ON LOCAL LIMIT THEOREMS AND BLACKWELL'S RENEWAL THEOREM FOR INDEPENDENT RANDOM VARIABLES

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Summary

Some types of local limit theorems for independent random variables are shown and the results obtained are applied to have generalizations of Blackwell's renewal theorem.

1. Introduction

Let $\{X_i,\ i=1, 2, \cdots\}$ be a sequence of independent, nonlattice random variables with finite means $\mathrm{E}\, X_i=\mu_i$ and finite variances $\mathrm{Var}\, X_i=\sigma_i^2$. Let us consider the renewal process generated by $\{X_i\}$. Set $S_n=\sum_{i=1}^{n} X_i$, $n\geqq 1$. The purpose of this paper is to discuss the sufficient conditions under which Blackwell's renewal theorem holds:

$$\lim_{x\to\infty}\sum_{n=1}^{\infty}\Pr\left\{x-\frac{h}{2}<S_n\leqq x+\frac{h}{2}\right\}=\frac{h}{\mu} \qquad (h>0)\,, \tag{1.1}$$

provided that

$$\mu=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mu_i$$

exists and is positive.

In the ordinary form of Blackwell's theorem, it is used to assume that $\{X_i\}$ is a sequence of independent and identically distributed random variables. The case in which X_i, $i=1, 2, \cdots$ are not necessarily identically distributed did not draw much attension except in a few papers [1], [2], where some sufficient conditions for the validity of (1.1) were given but they do not seem very satisfactory.

In the present paper, we are also going to study such sufficient conditions, but taking an approach quite different from those used in the papers cited above.

On the other hand, Cox and Smith [3] have shown the suggestive result that some local limit theorems for densities imply the renewal density theorem under some restrictions. In this connection, we shall first point out (Theorem 1) that, by the reasoning just same as in the proof of Cox and Smith, Blackwell's theorem follows from other type of local limit theorems. Therefore we shall, in this paper, give such a kind of local limit theorems (Theorems 2 and 3), and this would be the main part of the proof of Blackwell's renewal theorem for non-identically distributed case. In the last section, we shall give Blackwell's theorem for a sequence of independent random variables each of whose distribution functions is one of a finite number of distinct distribution functions.

2. Local limit theorems and Blackwell's renewal theorem

Let $\{X_i,\ i=1, 2, \cdots\}$ be a sequence of independent random variables with finite means μ_i and finite variances σ_i^2. Write $S_n=\sum_{i=1}^{n} X_i$, $A_n=\mathrm{E}\, S_n$, $B_n^2=\mathrm{Var}\, S_n$, $G_n(x)=\Pr\{S_n\leqq x\}$, for $h>0$ $I_h=(-h/2, h/2]$, $x+I_h=(x-h/2, x+h/2]$, $G_n(x+I_h)=\Pr\{x-h/2<S_n\leqq x+h/2\}$, and $p(x)=(2\pi)^{-1/2}\cdot \exp(-x^2/2)$, and denote the characteristic function of $(S_n-A_n)/B_n$ by $\theta_n(t)$.

THEOREM 1. *Suppose that* $\lim_{n\to\infty}(n+1)^{-1}\sum_{i=l}^{l+n}\mu_i=\mu>0$ *uniformly for* $l=1, 2, \cdots$, *and that* $B_n^2\sim C_1 n$ *for* $n\to\infty$, C_1 *being a positive constant. Fix* $h>0$ *arbitrarily. If*

$$\lim_{n\to\infty}\sup_{x}\left|\frac{B_n}{h}x^m G_n(xB_n+A_n+I_h)-x^m p(x)\right|=0 \tag{2.1}$$

hold for $m=0, 2$, *then we have Blackwell's renewal theorem,*

$$\lim_{x\to\infty}\sum_{n=1}^{\infty} G_n(x+I_h)=h/\mu\ .$$

The proof can be carried over in exactly the same way as Cox and Smith [3] derived the renewal density theorem from the local limit theorems for densities.

In order to have the sufficient conditions under which Blackwell's theorem holds, it suffices, in view of Theorem 1, to study the conditions for the local limit theorems (2.1). In the sequel, we assume that $\mathrm{E}\, X_i=0$ without loss of generality when the local limit theorems are considered, and investigate

$$\lim_{n\to\infty}\sup_{x}\left|\frac{B_n}{h}x^m G_n(xB_n+I_h)-x^m p(x)\right|=0 \qquad (m=0, 2)\,, \tag{2.2}$$

instead of (2.1).

For $m=0$, we have:

THEOREM 2. *Let $\{X_i\}$ be a sequence of independent random variables with* $\mathrm{E}\,X_i=0$ *and finite variances σ_i^2, and let $f_i(t)$ be the characteristic function of X_i. Suppose that:*

(I) *$\{X_i\}$ satisfies the Lindeberg condition,*

(II) *$B_n^2 \sim C_1 n$ as $n\to\infty$, C_1 being some positive constant, and*

(III) *for some $\varepsilon>0$, there exists a positive number $c=c(\varepsilon)<1$ in such a way that $|f_i(t)|\leqq c$ for $|t|\geqq\varepsilon$. Then we have*

$$\lim_{n\to\infty}\sup_{x}\left|\frac{B_n}{h}G_n(xB_n+I_h)-p(x)\right|=0\ . \tag{2.3}$$

This type of local limit theorem has first taken by Stone [4], [5] for the sequence of independent, identically distributed random variables. He actually has studied the case in which the random variables are multi-dimensional and $p(x)$ is the density of a stable law.

For the proof of Theorem 2, we note that $\{X_i\}$ admits the central limit theorem since X_i, $i=1,2,\cdots$ are independent random variables with the condition (I). Therefore, we can prove Theorem 2 in exactly the same way as Stone has shown his local limit theorems, by using the following lemma.

LEMMA 1. *Under the conditions* (II) *and* (III), *we have:*

(i) $|\theta_n(t)|\leqq\exp\{-\alpha_1 t^2\}$ *for* $|t|<\varepsilon B_n$ *and* $n\geqq N_1$, *and*

(ii) $|\theta_n(t)|\leqq\exp\{-\alpha_2 n\}$ *for* $|t|\geqq\varepsilon B_n$,

where α_1 and α_2 are some positive constants, and N_1 is a positive integer.

PROOF. We use a well-known inequality relation due to Cramér ([6], p. 26): *If $f(t)$ is a characteristic function such that $|f(t)|\leqq\kappa<1$ for all $|t|\geqq R$, then for $|t|<R$,*

$$|f(t)|\leqq 1-\frac{1-\kappa^2}{8R^2}t^2\ .$$

Since the condition (III) is no more than the condition of Cramér's relation, we have, for $|t|<\varepsilon$,

$$|f_i(t)|\leqq 1-\frac{1-c^2}{8\varepsilon^2}t^2\leqq\exp\{-\gamma t^2\}\ , \tag{2.4}$$

where $\gamma=(1-c^2)/8\varepsilon^2$. Hence, for $|t|<\varepsilon B_n$,

$$|\theta_n(t)|=\prod_{i=1}^{n}\left|f_i\left(\frac{t}{B_n}\right)\right|\leqq\exp\left\{-\frac{n\gamma t^2}{B_n^2}\right\}\ .$$

Taking account of the condition (II), we have (i). On the other hand, we can rewrite the condition (III) by

$$|f_i(t)| \leqq \exp\{-\alpha_2\} \tag{2.5}$$

for some positive constant α_2 and for $|t| \geqq \varepsilon$, from which (ii) is easily given. The lemma is completed.

For $m=2$ in (2.2), we have:

THEOREM 3. *In addition to the conditions in Theorem 2, we suppose that:*

(IV) X_i, $i=1, 2, \cdots$ *have the finite third moments. Then we have*

$$\lim_{n\to\infty} \sup_x \left| \frac{B_n}{h} x^2 G_n(xB_n + I_h) - x^2 p(x) \right| = 0 . \tag{2.6}$$

3. Proof of Theorem 3

Before proving Theorem 3, we shall show some lemmas.

LEMMA 2. *Under the single condition* (IV),

$$\int_{-\infty}^{\infty} \frac{B_n}{h} x^2 G_n(xB_n + I_h) dx = 1 + \frac{h^2}{3B_n^2}$$

for all n.

PROOF. By the definition,

$$\int_{-\infty}^{\infty} \frac{B_n}{h} x^2 G_n(xB_n + I_h) dx = \int_{-\infty}^{\infty} \frac{B_n}{h} x^2 \left[G_n\left(xB_n + \frac{h}{2}\right) - G_n\left(xB_n - \frac{h}{2}\right) \right] dx .$$

Integrating by parts and noting that X_i, $i=1, 2, \cdots$ have the finite third moments, we have that the last one is

$$\begin{aligned} -\frac{B_n}{3h} &\left[\int_{-\infty}^{\infty} x^3 dG_n\left(xB_n + \frac{h}{2}\right) - \int_{-\infty}^{\infty} x^3 dG_n\left(xB_n - \frac{h}{2}\right) \right] \\ &= -\frac{B_n}{3h} \left[-\frac{3h}{B_n} \int_{-\infty}^{\infty} x^2 dG_n(xB_n) - \frac{h^3}{B_n^3} \int_{-\infty}^{\infty} dG_n(xB_n) \right] \\ &= 1 + \frac{h^2}{3B_n^2} . \end{aligned}$$

The lemma is proved.

We put

$$\phi_n(t) = \int_{-\infty}^{\infty} e^{itx} \frac{B_n}{h} x^2 G_n(xB_n + I_h) dx$$

and

$$\phi(t)=\int_{-\infty}^{\infty} e^{itx}x^2p(x)dx .$$

The existence of $\phi_n(t)$ is assured by Lemma 2.

LEMMA 3. *Under the conditions* (I), (II), (III) *and* (IV), *we have*

$$\lim_{n\to\infty}\sup_t |\phi_n(t)-\phi(t)|=0 .$$

PROOF. For any $\eta>0$, there exists an R such that

$$\int_{-R}^{R} x^2p(x)dx>1-\frac{\eta}{2} ,$$

and it follows from Theorem 2 that there exists an integer $N_2(\eta)$ such that

$$\left|\frac{B_n}{h}G_n(xB_n+I_h)-p(x)\right|\leqq\frac{3\eta}{4R^3}$$

for $n\geqq N_2$. Then

$$\text{(3.1)}\quad \int_{-R}^{R}\frac{B_n}{h}x^2G_n(xB_n+I_h)dx>\int_{-R}^{R} x^2p(x)dx-\frac{3\eta}{4R^3}\int_{-R}^{R} x^2dx>1-\eta .$$

We have from Lemma 2 that for any $\eta>0$, there exists an $N_3(\eta)$ such that

$$\text{(3.2)}\quad \int_{-\infty}^{\infty}\frac{B_n}{h}x^2G_n(xB_n+I_h)dx<1+\eta$$

for $n\geqq N_3$. From (3.1) and (3.2), for $n\geqq N_4=\max(N_2, N_3)$,

$$\int_{|x|>R}\frac{B_n}{h}x^2G_n(xB_n+I_h)dx<2\eta .$$

Thus,

$$\begin{aligned}|\phi_n(t)-\phi(t)|&\leqq\left|\int_{-R}^{R} x^2\left[\frac{B_n}{h}G_n(xB_n+I_h)-p(x)\right]dx\right|\\&\quad+\left|\int_{|x|>R}\frac{B_n}{h}x^2G_n(xB_n+I_h)dx\right|+\left|\int_{|x|>R}x^2p(x)dx\right|\\&<\frac{3\eta}{4R^3}\int_{-R}^{R}x^2dx+2\eta+\frac{1}{2}\eta=3\eta .\end{aligned}$$

The lemma is completed.

LEMMA 4. *We have*

$$|\phi_n(t)|\leqq C_2|\theta_n''(t)|+\frac{C_3}{|t|}|\theta_n'(t)|+\left(\frac{C_4 h}{|t|B_n}+\frac{C_5}{t^2}\right)|\theta_n(t)|\ ,$$

where C_2, C_3, C_4 *and* C_5 *are positive constants independent of* n, t *and* h.

PROOF. Repeating the integration by parts, we have

$$\begin{aligned}
\phi_n(t)=&\int_{-\infty}^{\infty} e^{itx}\frac{B_n}{h}x^2 G_n(xB_n+I_h)dx \\
=&-\frac{1}{it}\int_{-\infty}^{\infty} e^{itx}\frac{B_n}{h}x^2\left[dG_n\left(xB_n+\frac{h}{2}\right)-dG_n\left(xB_n-\frac{h}{2}\right)\right] \\
&+\frac{2}{(it)^2}\int_{-\infty}^{\infty} e^{itx}\frac{B_n}{h}x\left[dG_n\left(xB_n+\frac{h}{2}\right)-dG_n\left(xB_n-\frac{h}{2}\right)\right] \\
&-\frac{1}{(it)^3}\int_{-\infty}^{\infty} e^{itx}\frac{B_n}{h}\left[dG_n\left(xB_n+\frac{h}{2}\right)-dG_n\left(xB_n-\frac{h}{2}\right)\right] \\
=&-\frac{1}{it}\left[-\frac{B_n}{h}e^{-ith/2B_n}(1-e^{ith/B_n})\theta_n''(t)\right. \\
&\left.+\frac{1}{i}e^{-ith/2B_n}(1+e^{ith/B_n})\theta_n'(t)+\frac{h}{4B_n}e^{-ith/2B_n}(1-e^{ith/B_n})\theta_n(t)\right] \\
&+\frac{2}{t^2}\left[\frac{B_n}{ih}e^{-ith/2B_n}(1-e^{ith/B_n})\theta_n'(t)-\frac{1}{2}e^{-ith/2B_n}(1+e^{ith/B_n})\theta_n(t)\right] \\
&+\frac{2B_n}{it^3h}e^{-ith/2B_n}(1-e^{ith/B_n})\theta_n(t)\ .
\end{aligned}$$

Therefore, noticing that

$$\left|\frac{1-e^{ith/B_n}}{ith/B_n}\right|\leqq 1\ ,$$

we have the required conclusion.

LEMMA 5. *Under the conditions* (II) *and* (III), *we have*:
(i) $|\theta_n'(t)|\leqq|t|\exp\{-\alpha_3 t^2\}$ *for* $|t|<\varepsilon B_n$ *and* $n\geqq N_5$,
(ii) $|\theta_n''(t)|\leqq(1+t^2)\exp\{-\alpha_4 t^2\}$ *for* $|t|<\varepsilon B_n$ *and* $n\geqq N_6$,
(iii) $|\theta_n'(t)|\leqq n^{1/2}\exp\{-(n-1)\alpha_2\}$ *for* $|t|\geqq\varepsilon B_n$, *and*
(iv) $|\theta_n''(t)|\leqq(1+n)\exp\{-(n-2)\alpha_2\}$ *for* $|t|\geqq\varepsilon B_n$,
where α_3 *and* α_4 *are some positive constants,* N_5 *and* N_6 *are some positive integers, and* α_2 *is the one determined in* (2.5).

PROOF. We have

(3.3) $$\theta_n'(t)=\frac{1}{B_n}\sum_{j=1}^{n}f_j'\left(\frac{t}{B_n}\right)\prod_{\substack{i=1\\ i\neq j}}^{n}f_i\left(\frac{t}{B_n}\right)\ ,$$

(3.4) $$\theta_n''(t)=\frac{1}{B_n^2}\left[\sum_{j=1}^{n}f_j''\left(\frac{t}{B_n}\right)\prod_{\substack{i=1\\ i\neq j}}^{n}f_i\left(\frac{t}{B_n}\right)+\sum_{k=1}^{n}f_k'\left(\frac{t}{B_n}\right)\right.$$

$$\cdot \sum_{\substack{j=1 \\ j \neq k}}^{n} f_j'\left(\frac{t}{B_n}\right) \prod_{\substack{i=1 \\ i \neq j \neq k}}^{n} f_i\left(\frac{t}{B_n}\right)\Bigg]$$

and

$$|f_i''(\cdot)| \leqq \sigma_i^2 , \qquad \left|f_i'\left(\frac{t}{B_n}\right)\right| \leqq \frac{|t|}{B_n}\sigma_i^2 .$$

Therefore we have

$$|\theta_n'(t)| \leqq |t| \max_{1 \leqq j \leqq n} \prod_{\substack{i=1 \\ i \neq j}}^{n} \left|f_i\left(\frac{t}{B_n}\right)\right|$$

and

$$|\theta_n''(t)| \leqq (1+t^2) \max_{1 \leqq j,k \leqq n} \prod_{\substack{i=1 \\ i \neq j \neq k}}^{n} \left|f_i\left(\frac{t}{B_n}\right)\right| .$$

It follows from Cramér's relation (2.4) that for $|t| < \varepsilon B_n$,

$$|\theta_n'(t)| \leqq |t| \exp\left\{-\frac{(n-1)\gamma t^2}{B_n^2}\right\}$$

and

$$|\theta_n''(t)| \leqq (1+t^2) \exp\left\{-\frac{(n-2)\gamma t^2}{B_n^2}\right\} ,$$

where γ is the one determined in (2.4). By the condition (II), we have (i) and (ii).

Further, noticing that $|f_i'(\cdot)| \leqq \int |x| dF_i(x) \leqq \left(\int x^2 dF_i(x)\right)^{1/2} \leqq \sigma_i$, and that $\left(\sum_{i=1}^{n} \sigma_i\right)^2 \leqq nB_n^2$ we have from (3.3) and (3.4) that

$$|\theta_n'(t)| \leqq n^{1/2} \max_{1 \leqq j \leqq n} \prod_{\substack{i=1 \\ i \neq j}}^{n} \left|f_i\left(\frac{t}{B_n}\right)\right|$$

and

$$|\theta_n''(t)| \leqq (1+n) \max_{1 \leqq j,k \leqq n} \prod_{\substack{i=1 \\ i \neq j \neq k}}^{n} \left|f_i\left(\frac{t}{B_n}\right)\right| .$$

It follows from (2.5) that for $|t| \geqq \varepsilon B_n$,

$$|\theta_n'(t)| \leqq n^{1/2} \exp\{-(n-1)\alpha_2\}$$

and

$$|\theta_n''(t)| \leqq (1+n) \exp\{-(n-2)\alpha_2\} ,$$

where α_2 is the one determined in (2.5). The lemma is thus proved.

LEMMA 6. *Under the conditions* (II) *and* (III), *we have*:

(i) $|\phi_n(t)|$ *is dominated by a nonnegative function* $Q(t)$ *with* $\int_{|t|\geqq 1} Q(t)dt<\infty$ *for* $|t|<\varepsilon B_n$ *and* $n\geqq N_7$, *where* N_7 *is some positive integer, and*

(ii) $|\phi_n(t)|=o(B_n^{-1})$ *uniformly for* $|t|\geqq\varepsilon B_n$ *as* $n\to\infty$.

PROOF. Put $N_7=\max(N_1, N_5, N_6)$, where N_1 and (N_5, N_6) are the ones determined in Lemma 1 and Lemma 5, respectively. From Lemma 1 (i), Lemma 4 and Lemma 5 (i), (ii), we have

$$|\phi_n(t)|\leqq C_2(1+t^2)\exp\{-\alpha_4 t^2\}+\frac{C_3}{|t|}|t|\exp\{-\alpha_3 t^2\}$$
$$+\left(\frac{C_4 h}{|t|B_n}+\frac{C_5}{t^2}\right)\exp\{-\alpha_2 t^2\}$$

for $|t|<\varepsilon B_n$ and $n\geqq N_7$. The right-hand side on the last inequality satisfies the requirements on $Q(t)$. We thus have (i).

When $|t|\geqq\varepsilon B_n$, we have (ii) from Lemma 1 (ii), Lemma 4 and Lemma 5 (iii), (iv). This completes the lemma.

We now define

$$K(x)=\frac{1}{2\pi}\left(\frac{\sin(x/2)}{x/2}\right)^2,$$

$$k(t)=\begin{cases}1-|t|, & \text{if } |t|\leqq 1,\\ 0, & \text{if } |t|>1,\end{cases}$$

and further, for any $a>0$,

$$K_a(x)=\frac{1}{a}K\left(\frac{x}{a}\right), \qquad k_a(t)=k(at).$$

Then we see that

$$\int_{-\infty}^{\infty}K(x)dx=1, \qquad \int_{-\infty}^{\infty}e^{itx}K(x)dx=k(t)$$

and

$$\int_{-\infty}^{\infty}K_a(x)dx=1, \qquad \int_{-\infty}^{\infty}e^{itx}K_a(x)dx=k_a(t).$$

For any $x>0$, $h>0$ and $a>0$, we put

$$W_n(x, h, a)=\int_{-\infty}^{\infty}K_{a/B_n}(x-y)\frac{B_n}{h}y^2G_n(yB_n+I_h)dy$$

$$=\int_{-\infty}^{\infty} K_{a/B_n}(y)\frac{B_n}{h}(x-y)^2G_n((x-y)B_n+I_h)dy ,$$

which is well-defided under the condition (IV) by Lemma 2. Noticing that

$$K_{a/B_n}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}k_{a/B_n}(t)dt ,$$

we have

$$W_n(x, h, a)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(x-y)t}k_{a/B_n}(t)\frac{B_n}{h}y^2G_n(yB_n+I_h)dtdy$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}k_{a/B_n}(t)\phi_n(t)dt .$$

Since $k_{a/B_n}(t)$ has the bounded support,

$$W_n(x, h, a)=\frac{1}{2\pi}\int_{|t|\leqq B_n/a} e^{-ixt}k_{a/B_n}(t)\phi_n(t)dt .$$

We now have the following lemma.

LEMMA 7. *Under the conditions* (I), (II), (III) *and* (IV), *we have that for any fixed finite number* $M>0$,

$$\lim_{n\to\infty} |W_n(x, h, a)-x^2p(x)|=0$$

uniformly for x *and* $M^{-1}\leqq h\leqq M$.

PROOF. We note that

$$|k_{a/B_n}(t)|\leqq 1 .$$

For a suitably fixed $A>0$, we put

$$J_1=\int_{|t|\leqq A} |k_{a/B_n}(t)\phi_n(t)-\phi(t)|dt ,$$

$$J_2=\int_{A<|t|<\epsilon B_n} |\phi_n(t)|dt ,$$

$$J_3=\int_{\epsilon B_n\leqq|t|\leqq B_n/a} |\phi_n(t)|dt$$

and

$$J_4=\int_{|t|>A} |\phi(t)|dt .$$

Then we have

$$|W_n(x, h, a)-x^2p(x)|\leqq J_1+J_2+J_3+J_4 .$$

For J_1,

$$J_1=\int_{|t|\leqq A} |k_{a/B_n}(t)||\phi_n(t)-\phi(t)|dt+\int_{|t|\leqq A} |k_{a/B_n}(t)-1||\phi(t)|dt .$$

By Lemma 3, the integrand in the first term tends to zero when $n\to\infty$ uniformly for $|t|\leqq A$. Making use of the fact that $|k_{a/B_n}(t)-1|=o(1)$ as $n\to\infty$ uniformly for $|t|\leqq A$, we see that the second term tends to zero when $n\to\infty$, because of the integrability of $\phi(t)$. Hence $J_1\to 0$ as $n\to\infty$.

We now consider J_2. By (i) in Lemma 6, it follows that for $n\geqq N_7$,

$$J_2\leqq\int_{A<|t|<\varepsilon B_n} Q(t)dt\leqq\int_{A<|t|} Q(t)dt .$$

Choosing A sufficiently large, we can make J_2 as small as we desire, since $Q(t)$ is integrable on the interval with $|t|\geqq 1$.

Next, we have, taking account of (ii) in Lemma 6,

$$J_3\leqq o(B_n^{-1})\int_{\varepsilon B_n\leqq|t|\leqq B_n/a} dt\leqq o(B_n^{-1})B_n/a .$$

It follows that $J_3\to 0$ as $n\to\infty$.

Finally, we can make J_4 as small as we desire, by choosing A sufficiently large, because of the integrability of $\phi(t)$.

Hence the proof of the lemma is completed.

We now turn to the proof of Theorem 3.

PROOF OF THEOREM 3. Let $h>0$ be fixed and $0<\delta<1$. If $|y|\leqq \delta h/2B_n$, then

$$(x-y)B_n+I_{h(1-\delta)}\subseteqq xB_n+I_h\subseteqq(x-y)B_n+I_{h(1+\delta)} ,$$

and consequently

$$G_n((x-y)B_n+I_{h(1-\delta)})\leqq G_n(xB_n+I_h)\leqq G_n((x-y)B_n+I_{h(1+\delta)}) . \tag{3.5}$$

Since $\int K(y)dy=1$, for any $\eta>0$, there exists a $\delta=\delta(\eta)>0$ such that

$$\left(\int_{|y|\leqq h/\delta} K(y)dy\right)^{-1}<1+\eta$$

and

$$\int_{|y|>h/\delta} K(y)dy<\eta . \tag{3.6}$$

We suppose that $M^{-1}\leqq h(1-\delta)$ and $h(1+\delta)\leqq M$ without loss of generality.

Now, by (3.5),

$$\begin{aligned}
&W_n(x, h(1+\delta), \delta^2/2)\\
&\quad\geqq\int_{|y|\leqq\delta h/2B_n} K_{\delta^2/2B_n}(y)\frac{B_n}{h(1+\delta)}(x-y)^2G_n((x-y)B_n+I_{h(1+\delta)})dy\\
&\quad\geqq\frac{B_n}{h(1+\delta)}G_n(xB_n+I_h)\int_{|y|\leqq\delta h/2B_n} K_{\delta^2/2B_n}(y)(x-y)^2dy\ .
\end{aligned}$$

Noting that $K_{\delta^2/2B_n}(y)$ is the nonnegative even function for y, we have

$$\begin{aligned}
\int_{|y|\leqq\delta h/2B_n} K_{\delta^2/2B_n}(y)(x-y)^2dy&=\int_{|y|\leqq\delta h/2B_n} K_{\delta^2/2B_n}(y)(x^2-2xy+y^2)dy\\
&=\int_{|y|\leqq\delta h/2B_n} K_{\delta^2/2B_n}(y)(x^2+y^2)dy\\
&\geqq x^2\int_{|y|\leqq\delta h/2B_n} K_{\delta^2/2B_n}(y)dy\ .
\end{aligned}$$

Change of variables gives us that

$$\int_{|y|\leqq\delta h/2B_n} K_{\delta^2/2B_n}(y)dy=\int_{|y|\leqq h/\delta} K(y)dy\ .$$

Hence

$$W_n(x, h(1+\delta), \delta^2/2)\geqq\frac{B_n}{h(1+\delta)}x^2G_n(xB_n+I_h)\int_{|y|\leqq h/\delta} K(y)dy\ .$$

On the other hand, it follows by Lemma 7 that for any $\eta>0$, there exists an $N_8(\eta, \delta(\eta))=N_8(\eta)$ such that

$$W_n(x, h(1+\delta), \delta^2/2)\leqq x^2p(x)+\eta$$

for $n\geqq N_8$. Therefore

$$\begin{aligned}
(3.7)\qquad \frac{B_n}{h}x^2G_n(xB_n+I_h)&\leqq(1+\delta)(x^2p(x)+\eta)\left(\int_{|y|\leqq h/\delta} K(y)dy\right)^{-1}\\
&<(1+\delta)(x^2p(x)+\eta)(1+\eta)<x^2p(x)+C_6\eta+C_7\delta\ ,
\end{aligned}$$

because of $\max_x x^2p(x)<\infty$.

We next estimate

$$\begin{aligned}
&W_n(x, h(1-\delta), \delta^2/2)\\
&\quad=\int_{|y|\leqq\delta h/2B_n} K_{\delta^2/2B_n}(y)\frac{B_n}{h(1-\delta)}(x-y)^2G_n((x-y)B_n+I_{h(1-\delta)})dy\\
&\qquad+\int_{|y|>\delta h/2B_n} K_{\delta^2/2B_n}(y)\frac{B_n}{h(1-\delta)}(x-y)^2G_n((x-y)B_n+I_{h(1-\delta)})dy\\
&\quad\equiv L_1+L_2\ ,\qquad \text{(say)}\ .
\end{aligned}$$

For L_2, we have by (3.7) that for $n\geqq N_8$,

$$L_2 \leqq \int_{|y|>\delta h/2B_n} K_{\delta^2/2B_n}(y)\,[(x-y)^2 p(x-y)+C_6\eta+C_7\delta]dy$$
$$\leqq C_8 \int_{|y|>\delta h/2B_n} K_{\delta^2/2B_n}(y)dy < C_8\eta \;,$$

because of (3.6). For L_1, by using (3.5), we have

$$L_1 \leqq \frac{B_n}{h(1-\delta)} G_n(xB_n+I_h) \int_{|y|\leqq\delta h/2B_n} K_{\delta^2/2B_n}(y)(x-y)^2 dy$$
$$= \frac{B_n}{h(1-\delta)} G_n(xB_n+I_h) \int_{|y|\leqq\delta h/2B_n} K_{\delta^2/2B_n}(y)(x^2+y^2) dy$$
$$\leqq \frac{B_n}{h(1-\delta)} x^2 G_n(xB_n+I_h) + \frac{B_n}{h(1-\delta)} \frac{\delta^2 h^2}{4B_n^2} \int_{|y|\leqq\delta h/2B_n} K_{\delta^2/2B_n}(y)dy$$
$$\leqq \frac{B_n}{h(1-\delta)} x^2 G_n(xB_n+I_h) + \frac{C_9}{B_n} \;.$$

Since $B_n \to \infty$ as $n\to\infty$, for any $\eta>0$, there exists an N_9 such that $C_9/B_n<\eta$ for $n \geqq N_9$. Hence, if $n\geqq N_{10}=\max(N_8, N_9)$, then

$$W_n(x, h(1-\delta), \delta^2/2) \leqq \frac{B_n}{h(1-\delta)} x^2 G_n(xB_n+I_h)+C_{10}\eta \;.$$

By Lemma 7, it follows that for $n\geqq N_8$,

$$W_n(x, h(1-\delta), \delta^2/2) \geqq x^2 p(x) - \eta \;,$$

and consequently for $n \geqq N_{10}$,

$$\frac{B_n}{h} x^2 G_n(xB_n+I_h) \geqq x^2 p(x) - C_{11}\eta - C_{12}\delta \;. \tag{3.8}$$

Since η and δ can be chosen arbitrarily small, the estimations (3.7) and (3.8) conclude the theorem.

We finally obtain the following generalization of Blackwell's renewal theorem for independent random variables.

COROLLARY 1. *Let* $\{X_i\}$ *be a sequence of independent random variables with finite means* μ_i *and finite variances* σ_i^2. *Suppose that* $\lim_{n\to\infty}(n+1)^{-1} \cdot \sum_{i=l}^{l+n} \mu_i = \mu > 0$ *uniformly for* $l=1, 2, \cdots$. *Then, under the conditions* (I), (II), (III) *and* (IV), *we have Blackwell's renewal theorem* (1.1).

4. The case of finitely distinct distribution functions

In this section, we shall give some results for a sequence of independent random variables each of whose distribution functions is one of a finite number of distinct distribution functions.

THEOREM 4. *Let* $\{X_i\}$ *be a sequence of independent random variables with* $\mathrm{E}\,X_i=0$ *and finite variances* $\sigma_i^2>0$. *Suppose that each of* X_i, $i=1, 2,\cdots$ *has one of the* r *possible nonlattice distribution functions* $\{F_k(x), 1\leqq k\leqq r\}$. *Then we have* (2.3). *If we add the condition* (IV) *that* X_i, $i=1, 2,\cdots$ *have the finite third moments, then we have* (2.6).

PROOF. In this case, it is clear that the central limit theorem holds, and that $B_n^2\sim C_1 n$. Denote the characteristic function of $F_k(x)$ by $\phi_k(t)$, $k=1, 2,\cdots, r$. We can easily see that for a sufficiently small $\varepsilon_k>0$,

$$(4.1) \qquad |\phi_k(t)|\leqq\exp\{-\sigma_k^2t^2/4\} \qquad \text{for } |t|<\varepsilon_k ,$$

and that for any $\varepsilon>0$ and $T>\varepsilon$, there exists a positive constant $c_k(\varepsilon, T)$ such that

$$(4.2) \qquad |\phi_k(t)|\leqq\exp\{-c_k\} \qquad \text{for } \varepsilon\leqq|t|\leqq T ,$$

since X_k is nonlattice.

Put $\varepsilon=\min\limits_{1\leqq k\leqq r}\varepsilon_k$ (>0) and $\sigma^2=\min\limits_{1\leqq k\leqq r}\sigma_k^2$ (>0). By (4.1), we have

$$(4.3) \qquad |\phi_k(t)|\leqq\exp\{-\sigma^2t^2/4\} \qquad \text{for } |t|<\varepsilon .$$

Put $c=\min\limits_{1\leqq k\leqq r}c_k$ (>0). It follows from (4.2) that

$$(4.4) \qquad |\phi_k(t)|\leqq\exp\{-c\} \qquad \text{for } \varepsilon\leqq|t|\leqq T .$$

Making use of (4.3) and (4.4), we conclude the required result in exactly the same manner as we have proved Theorem 3.

Blackwell's renewal theorem is given in the following form.

COROLLARY 2. *Suppose that a sequence* $\{X_i\}$ *of independent random variables with finite means* μ_i *satisfies the conditions in Theorem* 4 *except the condition that* $\mathrm{E}\,X_i=0$. *If* $\lim\limits_{n\to\infty}(n+1)^{-1}\sum\limits_{i=l}^{l+n}\mu_i=\mu>0$ *uniformly for* $l=1, 2,\cdots$, *then we have Blackwell's renewal theorem* (1.1).

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