BAYES EQUIVARIANT ESTIMATORS IN A CROSSED CLASSIFICATION RANDOM EFFECTS MODEL

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Summary

The Bayes equivariant estimators of the variance components in the two-way crossed classification random effects model with K ($K \geqq 1$) observations per cell are characterized under the usual assumptions of normality and independence of the random effects. An illustrative example of non-trivial Bayes equivariant estimators derived using a special prior distribution is provided. It is pointed out that for the squared error loss function every Bayes equivariant estimator of the residual variance component is inadmissible.

1. Introduction

Consider a two-way crossed classification random effects model with K observations per cell given by

$$(1) \qquad Y_{ijk}=\mu+a_i+b_j+t_{ij}+e_{ijk} \qquad (i=1,\cdots,I;\ j=1,\cdots,J;\ k=1,\cdots,K)\ ,$$

where $-\infty<\mu<\infty$ is a constant, and a_i, b_j, t_{ij} are random effects and e_{ijk} are random errors. We further assume that a_i, b_j, t_{ij}, and e_{ijk} are all independent and have normal distribution with zero means and respective variances σ_a^2, σ_b^2, σ_t^2, and σ_e^2 ($0\leqq\sigma_a^2, \sigma_b^2, \sigma_t^2, \sigma_e^2<\infty$). The parameters σ_a^2, σ_b^2, σ_t^2, and σ_e^2 are called the variance components. In this paper we characterize all the Bayes estimators of σ_e^2, σ_b^2, σ_t^2, and σ_a^2 which are translation invariant and scale preserving, i.e., if $f(Y_{111},\cdots, Y_{IJK})$ is an estimator of any one of the variance components and if the observations are subjected to any real affine group

$$G=\{Y_{ijk}\rightarrow\beta(Y_{ijk}+\alpha),\ \beta>0,\ -\infty<\alpha<\infty\}$$

then $f(Y_{111},\cdots, Y_{IJK})\rightarrow\beta^2 f(Y_{111},\cdots, Y_{IJK})$. These estimators are called Bayes equivariant estimators (b.e.e.) (see, e.g., Zacks [10]) and similar characterizations of b.e.e. for variance components have recently been obtained in the two and higher stage nested random effects models by Zacks [9] and Sahai [5].

The minimal sufficient statistic for $(\mu, \sigma_e^2, \sigma_t^2, \sigma_b^2, \sigma_a^2)$ with independent coordinates is given by $(\bar{Y}..., S_e^2, S_t^2, S_b^2, S_a^2)$ where $\bar{Y}...$ is the grand mean and S_a^2, S_b^2, S_t^2, and S_e^2 are the sums of squares corresponding to the random effects a_i, b_j, t_{ij}, and the random errors e_{ijk} (see, e.g., Box and Tiao [1], pp. 329–331).

Let $\nu_1 = IJ(K-1)$, $\nu_2 = (I-1)(J-1)$, $\nu_3 = J-1$, $\nu_4 = I-1$, $\nu = IJK-1$; $\rho_1 = \sigma_t^2/\sigma_e^2$, $\rho_2 = \sigma_b^2/\sigma_e^2$, $\rho_3 = \sigma_a^2/\sigma_e^2$; $\omega_1 = 1 + K\rho_1$, $\omega_2 = \omega_1 + IK\rho_2$, $\omega_3 = \omega_1 + JK\rho_2$; $\eta_1 = S_t^2/S_e^2$, $\eta_2 = S_b^2/S_e^2$, and $\eta_3 = S_a^2/S_e^2$. Then it is known and can be shown that $\eta = (\eta_1, \eta_2, \eta_3)$ is a maximal invariant statistic and $S_e^2 \sim \sigma_e^2 \chi^2[\nu_1]$, $S_t^2 \sim \sigma_e^2 \omega_1 \chi^2[\nu_2]$, $S_b^2 \sim \sigma_e^2 \omega_1 \chi^2[\nu_3]$, and $S_a^2 \sim \sigma_e^2 \omega_1 \chi^2[\nu_4]$ (see, e.g., Box and Tiao [1], p. 331). Further using the conditional distribution theory similar to Zacks [9], it can also be proved that $S_e^2 | \eta \sim (\sigma_e^2/\Delta)\chi^2[\nu]$, $S_t^2 | \eta \sim \eta_1(\sigma_e^2/\Delta)\chi^2[\nu]$, $S_b^2 | \eta \sim \eta_2(\sigma_e^2/\Delta)\chi^2[\nu]$, and $S_a^2 | \eta \sim \eta_3(\sigma_e^2/\Delta)\chi^2[\nu]$, where $\Delta = 1 + \eta_1\omega_1^{-1} + \eta_2\omega_2^{-1} + \eta_3\omega_3^{-1}$ and $X|Y$ denotes that X for given Y is distributed as Z.

2. Bayes equivariant estimators

Using a squared error loss function and following the Blackwell-Rao Lehmann-Scheffé's theorem [3] and Stein's sufficiency invariance theorem (see, e.g., Zacks [10], p. 79), we consider equivariant estimators that are functions of the minimal sufficient statistic. These estimators are called sufficiently equivariant. If the variance ratios are known then, as in Zacks [9], there exist best equivariant estimators of variance components; namely $\tilde{\sigma}_e^2 = (S_e^2 + S_t^2\omega_1^{-1} + S_b^2\omega_2^{-1} + S_a^2\omega_3^{-1})/(\nu+2)$, $\tilde{\sigma}_t^2 = \rho_1\tilde{\sigma}_e^2$, $\tilde{\sigma}_b^2 = \rho_2\tilde{\sigma}_e^2$, and $\tilde{\sigma}_a^2 = \rho_3\tilde{\sigma}_e^2$. When variance ratios are unknown, the uniformly minimum mean squared error equivariant estimators do not exist. Subjecting Y_{ijk} to a transformation in G, the minimal sufficient statistic is transformed to $\{\beta(\bar{Y}... + \alpha), \beta^2 S_e^2, \beta^2 S_t^2, \beta^2 S_b^2, \beta^2 S_a^2\}$. Thus all sufficiently translation invariant estimators of the variance components are functions only of $(S_e^2, S_t^2, S_b^2, S_a^2)$ and all sufficiently equivariant estimators of σ_e^2, σ_t^2, σ_b^2, and σ_a^2 can be written in the form $\tilde{\sigma}_e^2 = S_e^2 f_1(\eta)$, $\tilde{\sigma}_t^2 = S_t^2 f_2(\eta)$, $\tilde{\sigma}_b^2 = S_b^2 f_3(\eta)$, and $\tilde{\sigma}_a^2 = S_a^2 f_4(\eta)$. We choose the functions f_i's so that the estimators are Bayes against some prior distribution of $(\sigma_e^2, \rho_1, \rho_2, \rho_3)$. It should be noted that the Bayes equivariant estimators are not necessarily Bayes in the general sense (in which one minimizes the prior risk among all estimators). Further, it can be seen that the prior distribution of σ_e^2 does not play any role in the determination of the Bayes equivariant estimators.

Let ξ represent the trivariate distribution law of (ρ_1, ρ_2, ρ_3). Then using the distribution results of Section 1 and following the methods given in Zacks [9], it is readily derived that the b.e.e. of σ_e^2 is given by

$$(2)\qquad \frac{S_e^2}{(\nu+2)}\frac{\mathrm{E}^*[\{1+\eta_1\omega_1^{-1}+\eta_2\omega_2^{-1}+\eta_3\omega_3^{-1}\}^{-1}]}{\mathrm{E}^*[\{1+\eta_1\omega_1^{-1}+\eta_2\omega_2^{-1}+\eta_3\omega_3^{-1}\}^{-2}]},$$

where $\mathrm{E}^*[\cdot]$ designates the posterior expectation of the quantity in the brackets, given η and the prior distribution ξ of ρ_i's. Similarly the b.e.e. of σ_t^2 is given by

$$(3)\qquad \frac{S_e^2}{(\nu+2)}\frac{\mathrm{E}^*[\rho_1\{1+\eta_1\omega_1^{-1}+\eta_2\omega_2^{-1}+\eta_3\omega_3^{-1}\}^{-1}]}{\mathrm{E}^*[\{1+\eta_1\omega_1^{-1}+\eta_2\omega_2^{-1}+\eta_3\omega_3^{-1}\}^{-2}]},$$

and the expressions of the b.e.e. for σ_b^2 and σ_a^2 are obtained by replacing ρ_1 in (3) by ρ_2 and ρ_3 respectively.

Finally using the distribution results of Section 1 and the methodology similar to Klotz *et al.* [2], Stein [7], and Zacks [9], it can be proved that given any b.e.e. of σ_e^2, one can construct a non-equivariant estimator which has uniformly smaller mean squared error. This shows that all b.e.e. are inadmissible in the general class of all estimators of σ_e^2.

3. An illustrative example

Now we derive explicit expressions for some non-trivial Bayes equivariant estimators. For simplicity we consider the special case of the model (1) with one observation per cell. In this case there are only three components of variance, namely σ_a^2, σ_b^2, and σ_e^2. Redefine $\nu_1=(I-1)(J-1)$, $\nu_2=J-1$, $\nu_3=I-1$, $\nu=IJ-1$; $\rho_1=\sigma_b^2/\sigma_e^2$, $\rho_2=\sigma_a^2/\sigma_e^2$; $\omega_1=1+I\rho_1$, $\omega_2=1+J\rho_2$; $\eta_1=S_b^2/S_e^2$, and $\eta_2=S_a^2/S_e^2$. Then we note that $(\eta_1,\eta_2)\sim(\omega_1U_1,\omega_2U_2)$, where (U_1,U_2) has a bivariate inverted Dirichlet distribution (see, e.g., Tiao and Guttman [8]) given by

$$p(u_1,u_2)\propto u_1^{\nu_2/2-1}u_2^{\nu_3/2-1}(1+u_1+u_2)^{-\nu/2},$$

for $0\leqq u_1, u_2<\infty$ and zero elsewhere. Thus the joint density function of (η_1,η_2) given (ρ_1,ρ_2) is

$$(4)\qquad p(\eta_1,\eta_2|\rho_1,\rho_2)\propto\omega_1^{-\nu_2/2}\omega_2^{-\nu_3/2}\eta_1^{\nu_2/2-1}\eta_2^{\nu_3/2-1}(1+\eta_1\omega_1^{-1}+\eta_2\omega_2^{-1})^{-\nu/2},$$

for $0\leqq\eta_1$, $\eta_2<\infty$ and zero elsewhere. Let $\phi_1=\omega_1^{-1}$, $\phi_2=\omega_2^{-1}$ and assume that (ϕ_1,ϕ_2) has a prior density function $\xi(\phi_1,\phi_2)$. Also $0\leqq\rho_1$, $\rho_2<\infty$ implies that $0\leqq\phi_1$, $\phi_2\leqq1$. The posterior distribution of (ϕ_1,ϕ_2) can now be obtained by combining (4) with $\xi(\phi_1,\phi_2)$ and is given by

$$\xi(\phi_1,\phi_2|\eta_1,\eta_2)\propto\xi(\phi_1,\phi_2)\phi_1^{\nu_2/2}\phi_2^{\nu_3/2}(1+\eta_1\phi_1+\eta_2\phi_2)^{-\nu/2},$$

for $0\leqq\phi_1$, $\phi_2\leqq1$, and zero elsewhere, and where we have omitted factors which are constant or depend on the data and play no role on the analysis that follows. Now, for mathematical simplicity, we let the prior distribution $\xi(\phi_1, \phi_2)$ be uniform. Then the posterior distribution is given by

$$\xi(\phi_1, \phi_2 \mid \eta_1, \eta_2) \propto \phi_1^{\nu_2/2}\phi_2^{\nu_3/2}(1+\eta_1\phi_1+\eta_2\phi_2)^{-\nu/2} .$$

Now, to obtain the b.e.e. of σ_e^2, we have to evaluate the ratio of the posterior expectations of $(1+\eta_1\phi_1+\eta_2\phi_2)^{-1}$ and $(1+\eta_1\phi_1+\eta_2\phi_2)^{-2}$. The required ratio is readily obtained by noting that for $p>0$, $q>0$, $t-p-q>0$, $c>0$, and $d>0$

$$(5)\qquad \int_0^1\int_0^1 \frac{x^{p-1}y^{q-1}}{(1+cx+dy)^t}dxdy=\frac{\Gamma(p)\Gamma(q)\Gamma(t-p-q)}{c^p d^q \Gamma(t)}D_{c,d}(p, q, t-p-q) ,$$

where $D_{c,d}(l, m, n)$ represents the cumulative distribution function of the bivariate inverted Dirichlet distribution (Tiao and Guttman [8]), and $\Gamma(r)$ is the usual complete gamma function. Thus the b.e.e. of σ_e^2 against the chosen prior is given by

$$(6)\qquad \frac{S_e^2}{(\nu_1-2)}\frac{D_{\eta_1,\eta_2}(\nu_2/2+1, \nu_3/2+1, \nu_1/2-1)}{D_{\eta_1,\eta_2}(\nu_2/2+1, \nu_3/2+1, \nu_1/2)} .$$

To obtain the b.e.e. of σ_b^2, one further needs to evaluate the ratio of the posterior expectations of $\rho_1(1+\eta_1\phi_1+\eta_2\phi_2)^{-1}=\{\phi_1^{-1}(1+\eta_1\phi_1+\eta_2\phi_2)^{-1}-(1+\eta_1\phi_1+\eta_2\phi_2)^{-1}\}/I$ and $(1+\eta_1\phi_1+\eta_2\phi_2)^{-2}$. The required ratio is again obtained by using the integral identity (5) and then the b.e.e. of σ_b^2 is given by

$$(7)\qquad \frac{1}{I}\left[\frac{S_b^2}{\nu_2}\frac{D_{\eta_1,\eta_2}(\nu_2/2, \nu_3/2+1, \nu_1/2)}{D_{\eta_1,\eta_2}(\nu_2/2+1, \nu_3/2+1, \nu_1/2)} - \frac{S_e^2}{(\nu_1-2)}\frac{D_{\eta_1,\eta_2}(\nu_2/2+1, \nu_3/2+1, \nu_1/2-1)}{D_{\eta_1,\eta_2}(\nu_2/2+1, \nu_3/2+1, \nu_1/2)}\right] .$$

Similarly the b.e.e. of σ_a^2 is given by

$$(8)\qquad \frac{1}{J}\left[\frac{S_a^2}{\nu_3}\frac{D_{\eta_1,\eta_2}(\nu_2/2+1, \nu_3/2, \nu_1/2)}{D_{\eta_1,\eta_2}(\nu_2/2+1, \nu_3/2+1, \nu_1/2)} - \frac{S_e^2}{(\nu_1-2)}\frac{D_{\eta_1,\eta_2}(\nu_2/2+1, \nu_3/2+1, \nu_1/2-1)}{D_{\eta_1,\eta_2}(\nu_2/2+1, \nu_3/2+1, \nu_1/2)}\right] .$$

There is some resemblance in the forms of the b.e.e. (6), (7), and (8) and some of the formal Bayes estimators derived in Sahai and Ramirez-Martinez [6]. Further, since $D_{\eta_1,\eta_2}(\cdot, \cdot, \cdot)\to 1$ as $\eta_1\to\infty$, $\eta_2\to\infty$, we obtain that

$$\lim_{\substack{\eta_1\to\infty\\ \eta_2\to\infty}} \tilde{\sigma}_e^2=\frac{S_e^2}{\nu_1-2} , \qquad \lim_{\substack{\eta_1\to\infty\\ \eta_2\to\infty}} \tilde{\sigma}_b^2=\frac{1}{I}\left[\frac{S_b^2}{\nu_2}-\frac{S_e^2}{\nu_1-2}\right] ,$$

and

$$\lim_{\substack{\eta_1\to\infty \\ \eta_2\to\infty}} \tilde{\sigma}_a^2 = \frac{1}{J}\left[\frac{S_a^2}{\nu_3} - \frac{S_e^2}{\nu_1-2}\right].$$

Thus for large values of ν_1, these estimators are essentially equivalent to the analysis of variance estimators. The mean squared error properties of these and some other classical and Bayesian estimators have been studied in Sahai [5]. Numerical comparison of their mean squared error functions show that these estimators compare favorably with the analysis of variance estimators.

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