ASYMPTOTIC NORMALITY OF H-L ESTIMATORS BASED ON DEPENDENT DATA

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Abstract

Let $X_1, X_2, \cdots$ be a strictly stationary φ-mixing sequence of r.v.'s with a common continuous cdf F. Let θ be a location parameter of F. We prove the asymptotic normality of a class of Hodges-Lehmann estimators of θ under various regularity conditions on the mixing number φ and the underlying F. We also establish the asymptotic linearity of signed rank statistics in the parameter θ.

Our results also enable us to study the effect of φ-dependence on the asymptotic power of signed rank tests for testing $H_0 : \theta=0$ against $H_n : \theta=\theta_0 n^{-1/2}$, $\theta_0 \neq 0$.

Finally these results are shown to remain valid for strongly mixing processes $\{X_i\}$ also.

1. Summary and introduction

Let $X_1, X_2, \cdots$ be a sequence of r.v.'s with the common cdf F. Let θ be a location parameter of F. Using a class of signed rank statistics based on $\{X_i-\theta,\ 1\leqq i\leqq n\}$, Hodges and Lehmann [2] proposed a class of estimators of θ—called H-L estimators $\hat{\theta}_n$ here—when $\{X_i\}$ are iid. Among other things, they proved the asymptotic normality of $\hat{\theta}_n$ when $\{X_i\}$ are iid F and F satisfies some mild regularity conditions. If $\{X_i\}$ are dependent the definition of $\hat{\theta}_n$ can still be given the same way as is done in (2.5) of [2] (see Theorem 3.3 below), although the condition (D) of [2] may not be satisfied by the underlying signed rank statistics.

In the present paper we prove the asymptotic normality of a class of H-L estimators $\hat{\theta}_n$ when the underlying observations are strictly stationary and φ-mixing (see (2.1) below) such that the joint distribution of $(X_1, \cdots, X_n)$ is continuous and the common cdf F is symmetric and absolutely continuous with bounded uniformly continuous density f.

The class of estimators is generated by bounded, nondecreasing score functions ϕ such that $\phi(1/2)=0$.

We obtain this result by first studying the empirical cumulative based on signed ranks of $\{X_i-\theta n^{-1/2}, 1\leqq i\leqq n\}$, $|\theta|\leqq a$. In Theorem 3.1 below the weak convergence of these empirical cumulatives to a continuous Gaussian process for each fixed θ is established. In the same theorem we also establish *asymptotic uniform linearity* of these empirical cumulatives in $|\theta|\leqq a$ in probability. Both these results are established when F is not symmetric. Using these results and the fact that a signed rank statistic corresponding to general ϕ is a stochastic integral of these empirical cumulatives, Theorem 3.2 gives *asymptotic uniform linearity* of a class of signed rank statistics in $|\theta|\leqq a$ in probability. Lemma 3.5 establishes the asymptotic normality of signed rank statistics under the hypothesis of symmetry of F. The results of Theorem 3.2 and Lemma 3.5 entail the asymptotic normality of $\hat{\theta}_n$ in usual fashion.

The results of Theorem 3.2 and Lemma 3.5 could also be used to study the effect of φ-dependence not only on the asymptotic level of signed rank tests for testing the hypothesis $\theta=0$ (in the spirit of [4]) but also on the asymptotic power of these tests against the sequence of alternatives $\theta=\theta_0 n^{-1/2}$. As should be expected, it turns out if F is symmetric (and has some other properties) the effect of φ-dependence on the power can be measured by the asymptotic variance alone (see Section 4 below). It may be noted that Theorem 3.1 can also be used to establish a version of Theorem 3.2 when F is not symmetric.

Finally let us mention that our results remain valid for any sequence of r.v.'s $\{X_i\}$ for which Lemmas 2.1 and 3.5 can be proved.

2. Notations, assumptions and preliminaries

Let $X_1, X_2, \cdots$ be a sequence of strictly stationary r.v.'s defined on a probability space $(\Omega, \mathcal{A}, P)$. Let $\mathcal{B}(X_1, X_2, \cdots, X_m)$ and $\mathcal{B}(X_{m+n}, \cdots)$ be Borel σ-fields generated by the indicated r.v.'s. The sequence $\{X_i\}$ is such that for each $m\geqq 1$ and $n\geqq 1$ and for every

$$A \in \mathcal{B}(X_1, \cdots, X_m)\,, \qquad B \in \mathcal{B}(X_{m+n}, \cdots)$$

$$|P(AB)-P(A)P(B)|<\varphi(n)P(A) \tag{2.1}$$

where φ is a function on integers such that $\varphi(n)\downarrow 0$ as $n\uparrow\infty$.

C: Furthermore let $\{X_i\}$ be such that the joint distribution of $(X_1, X_2, \cdots, X_n)$ is continuous.

The condition C in particular implies that their marginal cdf F is continuous. Let θ be the location parameter of this distribution. Since our problem is to obtain asymptotic distribution of H-L estimator of θ and since signed rank statistics are invariant under location change, we may without loss of generality assume that the true location parameter is 0. All the probabilities in the sequel will be computed under this additional assumption unless otherwise specified. Define, for θ fixed in an interval $|\theta|\leqq a$,

$$H(x)=F(x)-F(-x)\ , \qquad 0\leqq x<\infty\,;\ =0,\ -\infty<x<0\ , \tag{2.2}$$

$$\begin{aligned}\mu_1(t,\theta)&=F(H^{-1}(t)+\theta)-F(\theta)\ , \qquad 0\leqq t\leqq 1\ ,\\ \mu_2(t,\theta)&=F(\theta)-F(-H^{-1}(t)+\theta)\ , \qquad 0\leqq t\leqq 1\ ,\end{aligned} \tag{2.3}$$

and

$$\mu(t,\theta)=\mu_1(t,\theta)-\mu_2(t,\theta)\ , \qquad 0\leqq t\leqq 1\ . \tag{2.4}$$

Let $R_i(\theta)$ be the rank of $|X_i-\theta|$ among $\{|X_j-\theta|,\ j=1,\cdots,n\}$ for each fixed θ. Introduce

$$S_n(t,\theta)=n^{-1}\sum_{i=1}^{n} I(R_i(\theta)\leqq tn)s(X_i-\theta)\ , \qquad 0\leqq t\leqq 1 \tag{2.5}$$

with $s(x)=I(x\geqq 0)-I(x\leqq 0)$ and $I(A)$=indicator of the set A.

Let ϕ be a function on [0, 1] and let $J(u)=\phi((u+1)/2)$, $0\leqq u\leqq 1$.

$$S_n(J,\theta)=n^{-1}\sum_{i=1}^{n} J\left(\frac{R_i(\theta)}{n}\right)s(X_i-\theta)=\int_0^1 J(t)dS_n(t,\theta)\ . \tag{2.6}$$

To standardize things we introduce

$$\begin{aligned}T_n(t,\theta)&=n^{1/2}(S_n(t,\theta)-\mu(t,\theta))\ , \qquad 0\leqq t\leqq 1\\ T_n(J,\theta)&=n^{1/2}(S_n(J,\theta)-\mu(\theta))\end{aligned} \tag{2.7}$$

with

$$\mu(\theta)=\int_0^1 J(t)d[\mu_1(t,\theta)-\mu_2(t,\theta)]\ . \tag{2.8}$$

To get a different representation of (2.7) and (2.5) we introduce for $|\theta|\leqq an^{-1/2}$ and $0\leqq t\leqq 1$

$$\begin{aligned}W_1(t,\theta)&=n^{-1/2}\sum_{i=1}^{n}[I(0<X_i-\theta\leqq H^{-1}(t))-\mu_1(t,\theta)]\\ W_2(t,\theta)&=n^{-1/2}\sum_{i=1}^{n}[I(-H^{-1}(t)<X_i-\theta\leqq 0)-\mu_2(t,\theta)]\ ,\end{aligned} \tag{2.9}$$

$$
\begin{aligned}
V(t,\theta) &= W_1(t,\theta)-W_2(t,\theta)\\
&= n^{-1/2}\sum_{i=1}^{n}[I(|X_i-\theta|\leqq H^{-1}(t))s(X_i-\theta)-\mu(t,\theta)]\\
W(t,\theta) &= W_1(t,\theta)+W_2(t,\theta)\\
&= n^{-1/2}\sum_{i=1}^{n}[I(|X_i-\theta|\leqq H^{-1}(t))-\mu_1(t,\theta)-\mu_2(t,\theta)]\,.
\end{aligned}
\tag{2.10}
$$

In all the above definitions and in the sequel for any distribution function G, $G^{-1}(t)=\inf\{x;\ G(x)\geqq t\}$ unless otherwise specified. Next let F_n denote the usual empirical cumulative of $\{X_i,\ i=1,\cdots,n\}$. Let

$$
H_{n\theta}(x)=F_n(x+\theta n^{-1/2})-F_n(-x+\theta n^{-1/2})\,,\qquad x\geqq 0\,. \tag{2.11}
$$

Then wp 1 we have for all $0\leqq t\leqq 1$, $|\theta|\leqq a$

$$
\begin{aligned}
T_n(t,\theta n^{-1/2}) &= V(H(H_{n\theta}^{-1}(t)),\theta n^{-1/2})\\
&\quad + n^{1/2}[\mu(H(H_{n\theta}^{-1}(t)),\theta n^{-1/2})-\mu(t,\theta n^{-1/2})]\,.
\end{aligned}
\tag{2.12}
$$

Our basic objective is to obtain asymptotic normality of H-L estimators when original r.v.'s satisfy (2.1). In order to achieve this goal we first obtain weak convergence of $T_n(t,\theta n^{-1/2})$, $0\leqq t\leqq 1$ to a continuous Gaussian process and then using this result and the representation (2.6) we achieve asymptotic normality of $T_n(J,\theta n^{-1/2})$ by establishing asymptotic linearity of $T_n(J,\theta n^{-1/2})$ for $|\theta|\leqq a$. Using asymptotic linearity one gets then asymptotic normality of H-L estimator. Of course all this is true under some conditions on the underlying distribution function and score function ψ.

About F we assume that FH^{-1} is uniformly differentiable which implies that there exists functions $\dot\mu_j$, $j=1,2$, such that for every $K>0$

$$
\lim_{n\to\infty}\ \sup_{|t-s|\leqq Kn^{-1/2}} n^{1/2}|\mu_j(t,0)-\mu_j(s,0)-(t-s)\dot\mu_j(s)|=0\,. \tag{2.13}
$$

Note that if F is symmetric about the origin then $F(H^{-1}(t))=(t+1)/2$, $\dot\mu_j\equiv 1/2$, $j=1,2$. In many of the following statements we will not assume that F is symmetric about the origin. However we will need F to have density f such that

$$
f \text{ is bounded and uniformly continuous}\,. \tag{2.14}
$$

In this case $\dot\mu_1=f(H^{-1})/[f(H^{-1})+f(-H^{-1})]$, $\dot\mu_2=1-\dot\mu_1$.

About the score function ψ we assume

$$
\text{(i) } \psi\uparrow,\qquad \text{(ii) } \psi(1/2)=0\,,\qquad \text{(iii) } \psi \text{ is bounded}\,. \tag{2.15}
$$

As to the solution of the problem, (2.12) gives clear idea as to what is needed for the proof of the weak convergence of $T_n(t, \theta n^{-1/2})$, $0 \leqq t \leqq 1$. Also from the definitions of W_1, W_2 and V it is clear that there is obvious relation between these entities and ordinary empirical cumulatives. Let us define

$$Z_n(t) = n^{1/2}[F_n(F^{-1}(t)) - t] , \qquad 0 \leqq t \leqq 1 . \tag{2.16}$$

Then wp 1 for all $0 \leqq t \leqq 1$ and all $|\theta| \leqq a$

$$\begin{aligned} W_1(t, \theta) &= Z_n(F(H^{-1}(t)+\theta)) - Z_n(F(\theta)) \quad \text{and} \\ W_2(t, \theta) &= Z_n(F(\theta)) - Z_n(F(-H^{-1}(t)+\theta)) . \end{aligned} \tag{2.17}$$

From [1] we recall the following

LEMMA 2.1. (= *Theorem* 22.1 *of* [1])
If $\{X_i\}$ *are strictly stationary* φ*-mixing such that*

$$\sum_{n=1}^{\infty} n^2 \varphi^{1/2}(n) < \infty \tag{2.18}$$

and if F *is continuous cdf then*

$$Z_n \Rightarrow_D Z \tag{2.19}$$

where Z *is a continuous Gaussian process on* $[0, 1]$ *with* $\mathrm{E}\, Z \equiv 0$ *and covariance kernel given by*

$$K(t, s) = t \wedge s + \sum_{k=1}^{n} \mathrm{Cov}\,[Y_1(t), Y_{k+1}(s)] + \sum_{k=1}^{\infty} \mathrm{Cov}\,[Y_1(s), Y_{k+1}(t)] , \qquad 0 \leqq t,\ s \leqq 1 \tag{2.20}$$

where

$$Y_j(t) = [I(F(X_j) \leqq t) - t] , \qquad j \geqq 1,\ 0 \leqq t \leqq 1 .$$

The absolute series in (2.20) *converges. Consequently for every* $\varepsilon > 0$

$$\lim_{\delta \to 0} \overline{\lim_{n \to \infty}} \mathrm{P}\,[\sup_{|t-s| \leqq \delta} |Z_n(t) - Z_n(s)| > \varepsilon] = 0 . \tag{2.21}$$

In the sequel we will say a random sequence "$Y_n = o_p(1)$" iff "$Y_n \to_p o$" as $n \to \infty$. Always limit will be taken when $n \to \infty$ unless otherwise stated. Furthermore $\|\cdot\|$ denotes the sup norm taken over all $0 \leqq t \leqq 1$ and all $|\theta| \leqq a$. Finally Φ will be cdf of standard normal r.v.

3. Some weak convergence results and asymptotic normality of H-L estimator

In this section we first prove asymptotic linearity of $S_n(t, \theta n^{-1/2})$ in $|\theta| \leqq a$ in probability uniformly in $0 \leqq t \leqq 1$. Using this and (2.6) one gets similar results for $S_n(J, \theta n^{-1/2})$, $|\theta|\leqq a$. Then these results together with asymptotic normality of $S_n(J, 0)$ in a usual fashion lead to asymptotic normality of $\hat{\theta}_n$—the H-L estimator as defined in [2]. We begin with

LEMMA 3.1. *Under the condition* (2.1) *with* $\sum_{n=1}^{\infty}\varphi^{1/2}(n)<\infty$ *and continuity of* F *we have*

$$\|H(H_{n\theta}^{-1}(t))-t\|=o_p(1) \tag{3.1}$$

where $H_{n\theta}$ *is given by* (2.11).

PROOF. We have

$$\|H(H_{n\theta}^{-1}(t))-t\|\leqq\sup_{x,\theta}|H_{n\theta}(x)-H_\theta(x)|+\sup_{x,\theta}|H_\theta(x)-H(x)|+O(n^{-1}) \tag{3.2}$$

where $H_\theta(\cdot)=H(\cdot+\theta n^{-1/2})$. Since F is continuous, clearly then

$$\sup_\theta\sup_x|H_\theta(x)-H(x)|\to 0\ . \tag{3.3}$$

On the other hand since $H_{n\theta}(\cdot)=F_n(\cdot+\theta n^{-1/2})-F_n(-\cdot+\theta n^{-1/2})$, to conclude (3.1) it will be enough to prove

$$F_n(x+\theta n^{-1/2})-F(x+\theta n^{-1/2})=o_p(1) \tag{3.4}$$

for each fixed $-\infty<x<+\infty$ and each fixed $|\theta|\leqq a$ because both F_n and F are monotone bounded functions of x and θ.

To prove (3.4) write

$$F_n(x+\theta n^{-1/2})-F(x+\theta n^{-1/2})=n^{-1}\sum_{i=1}^{n}\alpha_i\ ;$$
$$\alpha_i=I(X_i\leqq x+\theta n^{-1/2})-F(x+\theta n^{-1/2}),\ 1\leqq i\leqq n\ .$$

Now apply Lemma 1, p. 170 of [1] to α_1, α_{i+1} with $r=2=s$ to get

$$|\mathrm{Cov}\,(\alpha_1, \alpha_{i+1})|\leqq 2\varphi^{1/2}(i)\,\mathrm{E}\,\alpha_1^2\leqq 2^{-1}\varphi^{1/2}(i)\ , \qquad i\geqq 1\ . \tag{3.5}$$

Here we also used $E\alpha_1^2\leqq 1/4$. Using stationarity, (3.5) above and the fact that $\forall n\geqq 1$, $\sum_{i=1}^{n-1}\varphi^{1/2}(i)\leqq\sum_{i=1}^{\infty}\varphi^{1/2}(i)$, one gets

$$\mathrm{Var}\left(n^{-1}\sum_{i=1}^{n}\alpha_i\right)=n^{-1}\,\mathrm{E}\,\alpha_1^2+2n^{-2}\sum_{i=1}^{n-1}(n-i)\,\mathrm{Cov}\,(\alpha_1, \alpha_{i+1}) \tag{3.6}$$
$$\leqq n^{-1}\left[\frac{1}{4}+\sum_{i=1}^{\infty}\varphi^{1/2}(i)\right]$$

which in view of the fact that $\sum_{i=1}^{\infty}\varphi^{1/2}(i)<\infty$ yield (3.4). The proof of the lemma is terminated.

For any process $X(\cdot, \theta)$ let

$$\omega(X, \delta)=\sup_{|\theta|\leq a}\sup_{|t-s|\leq\delta}|X(t, \theta n^{-1/2})-X(s, \theta n^{-1/2})| .$$

LEMMA 3.2. *Under the conditions of Lemma* 2.1

(3.7) $\omega(W_j, \delta)=o_p(1)$ *as* $n\rightarrow\infty$ *and then* $\delta\rightarrow 0$, $j=1, 2$.

Hence in view of (2.10)

(3.8) $\omega(W, \delta)=o_p(1)$, $\omega(V, \delta)=o_p(1)$

as $n\rightarrow\infty$ *and then* $\delta\rightarrow 0$.

PROOF. The proof of (3.7) follows from (2.17), the fact that

$$\begin{aligned}&||F(H^{-1}(t)+\theta n^{-1/2})-F(H^{-1}(t))||\rightarrow 0 ,\\&||F(\theta n^{-1/2})-F(0)||\rightarrow 0 ,\end{aligned}\tag{3.9}$$

and $F(-H^{-1})$ and FH^{-1} are uniformly continuous and (2.21).

LEMMA 3.3. *Under the conditions of Lemma* 2.1

(3.10) $||W_j(t, \theta n^{-1/2})-W_j(t, 0)||=o_p(1)$, $j=1, 2$.

Hence in view of (2.10)

$$\begin{aligned}&||W(t, \theta n^{-1/2})-W(t, 0)||=o_p(1) \quad \textit{and}\\&||V(t, \theta n^{-1/2})-V(t, 0)||=o_p(1) .\end{aligned}\tag{3.11}$$

PROOF. The proof again follows from (3.9), (2.17) and (2.21).

LEMMA 3.4. *Under the conditions of Lemma* 2.1 *and* (2.14)

(3.12) $n^{1/2}[H(H_{n\theta}^{-1}(t))-t]=-W(t, 0)-\theta b(t, f)+o_p(1)$

uniformly in $|\theta|\leq a$ *and* $0\leq t\leq 1$. *Here*

(3.13) $b(t, f)=f(H^{-1}(t))-f(-H^{-1}(t))$, $0\leq t\leq 1$.

PROOF. We have wp 1 for all $0\leq t\leq 1$, $|\theta|\leq a$

$$\begin{aligned}n^{1/2}[H(H_{n\theta}^{-1}(t))-t]=&-W(H(H_{n\theta}^{-1}(t)), \theta n^{-1/2})\\&-n^{1/2}[H_\theta(H_{n\theta}^{-1}(t))-H(H_{n\theta}^{-1}(t))]+O(n^{-1/2})\end{aligned}\tag{3.14}$$

where $H_\theta(\cdot)=H(\cdot+\theta n^{-1/2})$.

Now observe that since f is bounded, we have

(3.15) $n^{1/2}||H_\theta(H^{-1}(t))-t-\theta n^{-1/2}b(t, f)||\rightarrow 0$.

Hence, uniformly in $0\leq t\leq 1$, $|\theta|\leq a$

$$(3.16) \qquad \begin{aligned} & n^{1/2}[H_\theta(H_{n\theta}^{-1}(t)) - H(H_{n\theta}^{-1}(t))] \\ &= n^{1/2}[H_\theta(H^{-1}(H(_{n\theta}^{-1}(t)))) - H(H_{n\theta}^{-1}(t))] \\ &= \theta b(H(H_{n\theta}^{-1}(t)), f) + o_p(1) \\ &= \theta b(t, f) + o_p(1) . \end{aligned}$$

The last equality follows from the uniform continuity of f and Lemma 3.1.

Next from (3.1), (3.8) and (3.11) we have

$$(3.17) \qquad W(H(H_{n\theta}^{-1}(t)), \theta n^{-1/2}) = W(t, \theta n^{-1/2}) + o_p(1) = W(t, 0) + o_p(1)$$

uniformly in $0 \leqq t \leqq 1$ and $|\theta| \leqq a$. Note that condition (2.18) implies the condition of Lemma 3.1. Combining (3.17) with (3.16) and (3.14) one has (3.12).

THEOREM 3.1. *Let $X_1, X_2, \cdots$ be a strictly stationary sequence of r.v.'s satisfying* (2.1) *and* (2.18) *and* C. *Let F have pdf 'f' such that* (2.14) *is satisfied and let $\mu_j(\cdot, 0)$, $j=1, 2$ satisfy* (2.13). *Then*

$$(3.18) \qquad \|T_n(t, \theta n^{-1/2}) - V(t, 0) + \dot{\mu}(t)\{W(t, 0) + \theta b(t, f)\}\| = o_p(1)$$

with

$$(3.19) \qquad \dot{\mu}(t) = b(t, f)/[f(H^{-1}(t)) + f(-H^{-1}(t))] , \qquad 0 \leqq t \leqq 1 .$$

Consequently

$$(3.20) \qquad \|T_n(t, \theta n^{-1/2}) - T_n(t, 0) + \theta \dot{\mu}(t) b(t, f)\| = o_p(1) .$$

Furthermore

$$(3.21) \qquad \{T_n(t, \theta n^{-1/2}),\ 0 \leqq t \leqq 1\} \Rightarrow_D \{Y(t, \theta),\ 0 \leqq t \leqq 1\}$$

where $Y(\cdot, \theta)$ is a continuous Gaussian process for each θ and $Y(t, \cdot)$ is linear for each t with

$$(3.22) \qquad \mathrm{E}\,(Y(\cdot, \theta)) = -\theta \dot{\mu}(\cdot) b(\cdot, f) ,$$

and

$$(3.23) \qquad \begin{aligned} C(t, s) &= \mathrm{Cov}\,[Y(t, \theta), Y(s, \theta)] \\ &= \lim_{n\to\infty} \mathrm{Cov}\,[V(t, 0) - \dot{\mu}(t) W(t, 0), V(s, 0) - \dot{\mu}(s) W(s, 0)] \\ & \qquad 0 \leqq t,\ s \leqq 1 . \end{aligned}$$

This limit always exists. In particular if F is symmetric about the origin then

$$(3.24) \qquad \|T_n(t, \theta n^{-1/2}) - T_n(t, 0)\| = o_p(1)$$

and

$$\{T_n(t, \theta n^{-1/2}),\ 0 \leqq t \leqq 1,\ |\theta| \leqq a\} \Rightarrow_D \{Y(t, 0),\ 0 \leqq t \leqq 1\} \tag{3.25}$$

with $C(t, s)$ *given by* (3.30) *below.*

PROOF. To begin with let us observe that F symmetric about the origin implies $b(\cdot, f)=0$ and hence (3.24) and (3.25) follow from (3.20) and (3.21) respectively. And (3.20) follows from (3.18) when applied to $T_n(\cdot, 0)$ processes.

In order to prove (3.18) we recall the decomposition (2.12). Now in view of (3.1), (3.8) and (3.11) we have

$$V(H(H_{n\theta}^{-1}(t)), \theta n^{-1/2}) = V(t, \theta n^{-1/2}) + o_p(1) = V(t, 0) + o_p(1) \tag{3.26}$$

uniformly in all $0 \leqq t \leqq 1$ and $|\theta| \leqq a$. Furthermore observe that in view of (2.13) and f being uniformly continuous and bounded we have

$$\sup_{|\theta| \leqq a} \sup_{|t-s| \leqq Kn^{-1/2}} n^{1/2} |\mu_j(t, \theta n^{-1/2}) - \mu_j(s, \theta n^{-1/2}) - (t-s)\dot{\mu}_j(s)| \to 0\ , \quad j=1, 2\ . \tag{3.27}$$

On the other hand in view of Lemmas 3.4, 2.1, and (2.17) we have for every $\varepsilon > 0$ there exists K_ε and $n_\varepsilon \ni n \geqq n_\varepsilon \Rightarrow$

$$\mathrm{P}\,[\| H(H_{n\theta}^{-1}(t)) - t \| \leqq K_\varepsilon n^{-1/2}] \geqq 1 - \varepsilon\ . \tag{3.28}$$

Using (3.28), (3.27) and Lemma 3.4 again one has

$$n^{1/2}[\mu(H(H_{n\theta}^{-1}(t)), \theta n^{-1/2}) - \mu(t, \theta)] = -[W(t, 0) + \theta b(t, f)]\dot{\mu}(t) + o_p(1) \tag{3.29}$$

uniformly in $|\theta| \leqq a$, $0 \leqq t \leqq 1$. Recall that $\mu = \mu_1 - \mu_2$ from (2.4). Combining (3.29) with (3.26) and (2.12) one has (3.18).

Next to prove (3.21) we observe that in view of (2.17) and (2.19) and uniform continuity of FH^{-1} we have

$$\begin{aligned} &V(\cdot, 0) - \dot{\mu}(\cdot)[W(t, 0) + \theta b(\cdot, f)] \\ &\quad = [W_1(\cdot, 0) - W_2(\cdot, 0)] - \dot{\mu}(\cdot)[W_1(\cdot, 0) + W_2(\cdot, 0)] - \theta\dot{\mu}(\cdot)b(\cdot, f) \\ &\quad = (1-\dot{\mu}(\cdot))Z_n(FH^{-1}(\cdot)) + (1+\dot{\mu}(\cdot))Z_n(F(-H^{-1}(\cdot))) \\ &\qquad -2Z_n(F(0)) - \theta\dot{\mu}(\cdot)b(\cdot, f) \\ &\quad \Rightarrow_D (1-\dot{\mu}(\cdot))Z(FH^{-1}(\cdot)) + (1+\dot{\mu}(\cdot))Z(F(-H^{-1}(\cdot))) \\ &\qquad -2Z(F(0)) - \theta\dot{\mu}(\cdot)b(\cdot, f) \\ &\quad \underset{\mathrm{df}}{=} Y(\cdot, \theta)\ . \end{aligned}$$

Clearly then $Y(\cdot, \theta)$ is a continuous Gaussian process for each θ and $Y(t, \cdot)$ is linear for each fixed t. Moreover $C(t, s)$ is the covariance function and the fact that the limit in (3.23) exists is due to the fact that the lim cov. of $Y(\cdot, \theta)$ can be expressed as a transform of K function of (2.20).

To verify (3.25) one observes that F symmetric about the origin implies that

$$F(0)=\frac{1}{2}\,, \qquad F(H^{-1}(t))=\frac{t+1}{2}\,, \qquad F(-H^{-1}(t))=\frac{1-t}{2}$$
$$\text{and} \quad \dot{\mu}(t)\equiv 0\,.$$

In this case

$$\begin{aligned}(3.30)\quad C(t, s)=\text{Cov}\Big[&Z\Big(\frac{t+1}{2}\Big)+Z\Big(\frac{1-t}{2}\Big)-2Z\Big(\frac{1}{2}\Big),\\ &Z\Big(\frac{s+1}{2}\Big)+Z\Big(\frac{1-s}{2}\Big)-2Z\Big(\frac{1}{2}\Big)\Big]\\ =&K\Big(\frac{t+1}{2}, \frac{s+1}{2}\Big)+K\Big(\frac{t+1}{2}, \frac{1-s}{2}\Big)-2K\Big(\frac{1+t}{2}, \frac{1}{2}\Big)\\ &+K\Big(\frac{1-t}{2}, \frac{1+s}{2}\Big)+K\Big(\frac{1-t}{2}, \frac{1-s}{2}\Big)-2K\Big(\frac{1-t}{2}, \frac{1}{2}\Big)\\ &-2K\Big(\frac{1}{2}, \frac{s+1}{2}\Big)-2K\Big(\frac{1}{2}, \frac{1-s}{2}\Big)+4K\Big(\frac{1}{2}, \frac{1}{2}\Big)\,.\end{aligned}$$

The proof is terminated.

COROLLARY 3.1. *If the conditions of the above theorem are satisfied and F is symmetric about the origin then*

$$(3.31)\qquad n^{1/2}\| S_n(t, \theta n^{-1/2})-S_n(t, 0)-\theta n^{-1/2}q(t)\|=o_p(1)\,,$$

where

$$(3.32)\qquad q(t)=2[f(H^{-1}(t))-f(0)]\,, \qquad 0\leqq t\leqq 1\,.$$

PROOF. The proof follows from (3.24) and the fact that

$$(3.33)\qquad n^{1/2}\| \mu(t, \theta n^{-1/2})-\mu(t, 0)-\theta n^{-1/2}q(t)\|\rightarrow 0\,.$$

Next we state and prove asymptotic linearity of sign rank statistics.

THEOREM 3.2. *Under the conditions of Theorem 3.1 and symmetry of F about the origin and (2.15) (ii), (iii) we have*

$$(3.34)\qquad \sup_{|\theta|\leqq a} n^{1/2}| S_n(J, \theta n^{-1/2})-S_n(J, 0)+n^{-1/2}\theta q(J)|=o_p(1)$$

with

$$(3.35)\qquad q(J)=2\int_0^1 f(H^{-1})dJ\mp\int_0^1 q(t)dJ(t)+2f(0)J(1)\,.$$

PROOF. From (2.6), after integrating by parts we have wp 1

$$S_n(J, \theta)=S_n(1, \theta)J(1)-\int_0^1 S_n(t, \theta)dJ(t) , \qquad |\theta|\leqq a .$$

Note that $S_n(1, \theta)=n^{-1}\sum_{i=1}^n s(X_i-\theta)=[1-2F_n(\theta)]$ wp 1, $|\theta|\leqq a$. Hence, for every $|\theta|\leqq a$, we have (using $J(0)=0$)

$$\begin{aligned} &n^{1/2}[S_n(J, \theta n^{-1/2})-S_n(J, 0)+n^{-1/2}\theta q(J)] \\ &\quad =2J(1)n^{1/2}[F_n(0)-F_n(\theta n^{-1/2})] \\ &\qquad -n^{1/2}\int_0^1 [S_n(t, \theta n^{-1/2})-S_n(t, 0)-n^{-1/2}\theta q(t)]dJ(t)+2\theta f(0)J(1) \\ &\quad =J(1)2[Z_n(F(0))-Z_n(F(\theta n^{-1/2}))]+2J(1)n^{1/2}[F(0)-F(\theta n^{-1/2})] \\ &\qquad +\theta f(0)J(1)-n^{1/2}\int_0^1 [S_n(t, \theta n^{-1/2})-S_n(t, 0)-n^{-1/2}\theta q(t)]dJ(t) \\ &\quad =o_p(1) . \end{aligned}$$

In the above, the last equality follows from (3.31), (2.21), the fact that F has bounded density and the fact that J is bounded. The proof is terminated.

LEMMA 3.5. *If $\{X_i\}$ are strictly stationary and φ-mixing satisfying condition C and $\sum_{n=1}^\infty \varphi^{1/2}(n)<\infty$ and F is symmetric about the origin and J is bounded then*

$$\lim_{n\to\infty} \mathrm{P}\,[n^{1/2}S_n(J, 0)\leqq x\sigma]=\Phi(x) , \qquad -\infty<x<+\infty \tag{3.36}$$

where

$$\sigma^2=\sigma_J^2+2\sum_{k=1}^\infty \mathrm{Cov}\,[\hat{Y}_1, \hat{Y}_{k+1}] , \qquad \sigma_J^2=\int_0^1 J^2(u)du \tag{3.37}$$

and

$$\hat{Y}_k=J(H(|X_k|))s(X_k) , \qquad k\geqq 1 .$$

The series in (3.37) is absolutely convergent.

PROOF. Since F symmetric$\Rightarrow\mu(\cdot, 0)=0$ and therefore from (2.12) we have

$$T_n(t, 0)=V(H(H_n^{-1}(t)))=V(t)+o_p(1) , \qquad 0\leqq t\leqq 1 \tag{3.38}$$

by Lemma 3.1 applied with $\theta=0$.

Again because of symmetry of F and because J is bounded we have

$$
\begin{aligned}
(3.39)\qquad n^{1/2}S_n(J,0)=T_n(J,0)&=\int_0^1 J(t)dT_n(t,0)\\
&=\int_0^1 J(t)dV(t)+o_p(1)\qquad \text{(from (3.38))},\\
&=n^{-1/2}\sum_{i=1}^{n} J(H(|X_i|))s(X_i)+o_p(1)\\
&=n^{-1/2}\hat{S}_n+o_p(1)\ .
\end{aligned}
$$

But $\hat{S}_n$ is the sum of strictly stationary φ-mixing bounded r.v.'s and from Theorem 20.1 of [1] it follows that

$$\mathcal{L}(n^{-1/2}\hat{S}_n)\to N(0,\sigma^2)$$

where $\sigma^2=\lim \operatorname{Var}(n^{-1/2}\hat{S}_n)$ and is given by (3.37) above. This and (3.39) proves (3.36). Convergence of absolute series in (3.37) follows from (3.6) and the assumption that $\sum_{n=1}^{\infty}\varphi^{1/2}(n)<\infty$. The proof is terminated.

Next, let $\hat{\theta}_n$ be the solution of the equation $S_n(J,\theta)=0$. Since ψ is $\uparrow$ and $\psi(1/2)=0$, we have $J\geqq 0$ and $\uparrow$ and it may be verified that $S_n(J,\theta)$ is monotone decreasing function of θ and hence $\hat{\theta}_n$ could be defined as average of the last θ for which $S_n(J,\theta)\geqq 0$ and the first θ for which $S_n(J,\theta)\leqq 0$. We have

THEOREM 3.3. *Under the conditions of Theorem* 3.2 *we have*

$$(3.40)\qquad P_\theta(n^{1/2}(\hat{\theta}_n-\theta)\leqq x\tau)\to\Phi(x)\ ,\qquad -\infty<x<+\infty$$

where

$$(3.41)\qquad \tau=q^{-1}(J)\sigma$$

and P_θ *indicates that probability is computed when* θ *is the true parameter.*

PROOF. The proof follows from Lemma 3.5, Theorem 3.2, the shift invariance property of $\hat{\theta}_n$ and the condition C in the usual fashion.

4. Some remarks

1. Combining Theorem 3.2 and Lemma 3.5 we observe that the power of the level α test that rejects the hypothesis $\theta=0$ against the sequence of alternatives $\theta=\theta_0 n^{-1/2}$, $\theta_0>0$, when $n^{1/2}S_n(J,\theta_0 n^{-1/2})<-k_\alpha$ approaches to

$$(4.1)\qquad 1-\Phi(k_\alpha-\theta_0 q(J)\sigma^{-1})$$

where by level we mean the "asymptotic level." In the case of in-dependence we have the power tending to

$$(4.2) \qquad 1-\Phi(k_\alpha-\theta_0 q(J)\sigma_J^{-1}\tau_1)\ ,$$

where $\tau_1=\sigma_J\sigma^{-1}$. Thus the effect of dependence on the power is just measured by τ_1. Observe that if X_i's are independent then $\tau_1=1$.

2. In [3] Gastwirth and Rubin proved the weak convergence of Z_n processes to a continuous Gaussian process when $\{X_i\}$ are strictly stationary strongly mixing under various conditions on the mixing number. Since this kind of result is what lead us up to the results of Section 3 above (except for Lemma 3.1 which could be proved from Lemma 2.1 also) we can conclude that the results of Section 3 of this paper remain valid under Gastwirth and Rubin conditions of [3] on X_i's. Our results are therefore also valid for their strictly stationary first order autoregressive processes. In [3] they have demonstrated that the first order autoregressive process is not φ-mixing in general but is strongly mixing.

3. Recently Sen [5] relaxed the conditions under which Lemma 2.1 above could be proved. He proved Lemma 2.1 with (2.18) replaced to $\sum_{n=1}^{\infty} n\varphi^{1/2}(n)<\infty$. Consequently the results of Section 3 above are valid under this weaker condition also.

4. Estimation of location parameter in two sample problem where one observes two independent sequences of strictly stationary processes both of which are either strongly mixing satisfying conditions of [3] or φ-mixing satisfying the condition given above can be handled in the same fashion similar to above.

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