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MULTIVARIATE DIGAMMA DISTRIBUTION

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Summary

Digamma distributions are extended to multivariate distributions and their properties are examined. The distributions are closely related to multivariate logarithmic series distributions and will be useful when observed frequency data have too long tail to be fitted by a multivariate logarithmic series distribution.

1. The univariate digamma and other distributions

In a previous paper (Sibuya [4]) the author introduced and studied a digamma distribution defined by

$$\Pr[X=x]=q(x;\alpha,\gamma)=\frac{1}{\psi(\alpha+\gamma)-\psi(\gamma)}\,\frac{(\alpha)_x}{x(\alpha+\gamma)_x}\,, \tag{1.1}$$

where

$$x=1,2,\cdots;\ \gamma>0,\ \alpha>-1\ (\alpha\neq 0),\ \alpha+\gamma>0,\ (\alpha)_x=\alpha(\alpha+1)\cdots(\alpha+x-1),$$

and $\psi(z)=d\log\Gamma(z)/dz$ is the digamma or the psi function.

The distribution (1.1) will be referred to as DGa(α,γ). In this paper we extend DGa(α,γ) to a multivariate distribution.

The digamma distribution is closely related to the logarithmic series distribution LSr(θ) defined by

$$\Pr[X=x;\theta]=\theta^x/(-\log(1-\theta))x\,,\qquad x=1,2,\cdots;\quad 0<\theta<1\,. \tag{1.2}$$

In fact, if α and γ of DGa(α,γ) increase indefinitely keeping $\alpha/(\alpha+\gamma)=\theta$ constant, then the limit distribution is LSr(θ). Conversely, if the parameter θ of LSr(θ) is a random variable with the density

$$C(\alpha,\gamma)(-\log(1-\theta))\theta^{\alpha-1}(1-\theta)^{\gamma-1}\,,\qquad 0<\theta<1\,, \tag{1.3}$$

which we shall call an end accented beta distribution, then the compounded distribution is DGa(α,γ). Digamma distributions can be used when logarithmic series distributions cannot be fitted to data since the tail of observed frequencies is very long.

Our multivariate digamma distribution will be closely related to a multivariate logarithmic series distribution, defined by

$$\Pr[\boldsymbol{X}=\boldsymbol{x};\boldsymbol{\theta}]=\frac{(x-1)!}{-\log(1-\theta)}\prod_{i=1}^{k}\frac{\theta_i^{x_i}}{x_i!}, \tag{1.4}$$

where

$$\boldsymbol{X}=(X_1,\cdots,X_k),\ \boldsymbol{x}=(x_1,\cdots,x_k),\ x=\sum_{i=1}^{k}x_i=1,2,\cdots,\ x_i=0,1,2,\cdots,$$

$$\boldsymbol{\theta}=(\theta_1,\cdots,\theta_k),\ \theta=\sum_{i=1}^{k}\theta_i,\ 0<\theta_i,\ 0<\theta<1.$$

We shall denote the distribution (1.4) by MLSr $(\boldsymbol{\theta})$. See Patil and Bildikar [3] and Chatfield, et al. [1] concerning MLSr $(\boldsymbol{\theta})$.

Throughout the paper a vector with k components is denoted by a corresponding bold-face letter, like $\boldsymbol{x}=(x_1,x_2,\cdots,x_k)$, and the sum of the components by a light-face letter without subscript, like $x=\sum_{i=1}^{k}x_i$.

The digamma distribution is also closely related to the inverse Pólya-Eggenberger distribution (or the general Waring, the negative binomial beta, or the type B3 generalized hypergeometric distribution (Sibuya and Shimizu [5]), $F(\alpha,\beta;\alpha+\beta+\gamma)$, defined by

$$\Pr[X=x;\alpha,\beta,\gamma]=\frac{\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)\Gamma(\gamma)}\frac{(\alpha)_x(\beta)_x}{(\alpha+\beta+\gamma)_x x!}, \tag{1.5}$$

$$x=0,1,2,\cdots;\quad \alpha,\beta,\gamma>0.$$

If the distribution (1.5) is zero truncated, and further if β tends to zero, then the limit distribution is DGa (α,γ).

Therefore, our multivariate digamma distribution will be closely related to the multivariate inverse Pólya-Eggenberger distribution, defined by

$$\begin{aligned}\Pr[\boldsymbol{X}=\boldsymbol{x};\boldsymbol{\alpha},\beta,\gamma]&=p(\boldsymbol{x};\boldsymbol{\alpha},\beta,\gamma)\\&=\frac{\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)\Gamma(\gamma)}\frac{(\beta,x)}{(\alpha+\beta+\gamma,x)}\prod_{i=1}^{k}\frac{(\alpha_i,x_i)}{x_i!},\end{aligned} \tag{1.6}$$

where

$$x_i=0,1,2,\cdots;\ x=1,2,\cdots;\ \alpha_i>0;\ \text{and } (\beta,x) \text{ is Appell's symbol,}$$

$$\text{i.e., } (\beta,x)=(\beta)_x=\beta(\beta+1)\cdots(\beta+x-1).$$

See Janardan and Patil [2] and Sibuya and Shimizu [5] concerning the distribution. In the latter paper the distribution (1.6) was referred to as $F(\boldsymbol{\alpha}; \beta; \alpha+\beta+\gamma)$.

With the digamma distribution DGa(α, γ) a trigamma distribution was introduced. It was defined by

$$\Pr[X=x]=q(x; \gamma)=\frac{1}{\psi'(\gamma)}\frac{(x-1)!}{x(\gamma)_x}, \tag{1.7}$$

where

$x=1, 2, \cdots; \gamma>0$, and $\psi'(z)=d\psi(z)/dz$ is the trigamma function.

The trigamma distribution of (1.7), denoted by TGa(γ), is a limit distribution of DGa(α, γ) when α tends to zero. The multivariate extension of this distribution will be also discussed.

2. Multivariate digamma distributions

DEFINITION. A k-variate digamma distribution is defined by

$$\begin{aligned}\Pr[\boldsymbol{X}=\boldsymbol{x}]&=q(\boldsymbol{x}; \boldsymbol{\alpha}, \gamma)\\ &=\frac{1}{\psi(\alpha+\gamma)-\psi(\gamma)}\frac{(x-1)!}{(\alpha+\gamma, x)}\prod_{i=1}^{k}\frac{(\alpha_i, x_i)}{x_i!},\end{aligned} \tag{2.1}$$

where

$$x_i=0, 1, 2, \cdots; \quad x=1, 2, \cdots; \quad \alpha_i>0; \quad \text{and} \quad \gamma>0.$$

This will be referred to as MDGa$(\boldsymbol{\alpha}, \gamma)$. Remark that the ranges of α_i's are restricted to positive values.

Distribution of sums, and the conditional distribution given sum. Since,

$$\sum_{\Sigma x_i=x}\prod_{i=1}^{k}\frac{(\alpha_i, x_i)}{x_i!}=\frac{(\alpha, x)}{x!}, \qquad \alpha_i>0; \quad x=1, 2, 3, \cdots$$

the distribution of the sum of components, $X=\sum X_i$, follows a univariate digamma distribution, DGa(α, γ). The conditional distribution of $\boldsymbol{X}$, given $X=x$, is a singular multivariate negative hypergeometric

$$\prod_{i=1}^{k}\frac{(\alpha_i, x_i)}{x_i!}\Big/\frac{(\alpha, x)}{x!}=\prod_{i=1}^{k}\binom{-\alpha_i}{x_i}\Big/\binom{-\alpha}{x}, \tag{2.2}$$

where

$$x_i=0, 1, 2, \cdots; \quad x=1, 2, \cdots; \quad \text{and} \quad \alpha_i>0.$$

Let $\{J_1, \cdots, J_l\}$ be a partition of $\{1, 2, \cdots, k\}$, that is $J_1\cup J_2\cup\cdots\cup J_l=\{1, 2, \cdots, k\}$ and $J_i\cap J_j=\phi$, $i\neq j$. Put

$$\sum_{i\in J_j} X_i = X_j^*, \quad \sum_{i\in J_j} x_i = x_j^* \quad \text{and} \quad \sum_{i\in J_j} \alpha_i = \alpha_j^* .$$

Since

$$\sum_{x_j^*=w} \prod_{i\in J_j} \frac{(\alpha_i, x_i)}{x_i!} = \frac{(\alpha_j^*, w)}{w!}$$

the simultaneous distribution of $(X_1^*, \cdots, X_l^*)$ is a l-variate digamma distribution, MDGa $(\alpha_1^*, \cdots, \alpha_l^*; \gamma)$, that is

$$\frac{1}{\psi(\alpha+\gamma)-\psi(\gamma)} \frac{(x-1)!}{(\alpha+\gamma, x)} \prod_{j=1}^{l} \frac{(\alpha_j^*, x_j^*)}{x_j^*!} ,$$

where

$$x_j^* = 0, 1, 2, \cdots; \quad \text{and} \quad x = \sum_{j=1}^{l} x_j^* = \sum_{i=1}^{k} x_i = 1, 2, \cdots .$$

Conditional distributions. Assume that $X_{l+1}=u_{l+1}, \cdots, X_k=u_k$ are given. The distribution of $(X_1, X_2, \cdots, X_l)$ under this condition is

$$c\frac{(u^{**}+x^*-1)!}{(\alpha+\gamma, u^{**}+x^*)} \prod_{i=1}^{l} \frac{(\alpha_i, x_i)}{x_i!} = c\frac{(u^{**}-1)!}{(\alpha+\gamma, u^{**})} \frac{(u^{**}, x^*)}{(\alpha+\gamma+u^{**}, x^*)} \prod_{i=1}^{l} \frac{(\alpha_i, x_i)}{x_i!} ,$$

where $u^{**} = \sum_{i=l+1}^{k} u_i$ and $x^* = \sum_{i=1}^{l} x_i$, and the second expression assumes $u^{**} \geqq 1$.

If $u^{**} \geqq 1$, then this is a multivariate Pólya-Eggenberger distribution, $F(\alpha_1, \cdots, \alpha_l; u^{**}; \alpha+\gamma+u^{**})$. If $u^{**}=0$, then this is a multivariate digamma distribution MDGa $\left(\alpha_1, \cdots, \alpha_l; \sum_{i=l+1}^{k} \alpha_i + \gamma\right)$. That is,

(2.3) $\Pr[(X_1, \cdots, X_l)=(x_1, \cdots, x_l) \mid X_{l+1}=u_{l+1}, \cdots, X_k=u_k]$

$$= \begin{cases} \dfrac{\Gamma(\alpha+\gamma)\Gamma(\alpha^{**}+\gamma+u^{**})}{\Gamma(\alpha^{**}+\gamma)\Gamma(\alpha+\gamma+u^{**})} \dfrac{(u^{**}, x^*)}{(\alpha+\gamma+u^{**}, x^*)} \displaystyle\prod_{i=1}^{l} \dfrac{(\alpha_i, x_i)}{x_i!} , \\ \qquad x_i = 0, 1, 2, \cdots, \text{ if } u^{**} \geqq 1, \\ \dfrac{1}{\psi(\alpha+\gamma)-\psi(\gamma)} \dfrac{(x^*-1)!}{(\alpha+\gamma, x^*)} \displaystyle\prod_{i=1}^{l} \dfrac{(\alpha_i, x_i)}{x_i!} , \\ \qquad x_i = 0, 1, 2, \cdots, \ x^* = 1, 2, \cdots, \text{ if } u^{**}=0, \end{cases}$$

where

$$\alpha^{**} = \sum_{i=l+1}^{k} = \alpha_i .$$

In particular, the conditional distribution of X_i, given all the values of the other variables $X_j=u_j$ $(j \neq i)$, is an inverse Pólya-Eggenberger distribution $F(\alpha_i, u_i^*; \alpha+\gamma+u_i^*)$, where $u_i^* = \sum_{j\neq i} u_j$, provided that

$u_i^*>0$, and is a digamma distribution DGa $(\alpha_i, \sum_{j\neq i}\alpha_i+\gamma)$, provided that $u_i^*=0$.

Marginal distributions. We get the marginal distribution of $(X_1, X_2, \cdots, X_l)$ from the above discussion by assuming now $X_1=x_1, \cdots, X_l=x_l$ be fixed and adding up the probabilities. The resulting probabilities are

$$\text{(2.4)} \quad \Pr[(X_1, X_2, \cdots, X_l)=(x_1, x_2, \cdots, x_l)] = \frac{1}{\psi(\alpha+\gamma)-\psi(\gamma)} \frac{(x^*-1)!}{(\alpha^*+\gamma, x^*)} \prod_{i=1}^{l} \frac{(\alpha_i, x_i)}{x_i!} ,$$

where

$$x_i=0, 1, 2, \cdots, \text{ and } x^*=\sum_{i=1}^{l} x_i \geq 1 \text{ is assumed.}$$

These are l-variate digamma probabilities with the parameters $(\alpha_1, \alpha_2, \cdots, \alpha_l; \gamma)$ multiplied by

$$\Pr\left[\sum_{i=1}^{l} X_i>0\right]=(\psi(\alpha^*+\gamma)-\psi(\gamma))/(\psi(\alpha+\gamma)-\psi(\gamma))<1 .$$

Hence, the marginal distribution of $(X_1, X_2, \cdots, X_l)$ is a 'modified' l-variate digamma distribution, with probabilities

$$\text{(2.5)} \quad \Pr[(X_1, X_2, \cdots, X_l)=(0, 0, \cdots, 0)] =(\psi(\alpha+\gamma)-\psi(\alpha^*+\gamma))/(\psi(\alpha-\gamma)-\psi(\gamma))$$

and (2.4) for $\sum_{i=1}^{l} X_i \geq 1$.

As a special case, the marginal distribution of X_i is a 'modified' digamma distribution

$$\text{(2.6)} \quad \Pr[X_i=x_i]=\begin{cases} (\psi(\alpha+\gamma)-\psi(\alpha_i+\gamma))/(\psi(\alpha+\gamma)-\psi(\gamma)) , & x_i=0 \\ \dfrac{1}{\psi(\alpha+\gamma)-\psi(\gamma)} \dfrac{(\alpha_i, x_i)}{x_i(\alpha_i+\gamma, x_i)} , & x_i=1, 2, \cdots. \end{cases}$$

Moments. Moments of a digamma distribution is obtained as follows. Under the condition $\sum_{i=1}^{k} X_i=x$, the factorial moments are given by

$$\begin{aligned} E\left[\prod_{i=1}^{k} X_i^{(r_i)} \middle| \sum_{i=1}^{k} X_i=x\right] &= \sum_{\Sigma x_i=x} \prod_{i=1}^{k} x_i^{(r_i)} \frac{(\alpha_i, x_i)}{x_i!} \Big/ \frac{(\alpha, x)}{x!} \\ &=\prod_{i=1}^{k} (\alpha_i, r_i) \sum_{\Sigma x_i=x} \frac{(\alpha_i+r_i, x_i-r_i)}{(x_i-r_i)!} \Big/ \frac{(\alpha, x)}{x!} \\ &=\frac{x^{(r)}}{(\alpha, r)} \prod_{i=1}^{k} (\alpha_i, r_i) , \end{aligned}$$

where

$$r=\sum_{i=1}^{k} r_i .$$

The sum $\sum_{i=1}^{k} X_i$ follows a digamma distribution as seen above and its moments are known:

$$\mathrm{E}\,[X^{(r)}]=\frac{(r-1)!}{\psi(\alpha+\gamma)-\psi(\gamma)}\,\frac{(\alpha,\,r)}{(\gamma-1)^{(r)}}\,,$$

provided that $\gamma>r$. Hence, unconditionally,

$$u(r_1,\,\cdots,\,r_k)=\mathrm{E}\left[\prod_{i=1}^{k} X_i^{(r_i)}\right]=\frac{1}{\psi(\alpha+\gamma)-\psi(\gamma)}\,\frac{(r-1)!}{(\gamma-1)^{(r)}}\prod_{i=1}^{k}(\alpha_i,\,r_i)\,.$$

For example,

$$\mathrm{E}\,[X_i]=\mu(1;\,i)=\frac{1}{\psi(\alpha+\gamma)-\psi(\gamma)}\,\frac{\alpha_i}{\gamma-1}=\mu\cdot\frac{\alpha_i}{\alpha}\,,$$

where

$$\mu=\mathrm{E}\left[\sum_{i=1}^{k} X_i\right],$$

and

$$\mathrm{V}\,[X_i]=\mu(1;\,i)\left[1+\frac{\alpha_i+1}{\gamma-2}-\mu(1;\,i)\right],$$

$$\mathrm{Cov}\,[X_i,\,X_j]=\mu\cdot\frac{\alpha_i\alpha_j}{\alpha}\left[\frac{1}{\gamma-2}-\frac{\mu}{\alpha}\right],$$

$$\mathrm{Cor}\,[X_i,\,X_j]=\frac{\xi-\mu}{\sqrt{((1+(\gamma-1)\alpha_i^{-1})\xi-\mu)((1+(\gamma-1)\alpha_j^{-1})\xi-\mu)}}\,,$$

where

$$\xi=\alpha/(\gamma-2)\,.$$

Shape. To see that a multivariate digamma is unimodal check

$$\frac{p(x_1,\,\cdots,\,x_i+1,\,\cdots,\,x_k;\,\boldsymbol{\alpha},\,\gamma)}{p(x_1,\,\cdots,\,x_i,\,\cdots,\,x_k;\,\boldsymbol{\alpha},\,\gamma)}=\frac{x}{\alpha+\gamma+x}\,\frac{\alpha_i+x_i}{x_i+1}<1\,,$$

if $(x_i+1)/x>(\alpha_i-1)/(\alpha+\gamma)$. For any $\boldsymbol{x}$ there is at least one i such that this inequality is satisfied since $(x_i+1)/x>x_i/x$, $\alpha_i/\alpha>(\alpha_i-1)/(\alpha+\gamma)$ and union of the regions such that $x_i/x\geqq\alpha_i/\alpha$ $(i=1,2,\cdots,k)$ covers the whole sample space. Therefore, there is at least one direction in which

$p(\boldsymbol{x}; \boldsymbol{\alpha}, \gamma)$ decreases. The mode is at $\boldsymbol{x}$ such that $\sum_{j=1}^{k} x_j = x_i = 1$ and $\alpha_i = \max(\alpha_1, \cdots, \alpha_k)$. The maximum probability is equal to $(\psi(\alpha+\gamma)-\psi(\gamma))^{-1}\alpha_i/(\alpha+\gamma)$.

3. Geneses

Model 1. If we truncate $\mathbf{0}$ of a multivariate inverse Pólya-Eggenberger distribution (1.6), and further if the parameter β tends to zero, then we get a multivariate digamma distribution:

$$\lim_{\beta\to 0} p(\boldsymbol{x}; \boldsymbol{\alpha}, \beta, \gamma)/(1-p(\mathbf{0}; \alpha, \beta, \gamma)) = q(\boldsymbol{x}; \boldsymbol{\alpha}, \gamma) . \tag{3.1}$$

The convergence is uniform in $\boldsymbol{x}$.

This is easily proved since the probabilities $p(\boldsymbol{x}; \boldsymbol{\alpha}, \beta, \gamma)$ can be expressed as a product of the distribution of $\sum_{i=1}^{k} X_i$, $F(\alpha, \beta; \alpha+\beta+\gamma)$, and the conditional distribution of $(X_1, X_2, \cdots, X_k)$, given $\sum_{i=1}^{k} X_i$, a singular multivariate negative hypergeometric distribution:

$$p(\boldsymbol{x}; \boldsymbol{\alpha}, \beta, \gamma) = \frac{\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)\Gamma(\gamma)} \frac{(\alpha, x)(\beta, x)}{(\alpha+\beta+\gamma, x)x!} \cdot \prod_{i=1}^{k} \frac{(\alpha_i, x_i)}{x_i!} \Big/ \frac{(\alpha, x)}{x_i!} . \tag{3.2}$$

The conditional distribution is unchanged by the truncation and the limit process, and the problem reduces to the univariate case.

One difference from the univariate case is that the negative value of α and α_i's are not allowed even the point $\mathbf{0}$ is truncated. This is because the conditional distribution of $\boldsymbol{X}$, given $\sum_{i=1}^{k} X_i = x$, is restricted only to a singular negative hypergeometric distribution. See Sibuya and Shimizu [5].

It should be recalled that a multivariate logarithmic series distribution MLSr$(\boldsymbol{\theta})$ is obtained by truncating the point $\mathbf{0}$ of a negative multinomial distribution

$$\frac{\Gamma\left(\beta+\sum_{i=1}^{k} x_i\right)}{\Gamma(\beta)\sum_{i=1}^{k} x_i!}\left(1-\sum_{i=1}^{k}\theta_i\right)^{\beta}\prod_{i=1}^{k}\theta_i^{x_i}, \qquad 0<\theta_i, \quad \sum_{i=1}^{k}\theta_i<1 \tag{3.3}$$

and making $\beta \to 0$. Our process generating the multivariate digamma is completely parallel with this. The fact that a multivariate inverse Pólya-Eggenberger is obtained by compounding the negative multinomial distribution by Dirichlet distribution suggests another genesis.

Model 2. If the parameter $\boldsymbol{\theta}$ of the multivariate logarithmic series distribution MLSr $(\boldsymbol{\theta})$ is a random vector with the probability density

$$(3.4)\qquad \frac{1}{C(\boldsymbol{\alpha}, \gamma)}(-\log(1-\theta))(1-\theta)^{\gamma-1}\sum_{i=1}^{k}\theta_i^{\alpha_i-1},$$

where

$$0<\theta_i,\quad \theta<1,\quad \alpha_i>0,\quad \gamma>0,\quad \text{and}$$

$$C(\boldsymbol{\alpha}, \gamma)=(\psi(\alpha+\gamma)-\psi(\gamma))\Gamma(\gamma)\prod_{i=1}^{k}\Gamma(\alpha_i)/\Gamma(\alpha+\gamma),$$

then the compounded distribution is a multivariate digamma distribution MDGa $(\boldsymbol{\alpha}, \gamma)$.

The compounder can be called an "end accented Dirichlet" distribution. It is easy to see that $\theta=\sum_{i=1}^{k}\theta_i$ is distributed as an end accented beta distribution, that the distribution of $(\theta_1/\theta, \cdots, \theta_k/\theta)$ is a Dirichlet distribution, and thus the normalizing factor is determined. See Sibuya [4].

Model 3. On the other hand a multivariate logarithmic series distribution MLSr $(\boldsymbol{\theta})$ is obtained from the 0-truncated Poisson distribution as follows. Let $\boldsymbol{X}=(X_1, \cdots, X_k)$ be independent Poisson distribution with means $\boldsymbol{\omega}=(\omega_1, \cdots, \omega_k)$, and consider the simultaneous distribution when the point $\boldsymbol{X}=\boldsymbol{0}$ is truncated:

$$(3.5)\qquad \Pr[\boldsymbol{X}=\boldsymbol{x}]=\frac{e^{-\omega}}{1-e^{-\omega}}\prod_{i=1}^{k}\frac{\omega_i^{x_i}}{x_i!},$$

$$x_i=0, 1, 2, \cdots;\quad x=1, 2, \cdots;\quad \omega_i>0.$$

If $\omega_i=\omega p_i$, where p_i's are constants such that $0<p_i$, $\sum_{i=1}^{k}p_i=1$, and ω is a random variable with the probability density

$$(3.6)\qquad (1-e^{-\omega})\omega^{-1}e^{-\omega\lambda}/\log(1+\lambda^{-1}),$$

then the compounded distribution of $\boldsymbol{X}$ is an MLSr $(\boldsymbol{\theta})$, with $\theta_i=p_i/1+\lambda$.

Combining this fact with the above discussion, we can generate a multivariate digamma distribution MDGa $(\boldsymbol{\alpha}, \gamma)$ from the $\boldsymbol{0}$-truncated Poisson (3.5) as follows.

Define a probability density function

$$(3.7)\qquad h(\boldsymbol{\omega}; \boldsymbol{\alpha}, \gamma)=\frac{1}{C(\boldsymbol{\alpha}, \gamma)}(1-e^{-\omega})\omega^{-\alpha}\left(\prod_{i=1}^{k}\omega_i^{\alpha_i-1}\right)\int_0^{\infty}e^{-\omega\lambda}\frac{\lambda^{\gamma-1}}{(1+\lambda)^{\alpha+\gamma}}d\lambda,$$

where

$\omega_i>0$, $\alpha_i>0$, $\gamma>0$ and $C(\boldsymbol{\alpha}, \gamma)$ is the function introduced in (3.4).

We define $p_i=\omega_i/\omega$, $i=1, 2, \cdots, k$, and transform $(\omega_1, \cdots, \omega_k)$ into $(\omega, p_1, \cdots, p_{k-1})$. Since

$$||\partial(\omega, p_1, \cdots, p_{k-1})/\partial(\omega_1, \cdots, \omega_k)||=1/\omega^{n-2},$$

$$\int\cdots\int \frac{e^{-\omega}}{1-e^{-\omega}} \frac{\prod_{i=1}^{k} \omega_i^{x_i}}{x_i!} h(\boldsymbol{\omega}; \boldsymbol{\alpha}, \gamma) \prod_{i=1}^{k} d\omega_i$$
$$=\frac{1}{C(\boldsymbol{\alpha}, \gamma)} \frac{1}{\prod_{i=1}^{k} x_i!} \int\cdots\int \omega^{x-1}e^{-\omega(1+\lambda)} \frac{\lambda^{\gamma-1}}{(1+\lambda)^{\alpha+\gamma}} \prod_{i=1}^{k} p_i^{\alpha_i+x_i-1} \prod_{i=1}^{k} dp_i d\omega d\lambda$$
$$=\frac{1}{C(\boldsymbol{\alpha}, \gamma)} \prod_{i=1}^{k} \frac{\Gamma(\alpha_i+x_i)}{x_i!} \frac{\Gamma(x)\Gamma(\gamma)}{\Gamma(\alpha+\gamma+x)},$$

and this is MDGa $(\boldsymbol{\alpha}, \gamma)$.

Model 4. In Section 2, it is stated that the distribution of the sum of components of an MDGa $(\boldsymbol{\alpha}, \gamma)$ variable $\boldsymbol{X}$, $\sum X_i=X$, follows a univariate digamma distribution DGa (α, γ), and the conditional distribution of $\boldsymbol{X}$, given $X=x$, is a singular multivariate negative hypergeometric distribution. Conversely, if the distribution of $\boldsymbol{X}$ is a singular multivariate negative hypergeometric distribution (2.2), and its parameter x is a DGa (α, γ) variable, $\alpha=\sum \alpha_i$, then $\boldsymbol{X}$ is an MDGa $(\boldsymbol{\alpha}, \gamma)$ variable.

This is similar to the multivariate inverse Pólya-Eggenberger distribution (1.6), which is obtained when the parameter x of a singular multivariate negative hypergeometric distribution (2.2) is an inverse Pólya-Eggenberger variable with probabilities (1.5), $\alpha=\sum \alpha_i$.

4. Limits

Case 1. A multivariate digamma distribution MDGa $(\boldsymbol{\alpha}, \gamma)$ is obtained by compounding a multivariate logarithmic series MLSr $(\boldsymbol{\theta})$ as shown in Section 3. Consider the case where the compounder degenerates to a distribution on a point, then MLSr $(\boldsymbol{\theta})$ is obtained as a limit of MDGa $(\boldsymbol{\alpha}, \gamma)$. Actually, let α_i's and γ increase infinitely keeping $\theta_i=\alpha_i/(\alpha+\gamma)$, $i=1, \cdots, k$ constant, then MDGa $(\boldsymbol{\alpha}, \gamma)$ probabilities converge to those of MLSr $(\boldsymbol{\theta})$.

The proof is simple, since it is known that the distribution of $\sum X_i$, DGa (α, γ), converges to LSr (θ) if α and γ increase infinitely keeping $\theta=\alpha/(\alpha+\gamma)$ constant, while the conditional distribution of $\boldsymbol{X}$, given $\sum X_i=x$, of (2.4) tends to a multinomial distribution

$$x! \prod_{i=1}^{k} p_i^{x_i}/x_i! , \tag{4.1}$$

if α_i's increase indefinitely keeping $\alpha_i/\alpha=p_i$, $i=1, 2, \cdots, k$, constant. If the parameter x of (4.1) is an LSr(θ) variable, then $\boldsymbol{X}$ follows MLSr(θ) with $\theta_i=\theta p_i$, $i=1, 2, \cdots, k$.

Case 2. If the parameters α_i's tend to zero, then the distribution of $\sum X_i$, DGa(α, γ), converges to a trigamma distribution TGa(γ) of (1.7). While the conditional distribution of $\boldsymbol{X}$, given $\sum X_i=x$, degenerates as follows provided that $\alpha_i/\alpha \to p_i$ $(\alpha_i \to 0)$.

$$(4.2) \qquad \prod_{i=1}^{k} \frac{(\alpha_i, x_i)}{x_i!} \bigg/ \frac{(\alpha, x)}{x!} \to \begin{cases} p_i, & x_i=x \text{ and } x_j=0,\ j\neq i, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the limit distribution degenerates on all axes, and on each axis

$$(4.3) \qquad \Pr[X_i=x]=p_i \Pr[X=x], \qquad x=1, 2, \cdots; \quad i=1, 2, \cdots, k.$$

This is a p_i-portion of the distribution of X, TGa(γ).

This multivariate trigamma distribution will be less useful because it is degenerated. It is, of course, obtained from a zero-truncated multivariate inverse Pólya-Eggenberger distribution. That is, in the limit process (3.1), if both β and α_i's tend to zero keeping $\alpha_i/\alpha=p_i$, then we get a multivariate trigamma distribution.

Another type of multivariate digamma. The limit to a multivariate trigamma distribution raises a question, what will be the limit distribution in (3.1) if α_i's tend to zero while β remains constant. In the univariate case α and β are symmetric and there is no such a question.

Similar calculation as above shows that

$$(4.4) \qquad \lim_{\alpha\to 0} p(\boldsymbol{x}; \alpha\boldsymbol{p}, \beta, \gamma)/(1-p(\boldsymbol{0}; \alpha\boldsymbol{p}, \beta, \gamma))$$
$$= \begin{cases} \dfrac{1}{\psi(\beta+\gamma)-\psi(\gamma)} \dfrac{(\beta)_x}{x(\beta+\gamma)_x} p_i, & x_i=x, \quad x_j=0, \quad j\neq i, \\ 0, & \text{otherwise.} \end{cases}$$

This is also degenerated on all axes, and obtained by splitting DGa(β, γ) into k portions with ratio p_i, $i=1, 2, \cdots, k$, so we have not included it in the family of multivariate digamma distributions. The distribution (4.4) tends to a multivariate trigamma if β also tends to zero.

5. Estimation

For the choice of estimators we need a deep study of candidate estimates. Here we state just possible simple estimates and the ML equation.

Simple estimates. Since the information on $\alpha=\sum\alpha_i$ and γ is concentrated in the observations of $\sum X_i$, let m be a sample mean of $\sum X_i$, p_1 be an observed relative frequency of $\sum X_i=1$, and be p_2 an observed relative frequency of $\sum X_i=2$. As the univariate case

$$\hat{\gamma}=\left(\frac{m}{p_1}-1\right)\Big/\left(\frac{m}{p_1}-\frac{p_1}{p_1-2p_2}\right) \quad \text{and} \quad \hat{\alpha}=\frac{2p_2}{p_1-2p_2}\hat{\gamma}-1$$

are reasonable simple estimators. Further, using a sample mean $m(i)$ of X_i, $i=1, 2, \cdots, k$, we have

$$\hat{\alpha}_i=\hat{\alpha}m(i)/m\ .$$

The ML equation. The maximum likelihood equation depends on the way a sample is observed. Suppose we have a random sample from a digamma population with v_r relative frequencies of observations such that $\sum x_i\geqq r$, $r=1, 2, \cdots$, and with v_{is} relative frequencies of observations such that $x_i\geqq s$, $s=1, 2, \cdots$, $i=1, 2, \cdots, k$, then we get the ML equations

$$\begin{cases} -\dfrac{\psi'(\alpha+\gamma)-\psi'(\gamma)}{\psi(\alpha+\gamma)-\psi(\gamma)}-\displaystyle\sum_{r=1}^{\infty}\frac{v_r}{\alpha+\gamma+r-1}=0\,, \\ \dfrac{\psi'(\gamma)}{\psi(\alpha+\gamma)-\psi(\gamma)}-\displaystyle\sum_{s=1}^{\infty}\frac{v_{is}}{\alpha_i+s-1}=0\,, \qquad i=1, 2, \cdots, k. \end{cases}$$

6. Concluding remark

We have seen that the multivariate extension of the digamma distributions is quite parallel to that of the logarithmic series distributions. The digamma distribution is expected to be useful as a supplement to the logarithmic series when observed frequency data have longer tail and the logarithmic series is not adequate to be fitted. Similarly, the multivariate digamma is expected to be useful as a supplement to the multivariate logarithmic series.

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