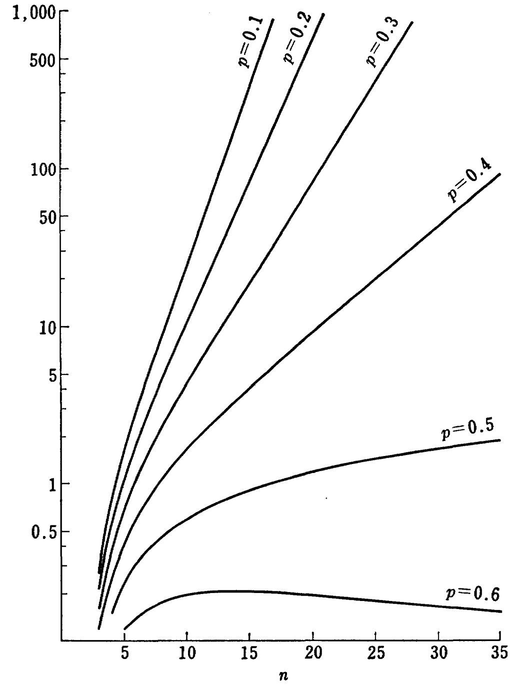

Fig. 2. Graphs of $\Delta(n)/n$.

In Figure 2, we show the graphs of $\Delta(n)/n$ for typical values of p. Since

$$\frac{\mu_1(n)-\mu_1^0(n)}{\mu_1^0(n)}=\frac{\Delta(n)}{n} \qquad (p\neq q)$$

and both $\mu_1(n)$ and $\mu_1^0(n)$ have the same sign, then if $p<q$, the additional risk rate of one-chance recovery strategy becomes seriously large. On the otherhand, in the favorable case to Peter, the additional merit of such strategy is not so large. Consequently, Peter should prefer the ordinary game to the game with one-chance recovery strategy.

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