# SOME NONPARAMETRIC TESTS AND SELECTION PROCEDURES FOR MAIN EFFECTS IN TWO-WAY LAYOUTS

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## Summary

In this paper a new distribution-free test based on $U$-statistics is offered for the hypothesis of "row effect." Asymptotic distribution of the test statistic is obtained under the null hypothesis and under translation alternatives. Asymptotic efficiency of this test relative to the classical analysis of variance test is the same as the asymptotic efficiency of Mann-Whitney test relative to the $t$-test. A selection procedure to choose the row with the highest "yield," based on the above test is developed.

## 1. Introduction

Literature on nonparametric tests for one-way layouts is quite extensive (e.g. [3], [14], [15], [19]). For the problem of main effect in two-way layout there is the well known Friedman's test which has a rather low asymptotic efficiency. It is applicable for the case of one observation per cell only. The test discussed below is useful for the case of multiple observations per cell and has a higher asymptotic efficiency than the Friedman's test. Consider the linear model

$$X_{ijk}=\mu+\alpha_i+\beta_j+\varepsilon_{ijk} \tag{1.1}$$

where $\mu$ is the general mean, $\alpha_i$ ($i=1,\cdots,r$) effect of $i$th row, $\beta_j$ ($j=1,\cdots,c$) effect of $j$th column and $\varepsilon_{ijk}$ ($k=1,\cdots,n_{ij}$, $n_{ij}\geqq 1$) is a random error component. We assume that $\varepsilon_{ijk}$ (for all $i$, $j$, $k$) are independent, identically distributed random variables with a common continuous distribution function $F$ with median zero. Without loss of generality we can assume that $\sum_{i=1}^{r}\alpha_i=\sum_{j=1}^{c}\beta_j=0$. Under this set up we wish to test the null hypothesis

$$H_0:\ \alpha_1=\cdots=\alpha_r\ , \tag{1.2}$$

i.e. the hypothesis of no "row effect."

## 2. The test statistics

Let $\phi(t)=1$ if $t>0$, $=1/2$ if $t=0$ and $=0$ otherwise. Define

$$U_{i,i',j}=\sum_{k=1}^{n_{ij}}\sum_{l=1}^{n_{i'j}}\phi(X_{ijk}-X_{i'jl})/n_{ij}n_{i'j}\ . \tag{2.1}$$

Note that due to the assumption about continuity of $F$, ties occur only with zero probability and can be ignored. Let

$$U_i=\sum_{\substack{i'=1\\ i'\neq i}}^{r}\sum_{j=1}^{c}U_{i,i',j}\ . \tag{2.2}$$

For the case $n_{ij}=n$ for all $i$ and $j$, we propose the following statistic for testing $H_0$:

$$S_1=\frac{12n}{r^2c}\sum_{i=1}^{r}(U_i-(r-1)c/2)^2\ . \tag{2.3}$$

More generally, if the number of observations per cell is not the same, let $n_{ij}=Np_{ij}$, $0<p_{ij}<1$, $\sum_{i,j}p_{ij}=1$. (Clearly when $n_{ij}=n$ $p_{ij}=1/rc$.) Further let $q_{ij}=p_{ij}^{-1}$ and

$$q_{i\cdot}=\sum_{j=1}^{c}p_{ij}^{-1}\ ,\qquad q_{\cdot\cdot}=\sum_{i=1}^{r}q_{i\cdot}\ ,\qquad q^*_{\cdot\cdot}=\sum_{i=1}^{r}q_{i\cdot}^{-1}\ . \tag{2.4}$$

The statistic proposed for this set-up is

$$S_2=\frac{12N}{r^2}\left[\sum_{i=1}^{r}(U_i-(r-1)c/2)^2/q_{i\cdot}-\left\{\sum_{i=1}^{r}(U_i-(r-1)c/2)/q_{i\cdot}\right\}^2\Big/q^*_{\cdot\cdot}\right]\ , \tag{2.5}$$

when $N=\sum_i\sum_j n_{ij}$. The test based on $S_1$ ($S_2$) consists of rejecting $H_0$ at a level of significance $\alpha$ if $S_1$ ($S_2$) exceeds a predetermined constant $S_{1\alpha}$ ($S_{2\alpha}$). We claim that $S_{1\alpha}$ and $S_{2\alpha}$ are free of $F$ under $H_0$ and hence the tests are distribution-free; for briefly

$$\mathrm{P}(S_2=s_2)=\sum_{*}\mathrm{P}(U_i=a_i,\ i=1,\cdots,r)$$

where $*$ denotes the sum over all distinct vectors $\boldsymbol{a}=(a_1,\cdots,a_r)'$ such that $U_i=a_i\ \forall\, i\Rightarrow S_2=s_2$. Also

$$\begin{aligned}&\mathrm{P}(U_i=a_i,\ i=1,\cdots,r)\\&\quad=\sum_{**}\mathrm{P}(U_{i,i',j}=a_{i,i',j},\ i'=1,\cdots,r,\ i'\neq i,\ j=1,\cdots,c,\ i=1,\cdots,r)\end{aligned}$$

where $**$ denotes the sum over all distinct sets of values $a_{i,i',j}$ such that $U_{i,i',j}=a_{i,i',j}\ \forall\, i,\ i',\ j\Rightarrow U_i=a_i\ \forall\, i$. Finally,

$$\begin{aligned}\mathrm{P}(U_{i,i',j}=a_{i,i',j}\ \forall\, i,\, i',\, j) \\ =\sum_{***} \mathrm{P}(\phi(X_{ij\alpha}-X_{i'j\beta})=a_{i,i',j,\alpha,\beta},\ \forall\, i,\, i',\, j,\ \alpha=1,\cdots,n_{ij}, \\ \beta=1,\cdots,n_{i'j}),\end{aligned}$$

where $***$ denotes the sum over all distinct sets of values $a_{i,i',j,\alpha,\beta}$ such that $\phi(X_{ij\alpha}-X_{i'j\beta})=a_{i,i',j,\alpha,\beta}\ \forall\, i,\, i',\, j,\, \alpha,\, \beta \Rightarrow U_{i,i',j}=a_{i,i',j}\ \forall\, i,\, i',\, j$. However this is nothing but a complete or partial ordering of the observations in each block. And when $H_0$ is true, all orderings within a block are equally likely. Hence when $H_0$ is true $S_2$ (and $S_1$) takes on different values with probabilities that are free of $F$. In the next section it is shown that both $S_1$ and $S_2$ have asymptotically, as $N\rightarrow\infty$, a chi-square distribution with $r-1$ degrees of freedom. Thus a large sample approximation for $S_{1\alpha}$ and $S_{2\alpha}$ is provided by the upper $\alpha$-point of the chi-square distribution with $r-1$ degrees of freedom. For the special case $r=2$ the use of $S_1$ ($S_2$) would be equivalent to applying normal approximation to van Elteren [8] statistic for comparison of two treatments. In this sense $S_1$ ($S_2$) could be considered as extensions of van Elteren's approach for the case of several treatments.

## 3. The asymptotic distribution under $H_0$

It can be easily seen that

$$\mathrm{E}_0\, U_{i,i',j}=1/2\ ,$$

$$\mathrm{V}_0\, U_{i,i',j}=(n_{ij}+n_{i'j}+1)/12 n_{ij} n_{i'j}\ ,$$

$$\mathrm{Cov}_0\,(U_{i,i',j},\, U_{i,i'',j})=1/12 n_{ij}\ ,$$

$$\mathrm{E}_0\, U_i=(r-1)c/2\ ,$$

$$\mathrm{V}_0\, U_i=\frac{1}{12}\sum_{j=1}^{c}\left[\frac{(r-1)^2}{n_{ij}}+\sum_{i'\neq i}^{r}\left(\frac{1}{n_{ij}}+\frac{1}{n_{ij}n_{i'j}}\right)\right],$$

and

$$\mathrm{Cov}_0\,(U_i,\, U_k)=\frac{1}{12}\sum_{j=1}^{c}\left[\sum_{i'\neq j,k}\frac{1}{n_{i'j}}-\frac{(r-2)}{n_{ij}}-\frac{(r-2)}{n_{kj}}-\frac{n_{ij}+n_{kj}+1}{n_{ij}n_{kj}}\right].$$

The computations involved are routine (see e.g. [18], [19]) and hence all details are omitted. Now we shall prove the assertion regarding the asymptotic distributions of $S_1$ and $S_2$ by invoking the properties of $U$-statistics. Note that $U_{i,i',j}$ for each $i$, $i'$ and $j$ is a two-sample generalized $U$-statistics (see Sukhatmé [20]) corresponding to $\phi$. Therefore as $N\rightarrow\infty$, $N^{1/2}(U_{i,i',j}-1/2)\ \forall\, i,\, i',\, j$ $(i'\neq i)$ have a limiting joint $r(r-1)c$-variate normal distribution with null mean vector and appropriate covariance matrix (Lemma 3.1 of [3]). Since $U_i$'s are linear combinations of $U_{i,i',j}$'s, $N^{1/2}(\boldsymbol{U}-((r-1)c/2)\boldsymbol{J})$, where $\boldsymbol{U}=(U_1,\cdots,U_r)'$ and $\boldsymbol{J}$ is a column vector of unit elements, has a limiting $r$-variate normal distribution with null mean vector and covariance matrix $\boldsymbol{\Sigma}=[\sigma_{ik}]$, where

$$\sigma_{ii}=\frac{1}{12}\sum_{j=1}^{c}\left\{(r-1)^2 q_{ij}+\sum_{i'\neq i} q_{i'j}\right\} \tag{3.1}$$

and for $i\neq k$

$$\sigma_{ik}=\frac{1}{12}\sum_{j=1}^{c}\{q_{\cdot j}-r(q_{ij}+q_{kj})\} . \tag{3.2}$$

It is easily seen that rows of $\boldsymbol{\Sigma}$ add up to zero. This is as expected since there is a linear constraint on $U_i$'s, namely

$$\sum_{i=1}^{r} U_i=r(r-1)c/2 .$$

Let $\boldsymbol{a}$ denote $N^{1/2}(\boldsymbol{U}-(r-1)(c/2)\boldsymbol{J})$ and let $\boldsymbol{a}'=(\boldsymbol{a}_0', a_r)$. Let $\boldsymbol{\Sigma}_0$ denote the limiting covariance matrix of $\boldsymbol{a}_0$. Then using the notation in (2.4), $\boldsymbol{\Sigma}_0$ can be expressed as

$$12\boldsymbol{\Sigma}_0=r^2\boldsymbol{D}_0-r(\boldsymbol{\Pi}_0\boldsymbol{J}_0'+\boldsymbol{J}_0\boldsymbol{\Pi}_0')+q_{\cdot\cdot}\boldsymbol{J}_0\boldsymbol{J}_0'$$

where $\boldsymbol{D}_0=\text{Diagonal}\,(q_{i\cdot},\ i=1,\cdots,r-1)$ and $\boldsymbol{\Pi}_0'=(q_{1\cdot},\cdots,q_{(r-1)\cdot})$ which can be inverted following [3]. After some simplification and using the fact that $\sum_i a_i=0$, we get

$$\boldsymbol{a}_0'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{a}_0=\frac{12}{r^2}\left\{\sum_{i=1}^{r} a_i^2/q_{i\cdot}-q_{\cdot\cdot}^{*\,-1}\left(\sum_{i=1}^{r} a_i/q_{i\cdot}\right)^2\right\} .$$

The right-hand side is precisely the statistic $S_2$, which is thus seen to have a limiting chi-square distribution with $(r-1)$ degrees of freedom.

If $n_{ij}=n,\ \forall\, i, j$, then $p_{ij}=1/rc$, $q_{i\cdot}=rc^2$, $q_{\cdot\cdot}=r^2c^2$ and the limiting covariance matrix of $\boldsymbol{a}_0$ is given by

$$12\boldsymbol{\Sigma}_0=r^2c^2[r\boldsymbol{I}_{(r-1)}-\boldsymbol{J}_0\boldsymbol{J}_0']$$

so that

$$\boldsymbol{\Sigma}_0^{-1}=\frac{12}{r^3c^2}[\boldsymbol{I}_{(r-1)}+\boldsymbol{J}_0\boldsymbol{J}_0'] .$$

and $\boldsymbol{a}_0'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{a}_0$ reduces to the statistic $S_1$. We have thus proved the following theorem.

THEOREM 3.1. *If in the model* (1.1), $\alpha_1=\cdots=\alpha_r$ *and* $n_{ij}=Np_{ij}$, *where* $0<p_{ij}<1$ *and* $\sum_{i,j} p_{ij}=1$, *then both the statistics* $S_1$ *and* $S_2$, *defined* *by* (2.3) *and* (2.5) *respectively, have the limiting chi-square distribution with* $r-1$ *degrees of freedom, as* $N\rightarrow\infty$.

## 4. Asymptotic distribution under translation alternatives

DEFINITION. A distribution function $F$ satisfies "Assumption A" if it possesses a continuous derivative $f$ and there exists a function $g$ such that for almost all $x(F)$ and for all sufficiently small $h$,

$$|(f(x+h)-f(x))/h|\leqq g(x)$$

and

$$\int_{-\infty}^{\infty} g(x)f(x)dx<\infty\ .$$

THEOREM 4.1. *Consider a sequence of Pitman alternatives given by*

$$H_N:\ X_{ijk}=\mu+N^{-1/2}\lambda_i+\beta_j+\varepsilon_{ijk}\,,\ \forall\, i,\, j,\, k \tag{4.1}$$

*where, not all* $\lambda_i$*'s are equal and* $\varepsilon$*'s behave as in* (1.1). *Also assume without loss of generality that* $\sum_i \lambda_i=0$. *Then as* $N\rightarrow\infty$, $S_2$ *has the limiting noncentral chi-square distribution with* $(r-1)$ *degrees of freedom and the noncentrality parameter*

$$12c^2(\psi(F))^2\left\{\sum_{i=1}^{r}\lambda_i^2/q_{i\cdot}-\left(\sum_{i=1}^{r}\lambda_i/q_{i\cdot}\right)^2 q_{\cdot\cdot}^{*\,-1}\right\}\ , \tag{4.2}$$

*where* $\psi(F)=\int_{-\infty}^{\infty} f^2(x)dx$.

PROOF.

$$\begin{aligned}
\mathrm{E}_N(U_{i,i',j})&=\mathrm{P}_N(X_{ijk}>X_{i'jl})\\
&=\mathrm{P}(\varepsilon_{ijk}+N^{-1/2}(\lambda_i-\lambda_{i'})>\varepsilon_{i'jl})\\
&=\int_{-\infty}^{\infty} F(x+N^{-1/2}(\lambda_i-\lambda_{i'}))dF(x)\\
&=\int_{-\infty}^{\infty}[F(x)+N^{-1/2}(\lambda_i-\lambda_{i'})f(x+h)]dF(x)\\
&\qquad\qquad\qquad(\text{for } |h|<N^{-1/2}|\lambda_i-\lambda_{i'}|)\\
&=1/2+N^{-1/2}(\lambda_i-\lambda_{i'})\int_{-\infty}^{\infty} f^2(x)dx\\
&\quad+\frac{N^{-1}}{2}(\lambda_i-\lambda_{i'})^2\int_{-\infty}^{\infty}\frac{f(x+h)-f(x)}{N^{-1/2}(\lambda_i-\lambda_{i'})}dF(x)\ .
\end{aligned}$$

Hence, in view of assumption A,

$$\mathrm{E}_N(U_{i,i',j})=1/2+N^{-1/2}(\lambda_i-\lambda_{i'})\psi(F)+O(N^{-1})\ .$$

It follows that

$$\mathrm{E}_N(U_i)=(r-1)c/2+N^{-1/2}rc\lambda_i\phi(F)+O(N^{-1}) . \tag{4.3}$$

Similarly, under assumption A,

$$\begin{aligned}\mathrm{E}_N\{\phi(X_{ijk}-X_{i'jl})\phi(X_{ijk'}-X_{i'jl'})\}\\ &=1/2+O(N^{-1/2}) \quad \text{if } k=k' \text{ and } l=l'\\ &=1/3+O(N^{-1/2}) \quad \text{if } k=k' \text{ or } l=l' \text{ but not both}\\ &=1/4+O(N^{-1/2}) \quad \text{if } k\neq k' \text{ and } l\neq l' .\end{aligned}$$

As a consequence, as $N\to\infty$, $NV_N(U_{i,i',j})$ and $N\,\mathrm{Cov}_N(U_{i,i',j}, U_{i,i'',j})$ converge to the corresponding values under $H_0$ (see Section 3) when $N\to\infty$. Consequently the covariance matrix of $\boldsymbol{a}$ converges to the matrix $\boldsymbol{\Sigma}$ given by (3.1) and (3.2). Thus it follows that $\boldsymbol{a}$ has the $r$-variate limiting normal distribution with mean vector $rc\phi(F)(\lambda_1,\cdots,\lambda_r)'$ and covariance matrix $\boldsymbol{\Sigma}$ and $\boldsymbol{a}_0'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{a}_0$ has the limiting noncentral chi-square distribution with $(r-1)$ degrees of freedom and the noncentrality parameter

$$r^2c^2(\phi(F))^2\boldsymbol{\lambda}_0'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\lambda}_0 \tag{4.4}$$

under the sequence of alternatives $\{H_N\}$, where $\boldsymbol{\lambda}'=(\boldsymbol{\lambda}_0', \lambda_r)=(\lambda_1,\cdots,\lambda_r)$. The expression (4.4) reduces to (4.2) as in Section 3.

COROLLARY 4.1. *Under* $\{H_N\}$, $S_1$ *has the limiting noncentral chi-square distribution with* $(r-1)$ *degrees of freedom and the noncentrality parameter*

$$12(\phi(F))^2\sum_{i=1}^{r}\lambda_i^2/r . \tag{4.5}$$

PROOF. Notice that $q_{i.}=rc^2$ when $n_{ij}=n\;\forall\; i, j$. Also $\sum_i \lambda_i=0$. The simplification follows immediately as in Section 3.

## 5. Asymptotic relative efficiency

*Case* 1. Let $n_{ij}=n\;\forall\; i, j$. In the classical ANOVA the procedure used for testing equality of row effects is to reject $H_0$ for large values of

$$\sum_{i=1}^{r}(x_{i..}-x_{...}/r)^2/nc\hat{\sigma}^2$$

where dots denote totals and $\hat{\sigma}^2$ is the error mean-square. Under $\{H_N\}$, this statistic has the limiting noncentral chi-square distribution with $(r-1)$ degrees of freedom and the noncentrality parameter

$$\sum_{i=1}^{r}\lambda_i^2/r\sigma^2 ,$$

where $\sigma^2$ is the variance of the distribution $F$. It follows that the asymptotic efficiency of $S_1$-test relative to this parametric test ($P$) is

$$e_{S_1,P}=12\sigma^2(\psi(F))^2 . \tag{5.1}$$

This is precisely the asymptotic efficiency of the Wilcoxon-Mann-Whitney test relative to the $t$-test. It is well known (see Hodges and Lehmann [12]), e.g., that it is bounded below (for all continuous distributions $F$ with finite variance) by 0.864, but that there is no upper bound. This expression, we note, is free of $r$ and $c$, and equals $3/\pi$ when $f$ is the density of a normal variate.

*Case* 2. More generally if the number of observations in the $(i, j)$-th cell is $n_{ij}$, the classical (least squares) test statistic has the limiting noncentral chi-square distribution with $(r-1)$ degrees of freedom and the noncentrality parameter (see Graybill [11], p. 298),

$$\sum_{i,j} p_{ij}\left(\lambda_i-\sum_{k=1}^{r}\left[p_{kj}\lambda_k\Big/\sum_i p_{ij}\right]\right)^2\Big/\sigma^2 ,$$

under the same sequence of alternatives $\{H_N\}$. As is immediately obvious, here the ratio of the two noncentrality parameters is not free of $\lambda_i$, $i=1,\cdots,r$. This makes the expression for asymptotic efficiency virtually impossible to interpret since it depends on the particular sequence of alternatives $\{H_N\colon \alpha_i=N^{-1/2}\lambda_i,\ i=1,\cdots,r\}$. Hence as an alternative we consider the test proposed by Yates [21] (see also Bancroft [1], p. 24). Here, consider the vector $\boldsymbol{Y}'=(\boldsymbol{Y}_0', Y_r)$ where $\bar{Y}_i=\bar{X}_i^*-\bar{X}^*$,

$$\bar{X}_i^*=\sum_{j=1}^{c}\left(\sum_{k=1}^{n_{ij}} X_{ijk}/n_{ij}\right)\Big/c \qquad \text{and} \qquad \bar{X}^*=\sum_{i=1}^{r}\bar{X}_i^*/r .$$

It is easy to check that

$$\mathrm{V}\,(Y_i)=\frac{\sigma^2}{Nr^2c^2}[(r^2-2r)q_{i\cdot}+q_{\cdot\cdot}]$$

and

$$\mathrm{Cov}\,(Y_i, Y_{i'})=\frac{\sigma^2}{Nr^2c^2}[-r(q_{i\cdot}+q_{i'\cdot})+q_{\cdot\cdot}] .$$

Notice that the elements of the covariance matrix of $\boldsymbol{Y}$ (say $\boldsymbol{B}(\sigma^2)$) differ from elements of the covariance matrix of $\boldsymbol{U}$ (see (3.1) and (3.2)) only by a constant multiple. Consider a procedure which rejects $H_0$ if $N\boldsymbol{Y}'B^{*-}\boldsymbol{Y}$ is too large, where $B^*=B(\hat{\sigma}^2)$ and $\hat{\sigma}^2$ is the error mean square or any consistent estimator of $\sigma^2$. This statistic simplifies to

$$\frac{Nc^2}{\hat{\sigma}^2}[\sum_i Y_i^2/q_{i\cdot}-(\sum_i Y_i/q_{i\cdot})^2/q_{\cdot\cdot}^*] . \tag{5.2}$$

It is easy to see that the "sum of squares" for rows given in Bancroft ([1], p. 25), is identical to the numerator of (5.2) and thus Yates' statistic is asymptotically equivalent to it. This statistic has in the limit the chi-square distribution with $(r-1)$ degrees of freedom. The distribution is central under $H_0$ and noncentral under the sequence of alternatives $\{H_N\}$, with the noncentrality parameter

$$\frac{c^2}{\sigma^2}\left[\sum_i \lambda_i^2/q_{i\cdot}-\left(\sum_i \lambda_i/q_{i\cdot}\right)^2/q^*_{\cdot\cdot}\right] .$$

Hence it follows that the asymptotic efficiency of the test based on $S_2$ relative to the Yates' test $(Y)$ is

$$e_{S_2,Y}=12\sigma^2(\psi(F))^2 . \tag{5.3}$$

## 6. Discussion

For the case with equal number of observations (say $n$) per cell, Lehmann [16] has proposed a test for the hypothesis of no row effect in two-way layouts. He has shown that the asymptotic efficiency of this test (say $L$-test) relative to the $F$-test is $12\sigma^2(\psi(F))^2$ which is precisely the efficiency of $S_1$-test relative to the $F$-test. It follows that asymptotic efficiency of $S_1$-test relative to $L$-test is

$$e_{S_1,L}=1 .$$

However, while $L$-test is only asymptotically distribution-free, $S_1$-test is exactly distribution-free.

On the other hand asymptotic efficiency of Friedman's test relative to $F$-test is $12\sigma^2(\psi(F))^2 r/(r+1)$, which is rather low, particularly for small values of $r$. One obvious way of improving this efficiency is to have more than one observation per cell. Consider the following generalization of Friedman's test for the case of $n$ observations per cell $(n\geqq 1)$. Let $U_i$, $i=1,\cdots,r$ be as defined in (2.2). The test rejects the null hypothesis of no row effect if

$$S_3=\frac{12n^2}{rc(1+nr)}\sum_{i=1}^{r}(U_i-(r-1)c/2)^2$$

is too large. Notice that for $n=1$, $S_3$ reduces to the Friedman's statistic. It can be shown, using central limit theorem, that as $c\to\infty$, for fixed $n$, $S_3$ has a limiting central chi-square distribution with $(r-1)$ degrees of freedom when the null hypothesis is true. Conover ([6], p. 273) has recently proposed another generalization which is equivalent to $S_3$ and which he has arrived at from the "mean rank" approach. It can be shown, after some computations similar to those in Sections

4 and 5 that the asymptotic efficiency of $S_3$ relative to $F$-test is

$$12\sigma^2(\psi(F))^2 nr/(1+nr) ,$$

for a sequence of alternatives $\{H_c\}$ given by $H_c : \alpha_i = c^{-1/2}\lambda_i,\ i=1,\cdots,r$. When the underlying distribution is normal this expression reduces to $3nr/\pi(1+nr)$. The following table gives values of this expression for some specific values of $n$ and $r$. Note that the case $n=1$ gives the values for Friedman's test and entries corresponding to increasing values of $n$ show the improvement due to $S_3$. The entries are symmetric with respect to $n$ and $r$ in view of the form of $e_{S_3,F}$.

Table 1 A.R.E. of the generalized Friedman test relative to $F$-test when the underlying distribution is normal

| $r$ \ $n$ | 1 | 2 | 3 | 4 | 5 | 10 |
|---|---|---|---|---|---|---|
| 2 | .637 | .764 | .819 | .849 | .868 | .910 |
| 3 | .716 | .819 | .859 | .881 | .895 | .924 |
| 4 | .764 | .849 | .881 | .899 | .910 | .932 |
| 5 | .796 | .868 | .895 | .910 | .918 | .936 |
| 10 | .868 | .910 | .924 | .932 | .936 | .945 |

Hodges and Lehmann [13] have remarked that the fact that Friedman's test only uses intrablock ranking and throws away any potential information in interblock comparisons (after adjustment for the block effect) may be responsible for the relatively low efficiency. Indeed substancial improvement in asymptotic efficiency is achieved ($3/\pi$ for normal distribution) by using their aligned rank tests. And this is without increasing the sampling cost, which initially seems remarkable. However, it has to be noted that the distribution-free property of the statistic is lost even though it is preserved asymptotically.

It is worth noting at this point that for many nonparametric procedures there is a fair agreement between the exact and asymptotic significance points even for very small sample sizes. Deshpandé [7], Bhapkar and Deshpandé [4] have demonstrated that for their $c$-sample location and scale procedures 3 to 4 observations per sample are enough to assure reasonable accuracy of the asymptotic approximations. Sukhatmé [20] in an earlier work found out that for his two-sample tests $n=3$ was "large enough." An inspection of the small sample significance points of Kruskal-Wallis test and Friedman's test reveals a similar phenomenon.

Finally notice that Friedman's test, Hodges-Lehmann's aligned rank test, $S_3$-test etc. assume that the number of columns, $c$, is large. We know from the results of the last section that use of tests based on

$S_1$ and $S_2$ results in high asymptotic efficiency, for *any* number of columns, provided of course that the number of observations in each cell is sufficiently large. In the light of what has been said in the earlier paragraph, it is expected that this number would have to be only moderately large for the approximations to be valid.

## 7. Selection problem

Consider the model

$$X_{ijk}=\mu+\alpha_i+\beta_j+\varepsilon_{ijk} \tag{7.1}$$

with the usual assumptions. Our problem is to decide which of the $\alpha_i$'s, $i=1,\cdots,r$, is the largest. Alternatively, let $\alpha_{[1]}\leqq\cdots\leqq\alpha_{[r]}$ be the ordered values of $\alpha_i$, $i=1,\cdots,r$. Then we wish to identify the row associated with $\alpha_{[r]}$ (the problem concerning $\alpha_{[1]}$ is analogous). In the "indifference zone" approach due to Bechhofer [2] it is assumed that

$$\alpha_{[r]}-\alpha_{[r-1]}\geqq\delta\ , \tag{7.2}$$

where $\delta$ is the threshold value prespecified by the experimentor such that when the above difference is less than $\delta$, he is indifferent to the choice between the rows associated with $\alpha_{[r]}$ or $\alpha_{[r-1]}$. The problem is to find the smallest number of observations in each row necessary so that for a given decision (selection) procedure

$$\mathrm{P}\,(\text{correct selection})\geqq\gamma\ , \tag{7.3}$$

where $1/r<\gamma<1$ and $\gamma$ is prespecified. Correct selection of course means that we select the row $s$ and $\alpha_s\geqq\alpha_i+\delta$ $i\neq s$. Commonly, the statistic employed for testing the null hypothesis, that all the parameters of interest ($\alpha$'s in our case) are equal, is used for developing a selection procedure. Bechhofer [2], for instance, considered the set-up of $K$ normal populations with common known variance. Towards identifying the population with the largest mean he proposed that we select that population which produced the largest sample mean. In such a one-way layout, Lehmann [17] pointed out that it is possible to use nonparametric test statistics in this fashion provided we require that the guarantee (7.3) need only be satisfied asymptotically in a certain sense. However, even with this weaker requirement, the procedures are not necessarily nonparametric, in the sense that the smallest number of observations from each group necessary to satisfy this requirement still depends on the underlying density. We shall propose two procedures in a two-way experimental layout based on nonparametric tests and compare them with their parametric counterparts.

## 8. Two procedures based on nonparametric tests

Suppose we have one observation per cell. Let $\phi_{i,i',j}$ equal 1 (1/2) if in the $j$th column, the $i$th row element (i.e. observation) is greater than (equal to) the element in the $i'$th row, and equal zero otherwise. Let

$$U_i = \sum_{i' \neq i}^{r} \sum_{j=1}^{c} \phi_{i,i',j}/c \ . \tag{8.1}$$

The $U_i$'s form the basis of Friedman's test since $c(U_i+1)$ gives the sum of ranks of observations in the $i$th row, when ranked within columns. (Because of the assumption of continuity ties occur only with zero probability.) We propose the following procedure related to these statistics.

$$\text{Select } s\text{th row if } U_s = \max_i U_i \ . \tag{8.2}$$

More generally, suppose now that we have $n_{ij}$ observations in the $(i, j)$-th cell. Let $\phi_{i,i',j,k,l}$ equal 1 if $X_{ijk} > X_{i'jl}$, equal 1/2 if $X_{ijk} = X_{i'jl}$, and equal zero otherwise. Let

$$\phi_{i,i',j} = \sum_{k=1}^{n_{ij}} \sum_{l=1}^{n_{i'j}} \phi_{i,i',j,k,l}/n_{ij}n_{i'j} \ .$$

Further, define

$$U_i = \sum_{i' \neq i}^{r} \sum_{j=1}^{c} \phi_{i,i',j} \ . \tag{8.3}$$

These $U_i$'s form the basis of the statistics $S_1$ and $S_2$ discussed in Sections 1–6. The selection procedure based on these, offered now, is also the same as in (8.2), viz.

$$\text{Select } s\text{th row if } U_s = \max_i U_i \ . \tag{8.4}$$

The probability of correct selection for these procedures is given by

$$\mathrm{P}\,(U_s > U_i \ \forall\, i \neq s \,|\, \alpha_s \geqq \alpha_i + \delta \ \forall\, i \neq s) \ . \tag{8.5}$$

As a competitor, the following 'parametric' procedure naturally suggests itself, for the case of one observation per cell.

$$\text{Select } s\text{th row if } \bar{X}_s = \max_i \bar{X}_i$$

where $\bar{X}_i$, $i=1, \cdots, r$, are means of all (say $c^*$) observations in the respective rows. In this case after manipulations similar to [5] we get

$$\text{Lim}\,\frac{c^*}{c}=\frac{r}{(r+1)}12\sigma^2(\psi(F))^2\,.$$

Hence, the asymptotic efficiency of the procedure (8.2) relative to the parametric procedure based on row means, is the same as the asymptotic efficiency of Friedman's test relative to the ANOVA test for the hypothesis of no row effect against location alternatives. Thus, the technique of comparison of two procedures proposed by Lehmann [17] for the one-way layout extends directly to the case of two-way layouts. The parametric procedure for the case of $n^*$ observations per cell ($N^* = rcn^*$) is

$$\text{Select } s\text{th row if } \bar{X}_s=\max_i \bar{X}_i$$

there the corresponding result is

$$\text{Lim}\,\frac{N^*}{N}=12\sigma^2(\psi(F))^2$$

which is precisely the asymptotic efficiency of the $S_1$-test relative to the ANOVA $F$-test. (See 5.1)

## 9. Problem of unequal samples

It was pointed out in Bhapkar and Goré [5] that the assumption of equal sample sizes in different groups is not necessary for comparing the performance of two selection procedures in one-way layout. In the particular case studied, it was demonstrated that the expression for asymptotic efficiency remains the same provided the frequencies in corresponding groups are in equal proportions.

Here, with the two-way layouts, let $n_{ij}=Np_{ij}$, and consider the selection procedure (8.4).

At first, it seems natural to consider a parametric procedure based on means of all observations in a row, as a competitor. However, that it is not a reasonable procedure, if $n_{ij}^*$ are not equal, is seen from the following.

$$\text{E}\left(\sum_{j=1}^{c}\sum_{k=1}^{n_{ij}^*} X_{ijk}\Big/\sum_{j=1}^{c} n_{ij}^*\right)=\mu+\alpha_i+\sum_j n_{ij}^*\beta_j\Big/\sum_j n_{ij}^*\,,$$

so that the differences in the row means do not reflect differences in $\alpha$'s alone. This difficulty can be eliminated by using unweighted means of cell means. Consider

$$\bar{X}_{ij}=\sum_k X_{ijk}/n_{ij}^*$$

and define

$$\bar{X}_i^* = \sum_{j=1}^{c} \bar{X}_{ij}/c \, .$$

Notice that $\mathrm{E}(\bar{X}_i^*) = \sum_j \mathrm{E}(\bar{X}_{ij})/c = \mu + \alpha_i$, under the usual assumption that $\sum_j \beta_j = 0$. We therefore use the procedure based on $\bar{X}_i^*$'s.

(9.1) Select $s$th row if $\bar{X}_s^* = \max_i \bar{X}_i^*$ .

Again it can be easily verified that efficiency of the procedure (8.4) relative to the one given by (9.1) is

$$\operatorname{Lim} \frac{N^*}{N} = 12\sigma^2(\psi(F))^2 \, ,$$

which is precisely the same as the asymptotic efficiency of the $S_2$-test relative to Yates' test. (See 5.3). One may again recall here the remarks made in Section 5 regarding the above asymptotic relative efficiency.

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