Ann. Inst. Statist. Math.
30 (1978), Part B, 479-498

A SCHEME OF ADAPTIVE CONTROL

MAKIO ISHIGURO

(Received Oct. 24, 1977; revised Sept. 7, 1978)

Abstract

A new scheme of adaptive control is proposed. This scheme does not require a priori knowledge of the structure of the plant to be controlled. The principal part of the scheme is a procedure which decides the order of the model of the plant. A criterion for the order determination is developed. Using this criterion, we can decide whether to keep the current controller or to adopt a new controller based on the information gathered during the operation of the system. The effectiveness of the scheme is illustrated by a numerical example.

1. Introduction

There is a vast literature about adaptive control systems (Wittenmark [10]). The widely shared assumption there is that the structure, or the parametrization of the plant to be controlled is known. For example, Alster and Bélanger [3] discussed dual control of a single-input-single-output plant:

$$y_k=\sum_{m=1}^{M} a_m(k)y_{k-m}+\sum_{l=1}^{L} b_l(k)u_{k-l}+\varepsilon_k\ , \tag{1}$$

where y_k is the output, u_k is the input and ε_k is a zero-mean white Gaussian sequence. They assumed that orders M and L are known. This assumption limits the applicability of the result to a rather restricted region.

Considering that the success of the (non-adaptive) optimal control theory in the real plant control is brought when a practical method to identify the order of the plant is developped (for example, Otomo et al. [7] and Otsu et al. [8]), it is highly desirable to realize an adaptive control system which has ability of order identification.

The purpose of this paper is to present an adaptive control scheme which determines the order of the model of the plant analyzing the input-output data of the plant. In next three sections we develop a

criterion for the order determination. The adaptive control scheme based on the criterion is given in Section 5.

2. Statement of the problem

We assume that the plant to be controlled can be expressed by

$$y_k=\sum_{m=1}^{M^*} a_m^* y_{k-m}+\sum_{l=1}^{L^*} b_l^* u_{k-l}+\varepsilon_k\ , \tag{2}$$

where ε_k is a zero-mean white Gaussian sequence of the variance σ^{*2}. For simplicity, it is assumed that the parameters of the plant is time-invariant. We replace the assumption that M^* and L^* are known with a weaker assumption that both of M^* and L^* are bounded above by a known constant M_0. The problem is to design a control system which minimizes the performance index:

$$J=\lim_{K\to\infty} \mathrm{E}\left\{\frac{1}{K}\sum_{k=1}^{K}(w_y y_k^2+w_u u_{k-1}^2)\right\}\ , \tag{3}$$

where w_y and w_u are given positive constants. It is assumed that only available information is the past observation $\{(u_{k-1}, y_k)\,|\,-T+1\leqq k\leqq 0\}$.

If the dynamics (2) is known, (3) is minimized by a control system

$$u_k=\sum_{m=1}^{M^*} c_m^* y_{k-m+1}+\sum_{l=1}^{L^*-1} d_l u_{k-l}\ , \tag{4}$$

where $\underline{c}^*=(c_1^*, c_2^*, \cdots, c_{M^*}^*)$ and $\underline{d}^*=(d_1^*, d_2^*, \cdots, d_{L^*-1}^*)$ are constant gain defined by the equations (see the Appendix):

$$\begin{aligned} c_m^* &= \begin{cases} g_1^* & (m=1) \\ \sum_{j=m}^{M^*} a_j^* g_{j-m+2}^* & (m=2, 3, \cdots, M^*) \end{cases} \\ d_l^* &= \sum_{j=l+1}^{L^*} b_j^* g_{j-l+1}^* \qquad (l=1, 2, \cdots, L^*-1) \end{aligned} \tag{5}$$

$\underline{g}_\infty^*=(g_1^*, g_2^*, \cdots, g_{M_0}^*)'$ is obtained as a limit of $\underline{g}_i$ which is defined by the iteration $(i=1, 2, 3, \cdots)$:

$$\begin{aligned} S_i &= (w_u+\tilde{\underline{b}}^{*\prime} P_{i-1}\tilde{\underline{b}}^*)^{-1} \\ T_i &= P_{i-1}-P_{i-1}\tilde{\underline{b}}^* S_i \tilde{\underline{b}}^{*\prime} P_{i-1} \\ P_i &= \Phi^{*\prime} T_i \Phi^* + Q \\ \underline{g}_i' &= -S_i \tilde{\underline{b}}^{*\prime} P_{i-1}\Phi^*\ , \end{aligned} \tag{6}$$

where $(')$ denotes the transpose. $M_0\times M_0$-matrices Φ^*, Q and M_0-vector

$\tilde{\underline{b}}^*$ are defined as follows.

$$\Phi^*=\begin{bmatrix} a_1^*, & 1, & 0, & \cdots & , & 0 \\ a_2^*, & 0, & 1, & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ a_{M^*}^*, & & & & 1, & 0 \\ 0, & & & & & 1 \\ \vdots & & & & & \\ 0, & 0, & & \cdots & , & 0 \end{bmatrix}$$

$$Q=\begin{bmatrix} w_x & & \\ & 0 & \\ & & \end{bmatrix}$$

$$\tilde{\underline{b}}^*=(b_1^*, b_2^*, \cdots, \underline{b}_{L^*}^*, 0, \cdots, 0)' .$$

The initial value P_0 is set equal to Q.

Now let us consider the situation where the orders of the plant is unknown. If M and L are specified, we can design a certainty equivalence control (Aoki [2]) by the procedure:

Step 1: Fit the model:

$$y_k=\sum_{m=1}^{M} a_m y_{k-m}+\sum_{l=1}^{L} b_l u_{k-1}+\varepsilon_k \tag{7}$$

to the data by the maximum likelihood method.

Step 2: Design a control system:

$$u_k=\sum_{m=1}^{M} \hat{c}_m y_{k-m+1}+\sum_{l=1}^{L-1} \hat{d}_l u_{k-1} \tag{8}$$

assuming that the estimated parameters of the plant are true.

Step 1 is done as follows. Assuming that ε_k is a zero-mean white Gaussian sequence of variance $\sigma_{M,L}^2$, the log likelihood of the model (7) with respect to the data is approximately given by

$$l(\underline{\omega}_{M,L}, \sigma_{M,L}^2)=-\frac{T}{2}\log \sigma_{M,L}^2-\frac{1}{2\sigma_{M,L}^2}\sum_{k=-T+1}^{0}(y_k-\underline{\omega}_{M,L}'\underline{z}_{k-1})^2 , \tag{9}$$

where $\underline{\omega}_{M,L}$ and $\underline{z}_{k-1}$ are $2M_0$-vectors defined by

$$\underline{\omega}_{M,L}=(a_1, a_2, \cdots, a_M, 0, \cdots, 0, \ b_1, b_2, \cdots, b_L, 0, \cdots, 0)'$$

and

$$\underbrace{\qquad\qquad}_{M_0} \qquad \underbrace{\qquad\qquad}_{M_0}$$

$$\underline{z}_{k-1}=(y_{k-1}, y_{k-2}, \cdots, y_{k-M_0}, \ u_{k-1}, u_{k-2}, \cdots, u_{k-M_0})'$$

respectively. Note that $y_{-T}, y_{-T-1}, \cdots, y_{-T-M_0}$, $u_{-T}, u_{-T-1}, \cdots, u_{-T-M_0+1}$

and u_{-T-M_0} are not observed, and assumed to be zero. Maximum likelihood estimates $\hat{\underline{\omega}}_{M,L}$ and $\hat{\sigma}^2_{M,L}$ are obtained by solving the likelihood equations:

$$\mathrm{E}_{M,L}\, A\, \mathrm{E}'_{M,L}\, \hat{\underline{\omega}}_{M,L} = \mathrm{E}_{M,L}\, \underline{\eta}$$

$$\hat{\sigma}^2_{M,L} = \frac{1}{T}(r - \hat{\underline{\omega}}'\underline{\eta})$$

which are derived from (9). $\mathrm{E}_{M,L}$, A, $\underline{\eta}$ and r defined by

$$\mathrm{E}_{M,L} = \begin{array}{c} \begin{array}{cc} M_0 & M_0 \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{array}\right] \end{array} \begin{array}{l} M \\ \\ \\ L \end{array}$$

$$A = \sum_{k=-T+1}^{0} \underline{z}_{k-1}\underline{z}'_{k-1}\,,$$

$$\eta = \sum_{k=-T+1}^{0} y_k \underline{z}_{k-1}$$

and

$$r = \sum_{k=-T+1}^{0} y_k^2$$

respectively. If $M \geqq M^*$ and $L \geqq L^*$, error covariance matrix $V_{M,L}$ of the estimate $\hat{\underline{\omega}}_{M,L}$ can be estimated. Especially, V_{M_0,M_0} is approximately given by

$$\hat{V}_{M_0,M_0} = \hat{\sigma}^2_{M_0,M_0} A^{-1}\,. \tag{10}$$

It can be also proved that $\hat{\sigma}^2_{M_0,M_0}$ and $\hat{\underline{\omega}}_{M_0,M_0}$ are asymptotically independent each other.

We will denote the fitted model (7) and the control system (8) by $M(\hat{\underline{\omega}}_{M,L}, \hat{\sigma}^2_{M,L})$ and $C(\hat{\underline{\theta}}_{M,L})$, respectively, where $\hat{\underline{\theta}}_{M,L}$ is a $2M_0$-vector defined by $\hat{\underline{\theta}}_{M,L} = (\hat{c}_1, \hat{c}_2, \cdots, \hat{c}_M, 0, \cdots, 0, \hat{d}_1, \hat{d}_2, \cdots, \hat{d}_{L-1}, 0, \cdots, 0)$. Applying the above procedure to every possible pair of M and L, we obtain a set of control systems $\{C(\hat{\underline{\theta}}_{M,L}) \mid 1 \leqq M \leqq M_0,\ 1 \leqq L \leqq M_0\}$. There arises a problem of how to select the best controller. An answer to this problem is given in the next section.

3. Criterion

The performance of each controller $C(\hat{\underline{\theta}}_{M,L})$ is a function of its parameters. Let this function be denoted by $J(\hat{\underline{\theta}}_{M,L}: \underline{\omega}^*, \sigma^*)$, where $\underline{\omega}^*$ is a $2M_0$-vector defined by

$$\underline{\omega}^* = (\overbrace{a_1^*, a_2^*, \cdots, a_{M^*}^*, 0, \cdots, 0}^{M_0}, \overbrace{b_1^*, b_2^*, \cdots, b_{L^*}^*, 0, \cdots, 0}^{M_0}) .$$

Apparently, the best choice is the one which minimizes $J(\hat{\underline{\theta}}_{M,L}; \underline{\omega}^*, \sigma^*)$. However $\underline{\omega}^*$ and σ^* are unknown and we must base the decision on some estimate of the value. One possibility is to replace $\underline{\omega}^*$ and σ^* by the estimates $\hat{\underline{\omega}}_{M_0,M_0}$ and $\hat{\sigma}_{M_0,M_0}$, respectively.

Let $\underline{\theta}^*_{M,L} \in \Theta_{M,L}$ be defined by the equation:

$$J(\underline{\theta}^*_{M,L}; \underline{\omega}^*, \sigma^*) = \min_{\underline{\theta} \in \Theta_{M,L}} J(\underline{\theta}; \underline{\omega}^*, \sigma^*) , \tag{11}$$

where $\Theta_{M,L}$ is the set of $2M_0$-vectors defined by

$$\Theta_{M,L} = \{(\theta_1, \theta_2, \cdots, \theta_{2M_0}) \mid \theta_{M+1} = \cdots = \theta_{M_0} = 0,\ \theta_{M_0+L} = \cdots = \theta_{2M_0} = 0\} .$$

Then, taking up to the 2nd order terms of the Taylor series expansion, $J(\hat{\underline{\theta}}_{M,L}; \hat{\underline{\omega}}_{M_0,M_0}, \hat{\sigma}_{M_0,M_0})$ can be approximated by

$$\begin{aligned}
(12)\quad & J(\hat{\underline{\theta}}_{M,L}; \hat{\underline{\omega}}_{M_0,M_0}, \hat{\sigma}_{M_0,M_0}) \\
&\simeq J(\underline{\theta}^*_{M,L}; \underline{\omega}^*, \sigma^*) + \frac{\partial J}{\partial \underline{\theta}}(\hat{\underline{\theta}}_{M,L} - \underline{\theta}^*_{M,L}) + \frac{\partial J}{\partial \underline{\omega}}(\hat{\underline{\omega}}_{M_0,M_0} - \underline{\omega}^*) \\
&\quad + \frac{\partial J}{\partial \sigma}(\hat{\sigma}_{M_0,M_0} - \sigma^*) + \frac{1}{2}(\hat{\underline{\theta}}_{M,L} - \underline{\theta}^*_{M,L})' \frac{\partial^2 J}{\partial \underline{\theta}^2}(\hat{\underline{\theta}}_{M,L} - \underline{\theta}^*_{M,L}) \\
&\quad + \frac{1}{2}(\hat{\underline{\omega}}_{M_0,M_0} - \underline{\omega}^*)' \frac{\partial^2 J}{\partial \underline{\omega}^2}(\hat{\underline{\omega}}_{M_0,M_0} - \underline{\omega}^*) \\
&\quad + \frac{1}{2}(\hat{\sigma}_{M_0,M_0} - \sigma^*)' \frac{\partial^2 J}{\partial \sigma^2}(\hat{\sigma}_{M_0,M_0} - \sigma^*) \\
&\quad + (\hat{\underline{\omega}}_{M_0,M_0} - \underline{\omega}^*)' \frac{\partial^2 J}{\partial \underline{\omega} \partial \underline{\theta}}(\hat{\underline{\theta}}_{M,L} - \underline{\theta}^*_{M,L}) \\
&\quad + (\hat{\sigma}_{M_0,M_0} - \sigma^*) \frac{\partial^2 J}{\partial \sigma \partial \underline{\theta}}(\hat{\underline{\theta}}_{M,L} - \underline{\theta}^*_{M,L}) \\
&\quad + (\hat{\sigma}_{M_0,M_0} - \sigma^*) \frac{\partial^2 J}{\partial \sigma \partial \underline{\omega}}(\hat{\underline{\omega}}_{M_0,M_0} - \underline{\omega}^*) ,
\end{aligned}$$

where $\partial J/\partial \underline{\theta}$, $\partial J/\partial \underline{\omega}$, $\partial^2 J/\partial \sigma \partial \underline{\theta}$ and $\partial^2 J/\partial \sigma \partial \underline{\omega}$ are row vectors whose jth elements are given by $\partial J/\partial \theta_j(\underline{\theta}^*_{M,L}; \underline{\omega}^*, \sigma^*)$, $\partial J/\partial \omega_j(\underline{\theta}^*_{M,L}; \underline{\omega}^*, \sigma^*)$, $\partial^2 J/\partial \sigma \partial \theta_j(\underline{\theta}^*_{M,L}; \omega^*, \sigma^*)$ and $\partial^2 J/\partial \sigma \partial \omega_j(\underline{\theta}^*_{M,L}; \omega^*, \rho^*)$ respectively. $\partial^2 J/\partial \underline{\theta}^2$, $\partial^2 J/\partial \underline{\omega}^2$ and $\partial^2 J/\partial \underline{\omega} \partial \underline{\theta}$ are $(2M_0 \times 2M_0)$-matrices whose i-j elements are given by

$$\frac{\partial^2 J}{\partial\theta_i\partial\theta_j}(\underline{\theta}^*_{M,L};\underline{\omega}^*,\sigma^*)\,,\quad \frac{\partial^2 J}{\partial\omega_i\partial\theta_j}(\underline{\theta}^*_{M,L};\underline{\omega}^*,\sigma^*)\quad\text{and}\quad \frac{\partial^2 J}{\partial\omega_i\partial\omega_j}(\underline{\theta}^*_{M,L};\underline{\omega}^*,\sigma^*)$$

respectively. Similarly, $J(\hat{\underline{\theta}}_{M,L};\underline{\omega}^*,\sigma^*)$ is approximated by

$$J(\hat{\underline{\theta}}_{M,L};\underline{\omega}^*,\sigma^*)\simeq J(\underline{\theta}^*_{M,L};\underline{\omega}^*,\sigma^*)+\frac{\partial J}{\partial\underline{\theta}}(\hat{\underline{\theta}}_{M,L}-\underline{\theta}^*)+\frac{1}{2}(\hat{\underline{\theta}}_{M,L}-\underline{\theta}^*_{M,L})'\frac{\partial^2 J}{\partial\theta^2}(\hat{\underline{\theta}}_{M,L}-\underline{\theta}^*_{M,L})\,. \tag{13}$$

Subtracting (13) from (12), we get

$$\begin{aligned}J(\hat{\underline{\theta}}_{M,L};\hat{\underline{\omega}}_{M_0,M_0},\hat{\sigma}_{M_0,M_0})&-J(\hat{\underline{\theta}}_{M,L};\underline{\omega}^*,\sigma^*)\\ \simeq\frac{\partial J}{\partial\underline{\omega}}(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)&+\frac{\partial J}{\partial\sigma}(\hat{\sigma}_{M_0,M_0}-\sigma^*)\\ &+\frac{1}{2}(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)'\frac{\partial^2 J}{\partial\underline{\omega}^2}(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)\\ &+\frac{1}{2}(\hat{\sigma}_{M_0,M_0}-\sigma^*)'\frac{\partial^2 J}{\partial\sigma^2}(\hat{\sigma}_{M_0,M_0}-\sigma^*)\\ &+(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)'\frac{\partial^2 J}{\partial\underline{\omega}\partial\underline{\theta}}(\hat{\underline{\theta}}_{M,L}-\underline{\theta}^*_{M,L})\\ &+(\hat{\sigma}_{M_0,M_0}-\sigma^*)\frac{\partial^2 J}{\partial\sigma\partial\underline{\theta}}(\hat{\underline{\theta}}_{M,L}-\underline{\theta}^*_{M,L})\\ &+(\hat{\sigma}_{M_0,M_0}-\sigma^*)\frac{\partial^2 J}{\partial\sigma\partial\underline{\omega}}(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)\,.\end{aligned} \tag{14}$$

We assume here that $\hat{\underline{\theta}}_{M,L}$ satisfies the equation:

$$J(\hat{\underline{\theta}}_{M,L};\underline{\omega}_{M_0,M_0},\hat{\sigma}_{M_0,M_0})=\min_{\underline{\theta}\in\Theta_{M,L}}J(\underline{\theta};\hat{\underline{\omega}}_{M_0,M_0},\hat{\sigma}_{M_0,M_0})\,.$$

Then, $\hat{\underline{\theta}}_{M,L}$ has to satisfy the $(M+L-1)$ equations:

$$\frac{\partial J}{\partial\theta_i}(\hat{\underline{\theta}}_{M,L};\hat{\underline{\omega}}_{M_0,M_0},\hat{\sigma}_{M_0,M_0})=0\qquad(1\leqq i\leqq M,\ M_0+1\leqq i\leqq M_0+L-1)\,. \tag{15}$$

Approximating the left-hand side of (15), we get

$$\frac{\partial^2 J}{\partial\theta_i\partial\underline{\theta}}(\hat{\underline{\theta}}_{M,L}-\underline{\theta}^*_{M,L})+\frac{\partial^2 J}{\partial\theta_i\partial\underline{\omega}}(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)=0\qquad(1\leqq i\leqq M,\ M_0+1\leqq i\leqq M_0+L-1)\,, \tag{16}$$

where $\partial^2 J/\partial\theta_i\partial\underline{\theta}$ and $\partial^2 J/\partial\theta_i\partial\underline{\omega}$ are row vectors whose jth elements are $\partial^2 J/\partial\theta_i\partial\theta_j(\underline{\theta}^*_{M,L};\underline{\omega}^*,\sigma^*)$ and $\partial^2 J/\partial\theta_i\partial\omega_j(\underline{\theta}^*_{M,L};\underline{\omega}^*,\sigma^*)$ respectively. Note that the equations:

$$\frac{\partial^2 J}{\partial\theta_i\partial\sigma}(\underline{\theta}^*_{M,L};\underline{\omega}^*,\sigma^*)=0 \qquad (1\leqq i\leqq M,\ M_0+1\leqq i\leqq M_0+L-1)$$

are derived directly from the assumption (11). Since $(\hat{\underline{\theta}}_{M,L}-\underline{\theta}^*_{M,L})\in\Theta_{M,L}$, (16) is equivalent to the equation:

$$\text{(17)}\qquad \mathrm{E}_{M,L-1}\frac{\partial^2 J}{\partial\underline{\theta}^2}\mathrm{E}'_{M,L-1}\mathrm{E}_{M,L-1}(\hat{\underline{\theta}}_{M,L}-\underline{\theta}^*_{M,L}) + \mathrm{E}_{M,L-1}\left(\frac{\partial^2 J}{\partial\underline{\omega}\partial\underline{\theta}}\right)'(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)=\underline{0}\ .$$

Solving the equation (17), we get

$$\mathrm{E}_{M,L-1}(\hat{\underline{\theta}}_{M,L}-\underline{\theta}^*_{M,L}) = -\left(\mathrm{E}_{M,L-1}\frac{\partial^2 J}{\partial\underline{\theta}^2}\mathrm{E}'_{M,L-1}\right)^{-1}\mathrm{E}_{M,L-1}\left(\frac{\partial^2 J}{\partial\underline{\omega}\partial\underline{\theta}}\right)'(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)\ .$$

The fact that $(\hat{\underline{\theta}}_{M,L}-\underline{\theta}^*_{M,L})\in\Theta_{M,L}$, implies that

$$\text{(18)}\qquad \hat{\underline{\theta}}_{M,L}-\underline{\theta}^*_{M,L}=-\mathrm{E}'_{M,L-1}\left(\mathrm{E}_{M,L-1}\frac{\partial^2 J}{\partial\underline{\theta}^2}\mathrm{E}'_{M,L-1}\right)^{-1} \cdot \mathrm{E}_{M,L-1}\left(\frac{\partial^2 J}{\partial\underline{\omega}\partial\underline{\theta}}\right)'(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)\ .$$

Substituting (18) into (14), we obtain

$$\begin{aligned}\text{(19)}\qquad & J(\hat{\underline{\theta}}_{M,L};\hat{\underline{\omega}}_{M_0,M_0},\hat{\sigma}_{M_0,M_0})-J(\hat{\underline{\theta}}_{M,L};\underline{\omega}^*,\sigma^*)\\ &\simeq\frac{\partial J}{\partial\underline{\omega}}(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)+\frac{\partial J}{\partial\sigma}(\hat{\sigma}_{M_0,M_0}-\sigma^*)\\ &\quad+\frac{1}{2}(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)'\frac{\partial^2 J}{\partial\underline{\omega}^2}(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)\\ &\quad+\frac{1}{2}(\hat{\sigma}_{M_0,M_0}-\sigma^*)\frac{\partial^2 J}{\partial\sigma^2}(\hat{\sigma}_{M_0,M_0}-\sigma^*)\\ &\quad-(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)'\left(\frac{\partial^2 J}{\partial\underline{\omega}\partial\underline{\theta}}\right)\mathrm{E}'_{M,L-1}\left(\mathrm{E}_{M,L-1}\frac{\partial^2 J}{\partial\underline{\theta}^2}\mathrm{E}'_{M,L-1}\right)^{-1}\\ &\quad\cdot\mathrm{E}_{M,L-1}\left(\frac{\partial^2 J}{\partial\underline{\omega}\partial\underline{\theta}}\right)'(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)\\ &\quad+(\hat{\sigma}_{M_0,M_0}-\sigma^*)\frac{\partial^2 J}{\partial\sigma\partial\underline{\omega}}(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)\ .\end{aligned}$$

Taking expectation of (19), we get the asymptotic equation:

$$\begin{aligned}\text{(20)}\qquad & \mathrm{E}\{J(\hat{\underline{\theta}}_{M,L};\underline{\omega}_{M_0,M_0},\hat{\sigma}_{M_0,M_0})\}-\mathrm{E}\{J(\hat{\underline{\theta}}_{M,L};\underline{\omega}^*,\sigma^*)\}\\ &\simeq\frac{1}{2}\operatorname{tr}\left[\frac{\partial^2 J}{\partial\underline{\omega}^2}V^*\right]+\frac{1}{2}\frac{\partial^2 J}{\partial\sigma^2}\mathrm{E}\{(\hat{\sigma}_{M_0,M_0}-\sigma^*)^2\}\end{aligned}$$

$$-\mathrm{tr}\left[\left(\frac{\partial^2 J}{\partial\underline{\omega}\partial\underline{\theta}}\right)\mathrm{E}'_{M,L-1}\left(\mathrm{E}_{M,L-1}\frac{\partial^2 J}{\partial\underline{\theta}^2}\mathrm{E}'_{M,L-1}\right)^{-1}\left(\frac{\partial^2 J}{\partial\underline{\omega}\partial\underline{\theta}}\right)'V^*\right]$$

where it is assumed that $\hat{\underline{\omega}}_{M_0,M_0}$ and $\hat{\sigma}_{M_0,M_0}$ are asymptotically consistent estimates of $\underline{\omega}^*$ and σ^* respectively, and independent of each other. V^* is the error covariance matrix which is by definition $V^*=\mathrm{E}\{(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)'(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)$.

(20) shows that $J(\hat{\underline{\theta}}_{M,L};\hat{\underline{\omega}}_{M_0,M_0},\hat{\sigma}_{M_0,M_0})$ is biased as an estimate of $J(\hat{\underline{\theta}}_{M,L};\underline{\omega}^*,\sigma^*)$. But, if a good estimate $B_{M,L}$ of the last terms of the right-hand side of (20) is available, a criterion defined by

$$\tilde{J}(\hat{\underline{\theta}}_{M,L};\hat{\underline{\omega}}_{M_0,M_0},\hat{\sigma}_{M_0,M_0})=J(\hat{\underline{\theta}}_{M,L};\hat{\underline{\omega}}_{M_0,M_0},\hat{\sigma}_{M_0,M_0})-B_{M,L} \tag{21}$$

will serve as an useful estimate of $J(\hat{\underline{\theta}}_{M,L};\underline{\omega}^*,\sigma^*)$. Note that the first and the second term of the right-hand side are common for all combinations of M and L, and need not be evaluated for the purpose of the comparison of the controller performance. We may use $\widehat{\partial^2 J/\partial\underline{\omega}\partial\underline{\theta}}=\partial^2 J/\partial\underline{\omega}\partial\underline{\theta}(\hat{\underline{\theta}}_{M_0,M_0};\hat{\underline{\omega}}_{M_0,M_0},\hat{\sigma}_{M_0,M_0})$, $\widehat{\partial^2 J/\partial\underline{\theta}^2}=\partial^2 J/\partial\underline{\theta}^2(\hat{\underline{\theta}}_{M_0,M_0};\hat{\underline{\omega}}_{M_0,M_0},\hat{\sigma}_{M_0,M_0})$ and $\hat{V}_{M_0,M_0}$, which is defined by (10) as estimates of $\partial^2 J/\partial\underline{\omega}\partial\underline{\theta}(\underline{\theta}^*_{M,L};\underline{\omega}^*,\sigma^*)$, $\partial^2 J/\partial\underline{\theta}^2\cdot(\underline{\theta}^*_{M,L};\underline{\omega}^*,\sigma^*)$ and V^* respectively, provided that the data length is sufficiently large. Thus we propose the criterion

$$\begin{aligned}\tilde{J}(\hat{\underline{\theta}}_{M,L};&\hat{\underline{\omega}}_{M_0,M_0},\hat{\sigma}_{M_0,M_0})\\ &=J(\hat{\underline{\theta}}_{M,L};\hat{\underline{\omega}}_{M_0,M_0},\hat{\sigma}_{M_0,M_0})+\mathrm{tr}\left[\left(\frac{\widehat{\partial^2 J}}{\partial\underline{\omega}\partial\underline{\theta}}\right)\mathrm{E}'_{M,L-1}\right.\\ &\quad\left.\cdot\left(\mathrm{E}_{M,L-1}\frac{\widehat{\partial^2 J}}{\partial\underline{\theta}^2}\mathrm{E}'_{M,L-1}\right)^{-1}\mathrm{E}_{M,L-1}\left(\frac{\widehat{\partial^2 J}}{\partial\underline{\omega}\partial\underline{\theta}}\right)'\hat{V}_{M_0,M_0}\right]\end{aligned} \tag{21}$$

as an answer to the problem. We choose the control system $C(\hat{\underline{\theta}}_{M,L})$ which minimizes (21) as the best one. We will show the numerical procedures to calculate the criterion in the next section.

4. Numerical procedure

Since the plant and the controller are assumed to be time-invariant, the performance index (3) has an equivalent expression:

$$J=\lim_{k\to\infty}\mathrm{E}\{w_y y_k^2+w_u u_k^2\}\ . \tag{22}$$

Then, $J(\hat{\underline{\theta}}_{M,L};\hat{\underline{\omega}}_{M_0,M_0},\hat{\sigma}_{M_0,M_0})$ can be evaluated if $\lim_{k\to\infty}\mathrm{E}\{y_k^2\}$ and $\lim_{k\to\infty}\mathrm{E}\{u_k^2\}$ are calculated. The system which is composed of the plant:

$$y_k=\sum_{m=1}^{M_0}\hat{a}_m y_{k-m}+\sum_{l=1}^{M_0}\hat{b}_l u_{k-l}+\varepsilon_i$$

and the controller:

$$u_k=\sum_{m=1}^{M}\hat{c}_m y_{k-m+1}+\sum_{l=1}^{L-1}\hat{d}_l u_{k-l}$$

is equivalent to the system:

$$\underline{z}_k=\psi_0\left\{\psi_1\underline{z}_{k-1}+\begin{pmatrix}\varepsilon_k\\0\\\vdots\\0\end{pmatrix}\right\}, \tag{23}$$

where $\underline{z}_k$ is a $2M_0$-vector defined in Section 2, and ψ_0 and ψ_1 are $(2M_0\times 2M_0)$-matrices defined by

$$\psi_0=\left(\begin{array}{cccc|cccc}1 & & & & & & & \\ & \cdot & & & & & 0 & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & 1 & & & & \\ \hline \hat{c}_1, & \hat{c}_2, & \cdots, & \hat{c}_M, & \hat{d}_1, & \hat{d}_2, & \cdots, & \hat{d}_{L-1}, \\ & & & & 1 & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ & & & & & & & 1, 0\end{array}\right) \tag{24}$$

and

$$\psi_1=\left(\begin{array}{cccc|cccc}\hat{a}_1, & \hat{a}_2, & \cdots, & \hat{a}_{M_0} & \hat{b}_1, & \hat{b}_2, & \cdots, & \hat{b}_{M_0} \\ 1 & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & 1, 0 & & & & \\ \hline & & & & 1 & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ & & & & & & & \cdot \\ & & & & & & & 1\end{array}\right) \tag{25}$$

respectively. Denoting $\psi_0\psi_1$ by ψ, (23) is expressed by

$$\underline{z}_k=\psi\underline{z}_{k-1}+\begin{pmatrix}\varepsilon_k\\0\\\vdots\\0\\c_1\varepsilon_k\\0\\\vdots\\0\end{pmatrix}. \tag{26}$$

From the assumption that ε_k is a zero-mean white Gaussian sequence whose variance is $\hat{\sigma}^2_{M_0, M_0}$, we obtain

$$\mathrm{E}\{\underline{z}_k\underline{z}_k'\} = \phi\,\mathrm{E}\{\underline{z}_{k-1}\underline{z}_{k-1}'\}\phi' + S\,, \tag{27}$$

where S is a $(2M_0 \times 2M_0)$-matrix defined by

$$S = \hat{\sigma}^2_{M_0, M_0}\left(\begin{array}{c|c} 1 & \hat{c}_1 \\ \hline \hat{c}_1 & \hat{c}_1^2 \end{array}\right).$$

If the limit $X = \lim_{k\to\infty} \mathrm{E}\{\underline{z}_k\underline{z}_k'\}$ exists, X is given as the solution of the equation

$$X = \phi X \phi' + S\,. \tag{28}$$

There are several methods to solve the equation of type (28), but the algorithm by Kitagawa [6] is especially suited for the present use. If X is obtained, $J(\hat{\underline{\theta}}_{M,L}\,;\,\hat{\underline{\omega}}_{M_0, M_0}, \hat{\sigma}_{M_0, M_0})$ is calculated by the equation

$$J(\hat{\underline{\theta}}_{M,L}\,;\,\hat{\underline{\omega}}_{M_0, M_0}, \hat{\sigma}_{M_0, M_0}) = w_y X_{11} + w_u X_{M_0+1, M_0+1}\,, \tag{29}$$

where X_{ij} denotes the i-j element of the matrix X. The derivatives of J can also be calculated. For example, $\widehat{\partial J/\partial\theta_1} = \partial J/\partial\theta_1(\hat{\underline{\theta}}_{M,L}\,;\,\hat{\underline{\omega}}_{M_0, M_0}, \hat{\sigma}_{M_0, M_0})$ is obtained as follows. Let $Y = \partial X/\partial\theta_1$, where $\partial/\partial\theta_1$ denotes the componentwise operation. Then Y is obtained by solving the equation

$$Y = \phi Y \phi' + \frac{\partial\phi}{\partial\theta_1} X \phi' + \phi X \left(\frac{\partial\phi}{\partial\theta_1}\right)' + \frac{\partial S}{\partial\theta_1}\,,$$

where X is the solution of (28). $\partial J/\partial\theta_1$ is given by

$$\frac{\partial J}{\partial\theta_1} = w_y Y_{11} + w_u Y_{M_0+1, M_0+1}\,.$$

2nd order derivatives such as $\widehat{\partial^2 J/\partial\theta_1\partial\theta_2}$ are calculated from Y and X in a similar fashion.

The order determination procedure is summarized as follows.

Step 1. Fit the model $M(\hat{\underline{\omega}}_{M_0, M_0}, \hat{\sigma}^2_{M_0, M_0})$ to the data $\{(u_{k-1}, y_k) \mid -T+1 \leqq k \leqq 0\}$ and compute the estimate of the error covariance matrix $\hat{V}_{M_0, M_0}$.

Step 2. Design the controller $C(\hat{\underline{\theta}}_{M_0, M_0})$.

Step 3. Compute the matrices

$$\widehat{\frac{\partial^2 J}{\partial\underline{\omega}\partial\underline{\theta}}} = \frac{\partial^2 J}{\partial\underline{\omega}\partial\underline{\theta}}(\hat{\underline{\theta}}_{M_0, M_0}\,;\,\hat{\underline{\omega}}_{M_0, M_0}, \hat{\sigma}_{M_0, M_0})$$

and

$$\widehat{\frac{\partial^2 J}{\partial \underline{\theta}^2}} = \frac{\partial^2 J}{\partial \underline{\theta}^2}(\hat{\underline{\theta}}_{M_0, M_0}; \hat{\underline{\omega}}_{M_0, M_0}, \hat{\sigma}_{M_0, M_0})$$

Step 4. For each pair of M and L,

a) Fit the model $M(\hat{\underline{\omega}}_{M,L}, \hat{\sigma}^2_{M,L})$ to the same data

b) Design the controller $C(\hat{\underline{\theta}}_{M,L})$

c) Compute the function $J(\hat{\underline{\theta}}_{M,L}; \hat{\underline{\omega}}_{M_0, M_0}, \hat{\sigma}_{M_0, M_0})$

d) Evaluate the criterion $\tilde{J}(\hat{\underline{\theta}}_{M,L}; \hat{\underline{\omega}}_{M_0, M_0}, \hat{\sigma}_{M_0, M_0})$.

Step 5. Select the pair (M, L) which minimizes the criterion $\tilde{J}(\hat{\underline{\theta}}_{M,L}; \hat{\underline{\omega}}_{M_0, M_0}, \hat{\sigma}_{M_0, M_0})$.

5. A scheme of adaptive control

The order determination procedure in the previous section is made adaptive by applying the procedure every T stages. But there are two problems to be considered. The first problem concerns with the problem of closed-loop operating data (Box and MacGregor [5]). Suppose that data $\{(u_{k-1}, y_k) | 1 \leqq k \leqq T\}$ is obtained from the plant (2) which is controlled by the controller (8). Then the matrix:

$$A = \sum_{k=1}^{T} \underline{z}_{k-1} \underline{z}'_{k-1}$$

becomes singular and the model fitting in the procedure is doomed to fail. This difficulty is avoided by adding a white Gaussian noise to the control input. How to decide the intensity of the noise is a difficult problem itself. But we can estimate the performance index of the controller:

$$u_k = \sum_{k=1}^{M} \hat{c}_m y_{k-m+1} + \sum_{l=1}^{L-1} \hat{d}_l u_{k-l} + \xi_k \tag{30}$$

where ξ_k is a zero-mean white Gaussian sequence of variance σ^2_ξ, by slightly modifying S in the equation (28). Then it is possible to set σ^2_ξ so that the control purpose is not too violated. Let the controller (30) and the performance index of the controller be denoted by $C(\hat{\underline{\theta}}_{M,L}, \sigma_\xi)$ and $J(\hat{\underline{\theta}}_{M,L}, \sigma_\xi; \underline{\omega}^*, \sigma^*)$ respectively. Then the adaptive order determination procedure at τth stage is given as follows,

Step 1. Fit the model $M(\hat{\underline{\omega}}_{M_0, M_0}, \hat{\sigma}^2_{M_0, M_0})$ to the data $\{(u_{k-1}, y_k) | (\tau-1)T + 1 \leqq k \leqq \tau T\}$ and compute the estimate of the error covariance matrix $\hat{V}_{M_0, M_0}$.

Step 2. Design the controller $C(\hat{\underline{\theta}}_{M_0, M_0})$, and choose $\hat{\sigma}_\xi$ so that $J(\hat{\underline{\theta}}_{M_0, M_0},$

$\hat{\sigma}_\xi$; $\underline{\hat{\omega}}_{M_0, M_0}$, $\hat{\sigma}_{M_0, M_0}$) is not too large compared to $J(\underline{\hat{\theta}}_{M_0, M_0}, 0; \underline{\hat{\omega}}_{M_0, M_0}, \hat{\sigma}_{M_0, M_0})$.

Step 3. Compute the matrices

$$\widehat{\frac{\partial^2 J}{\partial \underline{\omega} \partial \underline{\theta}}} = \frac{\partial^2 J}{\partial \underline{\omega} \partial \underline{\theta}} (\underline{\hat{\theta}}_{M_0, M_0}, \hat{\sigma}_\xi; \underline{\hat{\omega}}_{M_0, M_0}, \hat{\sigma}_{M_0, M_0}) \tag{31}$$

and

$$\frac{\partial^2 J}{\partial \underline{\theta}^2} = \frac{\partial^2 J}{\partial \underline{\theta}^2} (\underline{\hat{\theta}}_{M_0, M_0}, \hat{\sigma}_\xi; \underline{\hat{\omega}}_{M_0, M_0}, \hat{\sigma}_{M_0, M_0}) \ . \tag{32}$$

Step 4. For each pair of M and L,

a) Fit the model $M(\underline{\hat{\omega}}_{M, L}, \hat{\sigma}^2_{M, L})$ to the data.

b) Design the controller $C(\underline{\hat{\theta}}_{M, L})$.

c) Compute the function $J(\underline{\hat{\theta}}_{M, L}, \hat{\sigma}_\xi; \underline{\hat{\omega}}_{M_0, M_0}, \hat{\sigma}_{M_0, M_0})$.

d) Evaluate the criterion $\tilde{J}(\underline{\hat{\theta}}_{M, L}, \hat{\sigma}_\xi; \underline{\hat{\omega}}_{M_0, M_0}, \hat{\sigma}_{M_0, M_0})$.

Step 5. Select the pair (M, L) which minimizes the criterion:

$$\tilde{J}(\underline{\hat{\theta}}_{M, L}, \hat{\sigma}_\xi; \underline{\hat{\omega}}_{M_0, M_0}, \hat{\sigma}_{M_0, M_0}) \ .$$

Note that $\tilde{J}(\underline{\hat{\theta}}_{M, L}, \hat{\sigma}_\xi; \underline{\hat{\omega}}_{M_0, M_0}, \hat{\sigma}_{M_0, M_0})$ is defined by the right-hand side of (21), provided that $\widehat{\partial^2 J/\partial \underline{\omega} \partial \underline{\theta}}$ and $\widehat{\partial^2 J/\partial \underline{\theta}^2}$ are defined by (31) and (32) respectively.

Another problem arises from the following reflection. Suppose that a controller which is really optimal is in operation. If data are collected under this condition and a new controller is designed by the procedure, this new controller cannot be better than the current controller. To avoid the useless change of the controller, we should compare the performance of the new controller with that of the current controller and choose better one. This can be done as follows. Let the current controller has parameter $\underline{\theta}^+$. Then $J(\underline{\theta}^+, \hat{\sigma}_\xi; \underline{\hat{\omega}}_{M_0, M_0}, \hat{\sigma}_{M_0, M_0})$ and $J(\underline{\theta}^+, \hat{\sigma}_\xi; \underline{\omega}^*, \sigma^*)$ are approximately given by

$$\begin{aligned} J(\underline{\theta}^+, \hat{\sigma}_\xi; \underline{\hat{\omega}}_{M_0, M_0}, \hat{\sigma}_{M_0, M_0}) \\ \simeq J(\underline{\theta}^*_{M_0, M_0}, \hat{\sigma}_\xi; \underline{\omega}^*, \sigma^*) + \frac{\partial J}{\partial \underline{\theta}} (\underline{\theta}^+ - \underline{\theta}^*_{M_0, M_0}) + \frac{\partial J}{\partial \underline{\omega}} (\underline{\hat{\omega}}_{M_0, M_0} - \underline{\omega}^*) \\ + \frac{\partial J}{\partial \sigma} (\hat{\sigma}_{M_0, M_0} - \sigma^*) + \frac{1}{2} (\underline{\theta}^+ - \underline{\theta}^*_{M_0, M_0})' \frac{\partial^2 J}{\partial \underline{\theta}^2} (\underline{\theta}^+ - \underline{\theta}^*_{M_0, M_0}) \\ + \frac{1}{2} (\underline{\hat{\omega}}_{M_0, M_0} - \underline{\omega}^*)' \frac{\partial^2 J}{\partial \underline{\omega}^2} (\underline{\hat{\omega}}_{M_0, M_0} - \underline{\omega}^*) \\ + \frac{1}{2} (\hat{\sigma}_{M_0, M_0} - \sigma^*)' \frac{\partial^2 J}{\partial \sigma^2} (\hat{\sigma}_{M_0, M_0} - \sigma^*) \end{aligned} \tag{33}$$

$$+(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)'\frac{\partial^2 J}{\partial\underline{\omega}\partial\underline{\theta}}(\underline{\theta}^+-\underline{\theta}^*_{M_0,M_0})$$

$$+(\hat{\sigma}_{M_0,M_0}-\sigma^*)\frac{\partial^2 J}{\partial\sigma\partial\underline{\theta}}(\underline{\theta}^+-\underline{\theta}^*_{M_0,M_0})$$

$$+(\hat{\sigma}_{M_0,M_0}-\sigma^*)\frac{\partial^2 J}{\partial\sigma\partial\underline{\omega}}(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)$$

and

$$J(\underline{\theta}^+,\hat{\sigma}_\xi;\underline{\omega}^*,\sigma^*)\simeq J(\underline{\theta}^*_{M_0,M_0},\hat{\sigma}_\xi;\underline{\omega}^*,\sigma^*)+\frac{\partial J}{\partial\underline{\theta}}(\underline{\theta}^+-\underline{\theta}^*_{M_0,M_0})+\frac{1}{2}(\underline{\theta}^+-\underline{\theta}^*_{M_0,M_0})'\frac{\partial^2 J}{\partial\underline{\theta}^2}(\underline{\theta}^+-\underline{\theta}^*_{M_0,M_0}) \tag{34}$$

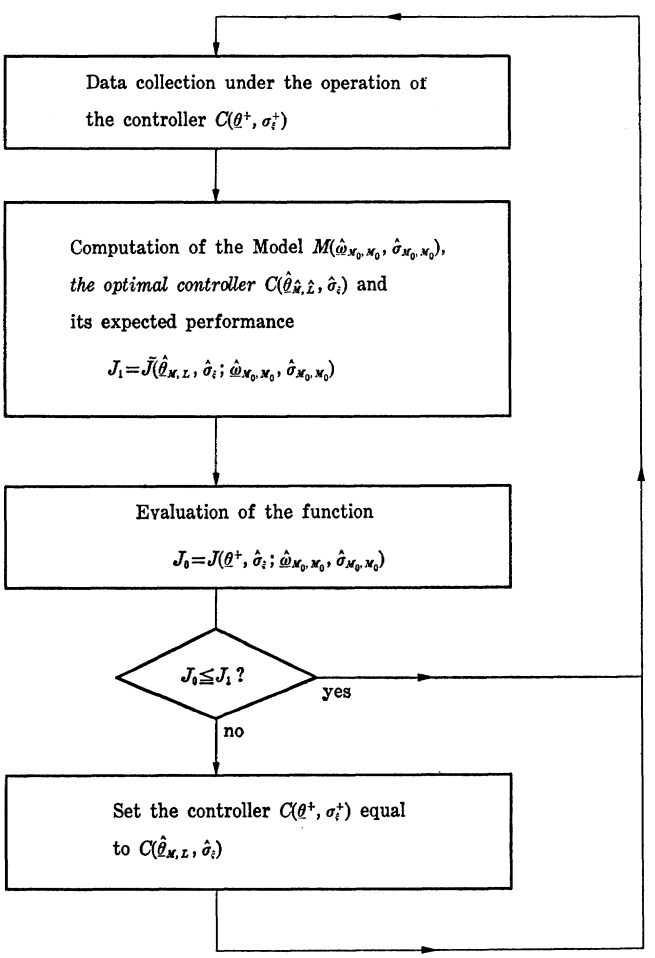

Fig. 1 The proposed adaptive control scheme

$$+(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)'\frac{\partial^2 J}{\partial\underline{\omega}\partial\underline{\theta}}(\underline{\theta}^+-\underline{\theta}^*_{M_0,M_0})$$

$$+(\hat{\sigma}_{M_0,M_0}-\sigma^*)\frac{\partial^2 J}{\partial\sigma\partial\underline{\theta}}(\underline{\theta}^+-\underline{\theta}^*_{M_0,M_0})$$

respectively.

Subtracting (34) from (33) we get

$$\begin{aligned}(35)\qquad & J(\underline{\theta}^+,\hat{\sigma}_\xi;\hat{\underline{\omega}}_{M_0,M_0},\hat{\sigma}_{M_0,M_0})-J(\underline{\theta}^+,\hat{\sigma}_\xi;\underline{\omega}^*,\sigma^*)\\ &\simeq\frac{\partial J}{\partial\underline{\omega}}(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)+\frac{\partial J}{\partial\sigma}(\hat{\sigma}_{M_0,M_0}-\sigma^*)\\ &\quad+\frac{1}{2}(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)'\frac{\partial^2 J}{\partial\underline{\omega}^2}(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)\\ &\quad+\frac{1}{2}(\hat{\sigma}_{M_0,M_0}-\sigma^*)'\frac{\partial^2 J}{\partial\sigma^2}(\hat{\sigma}_{M_0,M_0}-\sigma^*)\\ &\quad+(\hat{\sigma}_{M_0,M_0}-\sigma^*)\frac{\partial^2 J}{\partial\sigma\partial\underline{\omega}}(\hat{\underline{\omega}}_{M_0,M_0}-\underline{\omega}^*)\ .\end{aligned}$$

Taking expectations on both sides we obtain the asymptotic equation

$$\begin{aligned}(36)\qquad & \mathrm{E}\,\{J(\underline{\theta}^+,\hat{\sigma}_\xi;\hat{\underline{\omega}}_{M_0,M_0},\hat{\sigma}_{M_0,M_0})\}-J(\underline{\theta}^+,\hat{\sigma}_\xi;\underline{\omega}^*,\sigma^*)\\ &\simeq\frac{1}{2}\,\mathrm{tr}\left[\frac{\partial^2 J}{\partial\underline{\omega}^2}V^*\right]+\frac{1}{2}\frac{\partial^2 J}{\partial\sigma^2}\mathrm{E}\,\{(\hat{\sigma}_{M_0,M_0}-\sigma^*)^2\}\end{aligned}$$

where it is assumed that $\hat{\underline{\omega}}_{M_0,M_0}$ and $\hat{\sigma}_{M_0,M_0}$ are asymptotically consistent estimates of $\underline{\omega}^*$ and σ^*, and independent of each other. Comparing (36) with (20) it is concluded that $J(\underline{\theta}^+,\hat{\sigma}_\xi;\hat{\underline{\omega}}_{M_0,M_0},\hat{\sigma}_{M_0,M_0})$ can be used as an estimate of the performance of the current controller. Now we can present our adaptive control scheme in its final form. It is shown schematically in Fig. 1.

6. Numerical example

To illustrate the effectiveness of the scheme, the following plant is considered.

$$(37)\qquad y_k=\sum_{m=1}^{M^*}a_m^*y_{k-m}+\sum_{l=1}^{L^*}b_l^*u_{k-l}+\varepsilon_k$$

where ε_k is a zero-mean white Gaussian sequence of variance 1 and $M^*=L^*=4$. $\underline{a}^*=(a_1^*,a_2^*,a_3^*,a_4^*)'$ and $\underline{b}^*=(b_1^*,b_2^*,b_3^*,b_4^*)'$ are given as follows:

$$\underline{a}^*=(-0.6,\ -0.74,\ -0.18,\ -0.1)'$$

$$\underline{b}^*=(0.2,\ -0.66,\ 0.1,\ -0.08)'\ .$$

The performance index to be minimized is

$$J=\lim_{k\to\infty} \mathrm{E}\,\{10y_k^2+u_k^2\}\ .$$

Only a priori information assumed is that

$$1\leqq M^*\leqq 5 \qquad \text{and} \qquad 1\leqq L^*\leqq 5\ .$$

Data of length 500 is collected under the operation of the controller:

$$\begin{aligned} u_k=&-0.255y_k+0.098y_{k-1}-0.220y_{k-2}+0.111y_{k-3}\\ &+0.792u_{k-1}-0.554u_{k-2}+0.144u_{k-3}+\xi_k\ , \end{aligned} \tag{38}$$

where ξ_k is a zero-mean white Gaussian sequence of the variance 0.04. (38) is obtained as the optimal controller for the initial guess of the plant:

$$y_k=\sum_{m=1}^{4} a_m^+ y_{k-m}+\sum_{l=1}^{4} b_l^+ u_{k-l}+\varepsilon_k\ , \tag{39}$$

where $\underline{a}^+=(a_1^+, a_2^+, a_3^+, a_4^+)'$ and $\underline{b}^+=(b_1^+, b_2^+, b_3^+, b_4^+)'$ are given respectively by

$$\underline{a}^+=(-0.645,\ -0.521,\ -0.241,\ -0.243)'$$

and

$$\underline{b}^+=(-0.039,\ -0.379,\ 0.278,\ -0.315)'\ .$$

The performance index of the controller (38) is 16.443. How act the one cycle of the scheme in Fig. 1 is observed. Four cases are possible to occur.

Case I: New controller is judged not better than the current controller (38).

Case II: New controller is judged better than the current controller, and,

Table 1 Result of the experiment by the proposed procedure

Occurrence	Number of occurrence	Mean improvement of the performance
Case I	122	0.0
Case II-1	110	−1.074
Case II-2	18	1.654
Case II-3	0	—
Total	250	−0.354

II-1 : New controller is really better than the current controller.

II-2 : New controller can stabilize the plant but worse than the current controller.

II-3 : New controller cannot stabilize the plant.

The result of 250 simulation runs is summarized in Table 1, where $\hat{\sigma}_\xi^2$ is fixed at 0.04.

This result is compared with a controller selection scheme which is based on AIC, an information theoretic criterion which is introduced

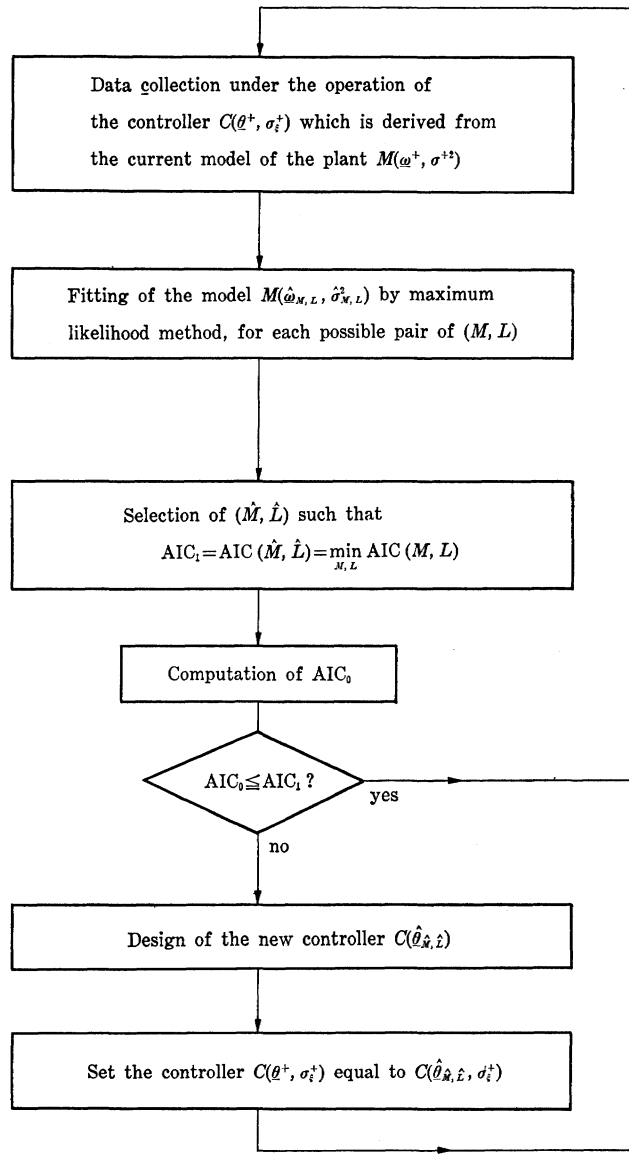

Fig. 2 Adaptive control scheme based on AIC

by Akaike [1] as a measure of the badness of the maximum likelihood estimate of a model. It is easy to show that AIC of the fitted model $M(\hat{\underline{\omega}}_{M,L}, \hat{\sigma}^2_{M,L})$ is given by

$$\mathrm{AIC}\,(M, L) = N \log \hat{\sigma}^2_{M,L} + 2(M+L+1)\ ,$$

where N denotes the length of the data. The AIC of the current model $M(\underline{\omega}^+, \sigma^{+2})$ is given by

$$\mathrm{AIC}_0 = N \log \hat{\sigma}^{+2} + 2$$

where $\hat{\sigma}^{+2}$ is the sample mean square error of the prediction of the output of the plant. The scheme based on AIC is shown in Fig. 2. The result by this scheme is summarized in Table 2. The data used are same with Table 1. Apparently these results show the effectiveness of our procedure. Table 2 shows that even if the model of the plant is judged better than the current model, the derived controller is not necessarily better than the current one. This is partly explained by the fact that the number of the parameters of the optimal controller of the plant (2) is less than the number of the parameters of the plant. This implies that a controller which is designed on a poor model of a plant sometimes shows a good performance. The procedure based on AIC will change the controller and fail in this case, where our scheme can avoid the useless change of the controller.

Table 2 Result of the experiment by AIC-procedure

Occurrence	Number of occurrence	Mean improvement of the performance
Case I	0	
Case II-1	179	−1.087
Case II-2	66	4.295
Case II-3	5	∞
Total	250	∞

7. Conclusion

A scheme of adaptive control which does not require a priori knowledge of the plant structure is presented. This scheme is based on a criterion which is a measure of the expected performance of the controller which is designed on a fitted model of a plant to be controlled. Though, the controller design is based on the certainty equivalence principle, the scheme is 'cautious' (Tse and Athans [9]) as a whole system. This feature is obtained from the fact that our criterion takes

into account the possible error in model fitting.

The plant dealt in this paper is a time-invariant single-input-single-output plant. However, it is easy to extend the result to multi-input-multi-output plants. It is also expected that the scheme will applicable to the slowly varying plants.

Acknowledgement

The present author is grateful to Dr. H. Akaike of the Institute of Statistical Mathematics for his valuable advices. He also wishes to thank Miss F. Tada for her help in programming. The work reported in this paper was partly supported by a grant from the Ministry of Education, Science and Culture.

Appendix

Design of the optimal controller

The plant (2) has an equivalent expression:

$$\text{(A-1)} \qquad \begin{aligned} \underline{z}_k &= \Phi^* \underline{z}_{k-1} + \tilde{\underline{b}}^* u_{k-1} + \tilde{\underline{\varepsilon}}_k \\ y_k &= (1, 0, 0, \cdots, 0) \underline{z}_k \end{aligned}$$

where $M_0 \times M_0$-matrix Φ^* and M_0-vectors $\tilde{\underline{b}}^*$ and $\tilde{\underline{\varepsilon}}_k$ are defined by

$$\Phi^* = \begin{pmatrix} a_1^*, & 1, & 0, & \cdots & , & 0 \\ a_2^*, & 0, & 1, & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ a_{M^*}^*, & & & & 1, & 0 \\ 0, & & & & & 1 \\ \vdots & & & & & \\ 0, & & & \cdots & , & 0 \end{pmatrix},$$

$$\tilde{\underline{b}}^* = (b_1^*, b_2^*, \cdots, b_{L^*}^*, 0, \cdots, 0)'$$

and

$$\tilde{\underline{\varepsilon}}_k = (\varepsilon_k, 0, 0, \cdots, 0)'$$

respectively. The performance index (3) is equivalent to

$$\text{(A-2)} \qquad J = \lim_{K \to \infty} \mathrm{E} \left\{ \frac{1}{K} \sum_{k=1}^{K} (\underline{z}_k' Q \underline{z}_k + e_u u_{k-1}^2) \right\},$$

where Q is an $M_0 \times M_0$-matrix defined by

$$Q=\begin{pmatrix} w_y & \\ & 0 \end{pmatrix}.$$

If the performance index is given by

$$J_K=\sum_{k=1}^{K}(\underline{z}_k'Q\underline{z}_k+w_u u_{k-1}^2)\ , \tag{A-3}$$

the optimal control $\{u_0, u_1, \cdots, u_{k-1}\}$ is calculated by the iteration ($i=1,\cdots,K$):

$$\begin{aligned} S_i&=(w_u+\tilde{\underline{b}}^{*\prime}P_{i-1}\tilde{\underline{b}}^*)^{-1}\\ T_i&=P_{i-1}-P_{i-1}\tilde{\underline{b}}^*S_i\tilde{\underline{b}}^{*\prime}P_{i-1}\\ P_i&=\Phi^{*\prime}T_i\Phi^*+Q\\ u_{K-i}&=-S_i\underline{b}^{*\prime}P_{i-1}\Phi^*\underline{z}_{K-i} \end{aligned} \tag{A-4}$$

where P_0 is set equal to Q. (A-4) is obtained by applying the Dynamic Programming technique (see, for example, Boudarel et al. [4]) to the plant (A-1) and the performance index (A-3). If the gain

$$\underline{g}_i'=-S_i\tilde{\underline{b}}^{*\prime}P_{i-1}\Phi^*$$

converges to some limit gain $\underline{g}_\infty'=(g_1^*, g_2^*, \cdots, g_{M_0}^*)'$ as i tends to the infinity, (A-2) is minimized by the control law

$$u_k=\underline{g}_\infty'\underline{z}_k\ . \tag{A-5}$$

It is easy to show that $\underline{z}_k$ can be expressed by

$$\underline{z}_k=\begin{pmatrix} 1, 0, & \cdot & \cdot & \cdot & 0 \\ 0, a_2^*, a_3^*, & \cdots, & a_{M^*}^*, & b_2^*, b_3^*, & \cdots, b_{L^*}^* \\ 0, a_3^*, & \cdots, & a_{M^*}^*, 0, & b_3^*, & \cdots, b_{L^*}^*, 0 \\ \vdots & & & \vdots & \\ a_{M^*}^* & & & \vdots & \\ & & & b_{L^*}^* & \\ 0 & & & & \end{pmatrix}\begin{pmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_{k-M^*+1} \\ u_{k-1} \\ \vdots \\ u_{k-L^*+1} \end{pmatrix}.$$

Then, (A-5) is equivalent to the control system:

$$u_k=\sum_{m=1}^{M^*}c_m^*y_{k-m+1}+\sum_{l=1}^{L^*-1}d_l^*u_{k-l}\ ,$$

where $\underline{c}^*=(c_1^*, c_2^*, \cdots, c_{M^*}^*)'$ and $\underline{d}^*=(d_1^*, \cdots, d_{L^*-1}^*)'$ are given by

$$c_m^* = \begin{cases} g_1^* & (m=1) \\ \sum_{j=m}^{M^*} a_j^* g_{j-m+2}^* & (m=2, 3, \cdots, M^*) \end{cases}$$

and

$$d^* = \sum_{j=l+1}^{L^*} b_j^* g_{j-l+1}^* \qquad (l=1, 2, \cdots, L^*-1)$$

respectively.

THE INSTITUTE OF STATISTICAL MATHEMATICS

REFERENCES

[1] Akaike, H. (1973). Information theory and an extension of the maximum likelihood principle, *2nd International Symposium on Information Theory*, Petrov, B. N. and Csaki, F. Eds., Akademiai Kiado, Budapest, 267-281.

[2] Aoki, M. (1967). *Optimization of Stochastic Systems*, Academic Press, New York.

[3] Alster, J. and Bélanger, P. R. (1974). A technique for dual adaptive control, *Automatica*, **10**, 627-634.

[4] Boudarel, R., Delmas, J. and Guichet, P. (1971). *Dynamic Programming and Its Application to Optimal Control*, Academic Press, New York.

[5] Box, G. E. P. and MacGregor, J. F. (1974). The analysis of closed-loop dynamic-stochastic systems, *Technometrics*, **16**, 391-398.

[6] Kitagawa, G. (1977). An algorithm for solving the matrix equation $X=FXF^t+S$, *Int. J. Control*, **25**, 745-753.

[7] Otomo, T., Nakagawa, T. and Akaike, H. (1972). Statistical approach to computer control of cement rotary kilns, *Automatia*, **8**, 35-48.

[8] Otsu, K., Horigome, M. and Kitagawa, G. (1976). On the prediction and stochastic control of ship's motion, *Ship Operation Automation* II, Pittkin, M., Roche, J. and Williams, T. J. Eds., North-Holland, Amsterdam, 69-76.

[9] Tse, E. and Athans, M. (1972). Adaptive stochastic control for class of linear systems, *IEEE Trans. Automat. Contr.*, AC-**17**, 38-51.

[10] Wittenmark, B. (1975). Stochastic adaptive control methods: a survey, *Int. J. Control*, **21**, 705-730.