SOME CRITERIA FOR UNIFORM ASYMPTOTIC EQUIVALENCE OF REAL PROBABILITY DISTRIBUTIONS

SADAO IKEDA

(Received Nov. 5, 1973)

Summary

Some criteria are given for type $(\boldsymbol{B})_d$ or uniform asymptotic equivalence of real probability distributions in terms of characteristic functions and of probability density functions. This work is a contribution to the asymptotic equivalence theory which has been developed by the present author [1], [2].

1. Introduction

Let $\{X_n;\ n=1, 2, \cdots\}$ and $\{Y_n;\ n=1, 2, \cdots\}$ be two sequences of k-dimensional real random variables, where k is fixed independently of n. Throughout the present article, it is assumed that X_n and Y_n are all absolutely continuous with respect to the usual Euclid-Lebesgue measure μ over $(R, \boldsymbol{B})$, R being the k-dimensional Euclidean space and $\boldsymbol{B}$ the usual Borel field of subsets of R. Let us denote the probability density functions (pdf.'s, in short) of X_n and Y_n by $f_n(x)$ and $g_n(x)$, respectively.

Let $\boldsymbol{C}$ be any given class of subsets of R belonging to $\boldsymbol{B}$. Two sequences $\{X_n\}$ and $\{Y_n\}$ are then said to be asymptotically equivalent in the sense of type $(\boldsymbol{C})_d$ and denoted briefly by $X_n \sim Y_n$ $(\boldsymbol{C})_d$ if it holds that $\sup\{|P^{X_n}(E)-P^{Y_n}(E)|:\ E\in \boldsymbol{C}\}\to 0$ as $n\to\infty$.

Let $\boldsymbol{M}$ be the class of all k-dimensional infinite intervals which are right-opened, $\boldsymbol{S}$ the class of all k-dimensional intervals which are left-closed and right-opened, and finally $\boldsymbol{A}$ be the class of all finite disjoint unions of the members of $\boldsymbol{S}$. It is then known [2] that two notions of asymptotic equivalence, type $(\boldsymbol{B})_d$ and type $(\boldsymbol{A})_d$, are mutually equivalent even in the case of unequal basic spaces, while type $(\boldsymbol{S})_d$ and type $(\boldsymbol{M})_d$ are mutually equivalent only in the case of equal basic spaces as in the present case. Type $(\boldsymbol{B})_d$ asymptotic equivalence of $\{X_n\}$ and $\{Y_n\}$ is sometimes called "uniform asymptotic equivalence" as in the title of the present article, and it is equivalent to the condition

$\int_R |f_n-g_n|dx \to 0$ as $n\to\infty$.

Several criteria for type $(\boldsymbol{B})_d$ asymptotic equivalence have been given by the present author [1], [2], [3] in terms of the pdf.'s. In the present article, some criteria are given in terms of characteristic functions (cf.'s, in short) in the following section.

A sharp discrepancy is recognized in strongness between type $(\boldsymbol{B})_d$ and type $(\boldsymbol{M})_d$, and a sufficient condition for these two types to be mutually equivalent is given in the final section in terms of pdf.'s.

2. Criteria for uniform asymptotic equivalence in terms of cf.'s

In the first place, we shall list up some of the well-known results on the integrability of pdf.'s and cf.'s.

LEMMA 1. *Let φ and ψ be cf.'s corresponding to pdf.'s f and g, respectively.*

(i) *If $\varphi\geqq 0$, then φ is integrable if and only if* $\sup\{f(x): x\in R\}$ *is bounded.*

(ii) *$|\varphi|$ is square-integrable if and only if f is square-integrable.*

(iii) *If $|\varphi|$ is integrable, then $|\varphi|^2$ is integrable.*

(iv) *If f is differentiable m times, and $f^{(s)}$, $s=1,\cdots,m$, are all integrable, then $|t^m||\varphi(t)|\to 0$ $(t\to\infty)$ and $|\varphi|$ is integrable.*

(v) *Let f and g be square-integrable. Then*

$$\int_R |f-g|^2dx=(2\pi)^{-k}\int_R |\varphi-\psi|^2dt .$$

Now, let φ_n and ψ_n be cf.'s of X_n and Y_n, respectively. In what follows, it is assumed that the integrals appeared there always exist.

We first prove the following

THEOREM 1. *Suppose that there exists a sequence $\{E_n\}$ of the members of $\boldsymbol{B}$ such that $P^{Y_n}(E_n)\to 1$ and*

$$\mu(E_n)\int_R |\varphi_n-\psi_n|dt\to 0 , \qquad (n\to\infty) . \tag{2.1}$$

Then it holds that

$$X_n\sim Y_n \ (\boldsymbol{B})_d , \qquad (n\to\infty) . \tag{2.2}$$

PROOF. By the inversion formula for pdf.'s, it holds immediately that

$$\int_{E_n} |f_n-g_n|dx \leqq (2\pi)^{-k}\mu(E_n)\int_R |\varphi_n-\psi_n|dt .$$

Since

$$\int_{\bar{E}_n} f_n dx \leqq \int_{\bar{E}_n} g_n dx + \int_{E_n} |f_n - g_n| dx ,$$

$\bar{E}_n$ being the complementary set of E_n, we have

$$\int_R |f_n - g_n| dx \leqq 2P^{Y_n}(\bar{E}_n) + (2\pi)^{-k}\mu(E_n) \int_R |\varphi_n - \psi_n| dt$$

for all n, from which it follows that

$$\int_R |f_n - g_n| dx \to 0 , \qquad (n \to \infty) . \tag{2.3}$$

This is equivalent to (2.2), and the proof of the theorem is completed.

It should be noted that we may replace the condition $P^{Y_n}(E_n) \to 1$ in the above theorem with $P^{X_n}(E_n) \to 1$.

The sequence $\{Y_n\}$ is said to have property $B(S)$ if it holds that for any given $\varepsilon > 0$ there exist a member E of S and a number $N > 0$ such that E is bounded and $\inf\{P^{Y_n}(E):\ n \geqq N\} > 1 - \varepsilon$.

The following corollaries are now immediate consequences of the above theorem.

COROLLARY 1.1. (a) *If in the above theorem* $\mu(E_n)$ *are uniformly bounded for all* n, *then the condition*

$$\int_R |\varphi_n - \psi_n| dt \to 0 , \qquad (n \to \infty) \tag{2.4}$$

is sufficient for (2.2).

(b) *If* $\{Y_n\}$ *has property* $B(S)$, *the condition* (2.4) *implies* (2.2).

COROLLARY 1.2. *If* Y_n *are all identical with* Y, *a continuous random variable with cf.* ψ *and pdf.* g, *then the condition*

$$\int_R |\varphi_n - \psi| dt \to 0 , \qquad (n \to \infty) \tag{2.5}$$

implies the uniform convergence of X_n *to* Y, *i.e.*,

$$X_n \to Y\ (\boldsymbol{B})_d , \qquad (n \to \infty) . \tag{2.6}$$

In the second place, we give another group of sufficient conditions for (2.2).

Since

$$\int_{E_n} |f_n - g_n| dx \leqq \left[\mu(E_n) \int_{E_n} |f_n - g_n|^2 dx\right]^{1/2}$$

for any given E_n, one can show easily the following theorem in a similar manner to the proof of the preceding theorem.

LEMMA 2. *Suppose that there exists a sequence* $\{E_n\}$ *of the members of* $\boldsymbol{B}$ *such that* $P^{Y_n}(E_n)\to 1$ *and*

$$\mu(E_n)\int_{E_n}|f_n-g_n|^2dx\to 0\,, \qquad (n\to\infty)\,. \tag{2.7}$$

Then, the condition (2.2) *holds true. In particular, if* $\mu(E_n)$ *are uniformly bounded for all* n, *then the condition*

$$\int_R |f_n-g_n|^2dx\to 0\,, \qquad (n\to\infty) \tag{2.8}$$

implies the condition (2.2).

The following example shows that the sole condition (2.8) does not necessarily imply (2.2).

Example 1. Let X_n and Y_n be uniform distributions over the intervals $(0, n)$ and $(0, 2n)$, respectively. It is then easily confirmed that $\int_R |f_n-g_n|^2dx=1/2n\to 0$, while $\int_R |f_n-g_n|dx=1$ for all n. Moreover, it is seen that $\int_R |\varphi_n-\psi_n|^2dt=\pi/n\to 0$, but $\int_R |\varphi_n-\psi_n|dt$ does not exist for each n.

The following lemma gives us a sufficient condition in order that (2.2) implies (2.8).

LEMMA 3. *If both* f_n *and* g_n *are uniformly bounded for all* n, *then the condition* (2.2) *implies* (2.8).

In this lemma, we can not drop the condition of uniform boundedness of any one of the sequences of pdf.'s, as will be seen by the following example.

Example 2. Let $f_n(x)=\sqrt{n}$ $(0\leqq x<1/n)$, $=(1-1/\sqrt{n})/(1-1/n)$ $(1/n \leqq x<1)$, $=0$ (elsewhere), while $g(x)=1$ $(0\leqq x<1)$, $=0$ (elsewhere). It is then clear that f_n and g are all square-integrable and $\int_R |f_n-g|^2dx \to 1$ as $n\to\infty$. On the other hand, $\int_R |f_n-g|dx\to 0$ as $n\to\infty$.

The following theorem is now straightforward from the preceding lemmas.

THEOREM 2. *Suppose that there exists a sequence* $\{E_n\}$ *of the members of* $\boldsymbol{B}$ *such that* $P^{Y_n}(E_n)\to 1$ *and*

$$\mu(E_n)\int_R |\varphi_n-\psi_n|^2 dt \to 0\ , \qquad (n\to\infty)\ . \tag{2.9}$$

Then the condition (2.2) *holds true.*

As immediate consequences of this theorem, we obtain the following corollaries.

COROLLARY 2.1. (a) *If, in the above theorem, $\mu(E_n)$ are uniformly bounded for all n, then the condition*

$$\int_R |\varphi_n-\psi_n|^2 dt \to 0\ , \qquad (n\to\infty) \tag{2.10}$$

implies (2.2). *If in addition both f_n and g_n are uniformly bounded for all n and x, the condition* (2.10) *is necessary and sufficient for* (2.2).

(b) *If the sequence $\{Y_n\}$ has property $B(S)$, then* (2.10) *implies* (2.2).

COROLLARY 2.2. *Under the same situation as in Corollary* 1.2, *the condition*

$$\int_R |\varphi_n-\psi|^2 dt \to 0\ , \qquad (n\to\infty) \tag{2.11}$$

implies the condition (2.6). *If f_n and g are uniformly bounded for all n and x, then the conditions* (2.6) *and* (2.11) *are mutually equivalent.*

In concluding the present section, it should be remarked that if at least one of X_n and Y_n has mean ξ_n and variance σ_n^2 for every n, then it is desirable to standardize X_n and Y_n in such a way as $\tilde{X}_n=(X_n-\xi_n)/\sigma_n$ and $\tilde{Y}_n=(Y_n-\xi_n)/\sigma_n$ before applying Theorems 1 and 2: Indeed, in such a case the sequences of new variables $\{\tilde{X}_n\}$ and $\{\tilde{Y}_n\}$ have property $B(S)$, and consequently the criterion (2.4) or (2.10) can be applied, which is in general slightly weaker than (2.1) or (2.9), respectively.

3. Asymptotic uniform continuity of pdf.'s and the uniform asymptotic equivalence

In this section, we first introduce a notion of asymptotic uniform continuity of a sequence of real-valued functions, and then give a sufficient condition for uniform asymptotic equivalence with the aid of this notion.

Let $\{A_n;\ n=1, 2,\cdots\}$ be any given sequence of members of $\boldsymbol{B}$, and let $\{h_n(x);\ n=1, 2,\cdots\}$ be a sequence of real-valued functions defined over R. $\{h_n\}$ is then said to be *asymptotically uniformly continuous with respect to* $\{A_n\}$ or *auc*$(\{A_n\})$ in short, if for any given $\varepsilon>0$ there can be found numbers $\delta=\delta(\varepsilon)>0$ and $N=N(\varepsilon)>0$ such that

$$\sup\{|h_n(x)-h_n(y)|:\ x, y \in A_n,\ |x-y|<\delta,\ n\geqq N\}<\varepsilon\ . \tag{3.1}$$

The following results on this property are then immediate.

LEMMA 4. *Let $\{p_n\}$ and $\{q_n\}$ be auc($\{A_n\}$) and $\{\lambda_n\}$ and $\{\mu_n\}$ be any given sequences of real numbers which are bounded uniformly for all n. Then the sequences $\{\lambda_n p_n+\mu_n q_n\}$ and $\{p_n q_n\}$ are both auc($\{A_n\}$). Furthermore, if $0<\rho\leqq|q_n|$ uniformly for all x in A_n and for all n, then the sequence $\{p_n/q_n\}$ is auc($\{A_n\}$).*

We now show the following lemma, which is essential in the discussion of this section.

LEMMA 5. *Let $\{A_n\}$ be a sequence of members of $\boldsymbol{S}$ such that $r(A_n)\geqq\eta$ for some $\eta>0$ and for all n, where $r(A)$ stands for the length of the shortest edge of the set A in $\boldsymbol{S}$. Suppose that $X_n\sim Y_n$ $(\boldsymbol{M})_d$ as $n\rightarrow\infty$ and $\{f_n\}$ and $\{g_n\}$ are both auc($\{A_n\}$). Then it holds that*

$$\sup\{|f_n(x)-g_n(x)|:\ x\in A_n\}\rightarrow 0\ ,\qquad (n\rightarrow\infty)\ . \tag{3.2}$$

PROOF. Suppose (3.2) be false. Then, we can assume without any loss of generality that there exist a number $\varepsilon>0$ and a sequence $\{x_n\}$ of points of $\{A_n\}$ such that $h_n(x_n)\geqq 2\varepsilon$ for all n, where $h_n=f_n-g_n$ for each n.

Since $\{h_n\}$ is auc ($\{A_n\}$) by the preceding lemma, there exist $\delta>0$ and $N>0$ such that

$$\sup\{|h_n(x)-h_n(y)|:\ x, y \in A_n,\ |x-y|<\delta,\ n\geqq N\}<\varepsilon\ .$$

Let us put $C_n=U(x_n, \delta)\cap A_n$, where $U(x_n, \delta)$ designates the δ-neighborhood of the point x_n. Then for any $x\in C_n$ it holds that $h_n(x)\geqq h_n(x_n)-|h_n(x_n)-h_n(x)|>\varepsilon$, or equivalently $f_n(x)>g_n(x)+\varepsilon$, provided $n\geqq N$.

Since x_n belongs to A_n, the set C_n always contains a member of $\boldsymbol{S}$, B_n say, such that $r(B_n)=\min\{\delta/\sqrt{k}, \eta\}$ for every $n\geqq N$. Then it follows that

$$P^{X_n}(B_n)-P^{Y_n}(B_n)\geqq\varepsilon\min\{(\delta/\sqrt{k})^k, \eta^k\}$$

for all $n\geqq N$, which contradicts the assumption of the lemma. This completes the proof.

The following theorem is now immediate from the above lemma.

THEOREM 3. *Suppose that the following conditions are satisfied:*
(i) $X_n\sim Y_n$ $(\boldsymbol{M})_d$, $(n\rightarrow\infty)$.
(ii) *For any given $\varepsilon>0$ there exist a sequence $\{E_n\}$ of the members*

of $\boldsymbol{S}$, *and positive numbers* M, N *and* η *such that* $r(E_n)>\eta$, $\mu(E_n)\leqq M$ *and* $P^{Y_n}(E_n)>1-\varepsilon$ *for all* $n\geqq N$.

(iii) $\{f_n\}$ *and* $\{g_n\}$ *are both* *auc*$(\{E_n\})$.

Then it holds that $X_n\sim Y_n$ $(\boldsymbol{B})_d$ *as* $n\to\infty$.

PROOF. By the assumptions (i) and (ii) above, we can assume that $P^{X_n}(E_n)>1-\varepsilon$ for all $n\geqq N$. Then it holds that

$$\int_R |f_n-g_n|\,dx\leqq\mu(E_n)\sup\{|f_n-g_n|:\ x\in E_n\}+2\varepsilon$$

for all $n\geqq N$, from which follows the theorem immediately.

We state the following corollaries without proofs, which are straightforward from the above theorem.

COROLLARY 3.1. *If* $\{Y_n\}$ *has property* $B(\boldsymbol{S})$ *and both* $\{f_n\}$ *and* $\{g_n\}$ *are* *auc*$(\{E\})$ *for every bounded subset* E *belonging to* $\boldsymbol{S}$, *then the condition* $X_n\sim Y_n$ $(\boldsymbol{M})_d$ *implies* $X_n\sim Y_n$ $(\boldsymbol{B})_d$ *as* $n\to\infty$.

COROLLARY 3.2. *If* Y_n *are all identical with some* Y, *whose pdf.* $g(x)$ *being continuous over* R, *and if* $\{f_n\}$ *is* *auc*$(\{E\})$ *for every bounded subset* E *belonging to* $\boldsymbol{S}$, *then* $X_n\sim Y$ $(\boldsymbol{M})_d$ *implies* $X_n\sim Y$ $(\boldsymbol{B})_d$ *as* $n\to\infty$.

In concluding the present section, some remarks should be given.

In practical applications, property of asymptotic uniform continuity of sequences of pdf.'s is in general not easy to check whether it is satisfied or not. The following lemma gives us a sufficient condition for the property.

LEMMA 6. *Let* $\{h_n(x)\}$ *be a sequence of real valued functions defined over* R. *If* $h_n(x)$ *is differentiable over a domain* G_n *containing* A_n *for every* n, *and* dh_n/dx *are bounded uniformly for all* x *in* G_n *and for all* n, *then the sequence* $\{h_n\}$ *is* *auc*$(\{A_n\})$.

The proof of this lemma is easy and will be omitted.

In the second place, it should be noted that the above theorem gives us a criterion for an asymptotic expansion

$$X_n=Y_n+Z_n\,,\qquad (n=1,2,\cdots) \tag{3.3}$$

to be of type $(\boldsymbol{B})_d$. It has been known [2] that if $\{Y_n\}$ has property $C(\boldsymbol{S})$ and $Z_n\to 0$ (i.p.), then $X_n\sim Y_n$ $(\boldsymbol{M})_d$ as $n\to\infty$, i.e., the above expansion is of type $(\boldsymbol{M})_d$. Here the sequence $\{Y_n\}$ is said to have property $C(\boldsymbol{C})$, $\boldsymbol{C}$ being any given subclass of $\boldsymbol{B}$, if for any given $\varepsilon>0$ there exist $\delta>0$ and $N>0$ such that

$$\sup\{P^{Y_n}(E):\ E\in\boldsymbol{C},\ \mu(E)<\delta,\ n\geqq N\}<\varepsilon\,.$$

It is easily verified that the conditions (ii) and (iii) of Theorem 3 simultaneously imply the property $C(\boldsymbol{B})$, and hence $C(\boldsymbol{S})$, of both $\{X_n\}$ and $\{Y_n\}$. Consequently, it is concluded that the conditions (ii) and (iii) of the theorem and the condition $Z_n \to 0$ (i.p.) imply $X_n \sim Y_n$ $(\boldsymbol{B})_d$, i.e., type $(\boldsymbol{B})_d$ asymptotic expansion for (3.3).

In passing, it is noted that if the condition (iii) of the theorem were replaced by property $C(\boldsymbol{B})$ of both $\{X_n\}$ and $\{Y_n\}$, however, the result of the theorem is no longer true, as will be seen by the following example.

Example 3. Let $\Omega=[0, 1)$ and let P be the Wiener measure over Ω. Put $Y(\omega)=\omega$ for every ω in Ω. Then Y is distributed as a uniform distribution on [0, 1), whose pdf. being $g(x)=1$ $(0\leqq x<1)$, $=0$ (elsewhere).

On the other hand, we define X_n as follows: Dividing the interval [0, 1) into n intervals of equal length, we put $I_{ni}=[i/n, (i+1)/n)$; further, let us divide each I_{ni} into two subintervals of equal length $I'_{ni}=[i/n, (2i+1)/2n)$ and $I''_{ni}=[(2i+1)/2n, (i+1)/n)$, $i=0, 1, \cdots, n-1$. For an arbitrarily chosen $\alpha>0$, let us put

$$X_n(\omega)=\begin{cases} i/n+(\omega-i/n)/(1+\alpha)\,, & (\omega\in I'_{ni})\,,\\ (i+1)/n-(1+2\alpha)[(i+1)/n-\omega]/(1+\alpha)\,, & (\omega\in I''_{ni})\,,\end{cases}$$

for $i=0, 1, \cdots, n-1$; $n=1, 2, \cdots$.

Then it holds that $|X_n(\omega)-Y(\omega)|\leqq 1/2n$ for all ω in Ω, and therefore $X_n\to Y$ (a.c.) as $n\to\infty$, which implies that $X_n\to Y$ (i.l.) as $n\to\infty$. This convergence is of type $(\boldsymbol{M})_d$, because the limiting distribution is absolutely continuous.

Since the pdf. of X_n is given by

$$f_n(x)=\begin{cases} 1+\alpha\,, & (i/n\leqq x<i/n+1/2n(1+\alpha))\,,\\ (1+\alpha)/(1+2\alpha)\,, & (i/n+1/2n(1+\alpha)\leqq x<(1+i)/n)\,,\end{cases}$$

$i=0, 1, \cdots, n-1$, it is evident that $X_n\to Y$ $(\boldsymbol{B})_d$ does not hold. It is also easy to see that $\{X_n\}$ has property $C(\boldsymbol{B})$, but it is not auc $(\{A_n\})$ for any sequence $\{A_n\}$ of the members of $\boldsymbol{S}$ such that $P^Y(A_n)\to 1$ as $n\to\infty$.

SOKA UNIVERSITY

REFERENCES

[1] Ikeda, S. (1963). Asymptotic equivalence of probability distributions with applications to some problems of asymptotic independence, *Ann. Inst. Statist. Math.*, **15**, 87-116.

[2] Ikeda, S. (1968). Asymptotic equivalence of real probability distributions, *Ann. Inst. Statist. Math.*, **20**, 339-361.

[3] Ikeda, S. and Matsunawa, T. (1972). On the uniform asymptotic joint normality of sample quantiles, *Ann. Inst. Statist. Math.*, **24**, 33-52.