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ON A FUNCTIONAL EQUATION IN THE THEORY OF LINEAR STATISTICS

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1. Introduction

Let X_1 and X_2 be independently and identically distributed non-degenerate random variables such that $a_1X_1+a_2X_2$ has the same distribution as X_1 where a_1 and a_2 are fixed real numbers which satisfy $0<|a_1|, |a_2|<1$ and $a_1^2+a_2^2=1$. In [3], [4] Linnik showed that X_1 is then normally distributed with mean zero. This characterization problem leads to the functional equation

$$H(x)=a_1^2H(x+A_1)+a_2^2H(x+A_2)\,, \qquad x\geqq x_0\,, \tag{1}$$

where A_1 and A_2 are positive numbers and H is a non-negative function such that

$$e^{-\lambda x}H(x) \text{ is non-increasing for some } \lambda>0\,. \tag{2}$$

Ramachandran and Rao [5], [6] and Shimizu [7] extended Linnik's result and were led to the functional equation

$$H(x)=\sum_1^\infty p_iH(x+A_i)\,, \qquad x\geqq x_0\,, \tag{3}$$

which was solved under (2) and the following conditions

$$\inf_{1\leqq i<\infty} A_i>0\,, \tag{4}$$

$$\sum_1^\infty p_i=1\,; \qquad p_i\geqq 0,\ \forall i\geqq 1 \tag{5}$$

and

$$1<\sum_1^\infty p_i \exp(\delta A_i)<\infty \qquad \text{for some } \delta>0\,. \tag{6}$$

The methods of proof in [5], [6] and [7] were essentially adaptions of Linnik's original proof and were long and difficult. In [1] Davies and Shimizu gave an elementary and, in the finite case, simple solution of

(3) and were able to improve slightly on the results of Ramachandran and Rao. The functional equation (3) was further generalized by Shimizu [8] who considered the integral equation

$$H(x)=\int_{(0,\infty)} H(x+y)G(dy) \tag{7}$$

where G is a function with total variation of at most 1 and H is non-negative. Shimizu's solution was obtained under even less restrictive conditions with (2) being replaced by

$$\sup_{0\leqq y<\eta} H(x+y)\leqq C(\eta)H(x)<\infty \tag{8}$$

for all $x\geqq x_0$ where $\eta>0$ and $\lim_{\eta\downarrow 0} C(\eta)=1$.

The condition (4) was dropped entirely. It is worth noting that (7) is similar to the integral equation

$$H(x)=\int_{-\infty}^{\infty} H(x-y)G(dy) \tag{9}$$

where G is a distribution function. This functional equation is of some importance in renewal theory. In [2] Feller shows that if G is not concentrated on a lattice then any bounded continuous solution of (9) is given by $H(x)\equiv$constant whilst if G is concentrated on a lattice of span ρ then any bounded solution of (9) satisfies $H(x+\rho)=H(x)$ for all x. Shimizu's solution of (7) is similar to the solution of (9) given by Feller once the boundedness of H has been established. In fact the most difficult part of Shimizu's paper is concerned with precisely this aspect of the problem, his proof being based on an idea used in [1] which in its turn constituted the most difficult part of that paper. Here we give a short and simple proof of the boundedness of any non-negative solution of (7) which satisfies

(10) there exists an $\eta>0$ and a $K>0$ such that

$$\sup_{0\leqq y<\eta} H(x+y)\leqq KH(x) \qquad \text{for all } x\geqq x_0\,,$$

this condition being less restrictive than (8).

2. Statement and proof of Theorem 1

THEOREM 1. *Let G be a distribution function on $(0,\infty)$ and H a finite non-negative measurable function defined for all $x\geqq x_0$ which satisfies the functional inequality*

$$H(x)\geqq\int_{(0,\infty)} H(x+y)G(dy)\,, \qquad \forall\, x\geqq x_0\,. \tag{11}$$

If

$$\int_{x_0}^{x} H(u)du < \infty \qquad \textit{for all } x \geqq x_0 \tag{12}$$

then

$$\int_{x_0}^{\infty} \exp(-\lambda u) H(u) du < \infty$$

for all $\lambda > 0$.

PROOF. As $\int_{(0,\infty)} G(dy) = 1$ there exists for each $\lambda > 0$ a $K = K(\lambda)$ such that

$$\int_{(0,K)} \exp(\lambda y) G(dy) \geqq 1 . \tag{13}$$

From (11) and Fubini's theorem we obtain

$$\begin{aligned} &\int_{x_0}^{x} \exp(-\lambda u) H(u) du \\ &\quad \geqq \int_{(0,K)} \left(\int_{x_0}^{x} \exp(-\lambda u) H(u+y) du \right) G(dy) \\ &\quad = \int_{(0,K)} \exp(\lambda y) \left(\int_{x_0+y}^{x+y} \exp(-\lambda u) H(u) du \right) G(dy) \end{aligned}$$

which implies

$$\begin{aligned} &\int_{(0,K)} \exp(\lambda y) \left(\int_{x_0}^{x+y} \exp(-\lambda u) H(u) du \right) G(dy) - \int_{x_0}^{x} \exp(-\lambda u) H(u) du \\ &\quad \leqq \int_{(0,K)} \exp(\lambda y) \left(\int_{x_0}^{x_0+y} \exp(-\lambda u) H(u) du \right) G(dy) = A_0 < \infty \end{aligned}$$

by (12). This together with (13) yields

$$\int_{(0,K)} \exp(\lambda y) \left(\int_{x}^{x+y} \exp(-\lambda u) H(u) du \right) G(dy) \leqq A_0$$

and if $\eta > 0$ is such that $G(K) - G(\eta) > 0$ we obtain

$$\int_{x}^{x+\eta} \exp(-\lambda u) H(u) du < A_1 < \infty$$

for all $x \geqq x_0$. This implies

$$\int_{x_0+n\eta}^{x_0+(n+1)\eta} \exp(-2\lambda u) H(u) du < A_2 \exp(-\lambda \eta n)$$

for all $n \geqq 0$ and hence

$$\int_{x_0}^{\infty} \exp(-2\lambda u)H(u)du$$
$$=\sum_{n=0}^{\infty} \int_{x_0+n\eta}^{x_0+(n+1)\eta} \exp(-2\lambda u)H(u)du < A_2 \sum_{n=0}^{\infty} \exp(-\lambda\eta n) < \infty .$$

As $\lambda > 0$ was arbitrary this completes the proof of the theorem.

3. Statement and proof of Theorem 2

This section contains the much simplified proof of Lemma 2 of [1].

THEOREM 2. *Let G be a distribution function on* $(0, \infty)$ *such that*

$$1 < \int_{(0,\infty)} \exp(2\delta x)G(dx) < \infty \tag{14}$$

for some $\delta > 0$ and let H be a non-negative function such that $\exp(-\delta x)\cdot H(x)$ *is non-increasing for all $x \geqq x_0$. If H satisfies the function inequality* (10) *then H is bounded.*

PROOF. Arguing as in the proof of Theorem 1 we have for all $x > x_0$

$$\int_{(0,\infty)} \left(\int_x^{x+y} H(u)du\right)G(dy)$$
$$\leqq \int_{(0,\infty)} \left(\int_{x_0}^{x_0+y} H(u)du\right)G(dy)$$
$$\leqq \int_{(0,\infty)} \exp(\delta(x_0+y))\left(\int_{x_0}^{x_0+y} \exp(-\delta u)H(u)du\right)G(dy)$$
$$\leqq \exp(-\delta x_0)H(x_0) \int_{(0,\infty)} y \exp(\delta(x_0+y))G(dy) = A_3 < \infty$$

by (14) and the fact that $\exp(-\delta x)H(x)$ is non-increasing. This implies

$$\int_{(0,K)} \left(\int_x^{x+y} \exp(\delta u)\exp(-\delta u)H(u)du\right)G(dy) \leqq A_3$$

for all $K > 0$ and hence, again as $\exp(-\delta x)H(x)$ is non-increasing, we obtain

$$\int_{(0,K)} \exp(\delta x)\exp(-\delta(x+K))H(x+K)G(dy) \leqq A_3$$

for all $x > x_0$. On choosing K such that $G(K) - G(0) > 0$ we obtain $H(x+K) \leqq A_4 < \infty$ for all $x \geqq x_0$ and, once again as $\exp(-\delta x)H(x)$ is non-increasing, we may conclude that H is bounded.

4. Statement and proof of Theorems 3 and 4

If we combine Theorems 1 and 2 we are able to give an improved version of Shimizu's Theorem 1 of [8] which also has the advantage of a simplified proof.

THEOREM 3. *Let G be a distribution function defined on $(0, \infty)$ and such that*

$$1<\int_{(0,\infty)} \exp(2\delta x)G(dy)<\infty \tag{15}$$

for some $\delta>0$. Then any non-negative measurable function H defined on $(0, \infty)$ which satisfies (10) and the integral inequality

$$H(x)\geqq\int_{(0,\infty)} H(x+y)G(dy)\,, \qquad x\geqq x_0 \tag{16}$$

is bounded.

PROOF. As H is measurable Theorem 1 implies that $\bar{H}(x)=\exp(\delta x)\cdot\int_x^\infty \exp(-\delta u)H(u)du$ is well defined and finite for all $x\geqq x_0$. Furthermore it is easily checked that $\bar{H}$ also satisfies the integral inequality (16). As $\exp(-\delta x)\bar{H}(x)$ is non-increasing it follows from Theorem 2 that $\bar{H}$ is bounded. Thus

$$\exp(\delta x)\int_x^\infty \exp(-\delta u)H(u)du<A<\infty \tag{17}$$

for all $x\geqq x_0$. We define $M(a)=\sup_{x_0\leqq x<a} H(x)$. Then there exists for all a sufficiently large an $x_1(a)$, $x_0\leqq x_1(a)\leqq a-\eta/2$ such that $\sup_{x_1\leqq x\leqq x_1+\eta/2} H(x)=M(a)$ where η is as in (10). If x satisfies $x_1-\eta/2\leqq x\leqq x_1$ we have

$$M(a)=\sup_{x_1\leqq u\leqq x_1+\eta/2} H(u)\leqq\sup_{0\leqq y\leqq\eta} H(x+y)\leqq KH(x)$$

by (10). Here we have assumed that $x_0\leqq x_1-\eta/2$ but if $x_1\leqq x_0+\eta/2$ for all a it follows immediately from (10) that H is bounded. We therefore obtain

$$\begin{aligned}\infty>A&\geqq\exp(\delta(x_1-\eta/2))\int_{x_1-\eta/2}^{x_1}\exp(-\delta u)H(u)du\\&\geqq\exp(\delta(x_1-\eta/2)-\delta x_1)M(a)/K\end{aligned}$$

which yields $M(a)\leqq AK\exp(\delta\eta/2)$ for all sufficiently large a. This implies that H is bounded.

In certain cases which are of special interest in applications we are able to dispense with the condition (10).

THEOREM 4. *Let G be a distribution function on* $(0, \infty)$ *which satisfies* (15) *and let Δ_G be the set of points of increase of G. Then for any non-negative measurable solution H of the integral equation*

$$H(x)=\int_0^\infty H(x+y)G(dy) \tag{18}$$

which also satisfies (12) *we have*

$$H(x+\omega)=H(x) \qquad a.e. \tag{19}$$

for each $\omega \in \Delta_G$.

PROOF. As in the proof of Theorem 3 $\bar{H}(x)=\exp(\delta x)\int_x^\infty \exp(-\delta u) \cdot H(u)du$ is well defined it is easily checked that $\bar{H}(x)$ satisfies (18). Furthermore as in the proof of Theorem 3 $\bar{H}(x)$ is bounded. Theorem 2 of Shimizu [8] implies $\bar{H}(x+\omega)=\bar{H}(x)$ for all $\omega \in \Delta_G$. This yields

$$\begin{aligned}\exp(\delta x)&\int_x^\infty \exp(-\delta u)H(u)du \\ &=\exp(\delta(x+\omega))\int_{x+\omega}^\infty \exp(-\delta u)H(u)du \\ &=\exp(\delta x)\int_x^\infty \exp(-\delta u)H(u+\omega)du\ .\end{aligned}$$

Thus $\int_x^\infty \exp(-\delta u)(H(u)-H(u+\omega))du=0$ for all $x \geqq x_0$ and all $\omega \in \Delta_G$. On differentiating with respect to x the result follows immediately from Lebesgue's theorem. Finally, in order to obtain (19) for all x it is sufficient to assume that H satisfies some continuity condition such as continuity from the right. It may be easily shown by means of a counterexample that (19) does not in general hold for all x.

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