NOTE ON BALANCED FRACTIONAL 2^m FACTORIAL DESIGNS OF RESOLUTION $2l+1$

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Abstract

It is shown that the characteristic roots of the information matrix of a balanced fractional 2^m factorial design T of resolution $2l+1$ are the same as those of its complementary design $\bar{T}$. Necessary conditions for the existence of such a design T are also given.

1. Introduction

As an important subclass of fractional designs, balanced designs were first introduced by Chakravarti [1]. They are the generalizations of orthogonal designs which make the estimates of various effects of interest mutually uncorrelated. Balanced designs are flexible in the number of assemblies or treatment combinations, whereas orthogonal designs require much more than the desirable number of assemblies. Such designs which are practical are now becoming more popular.

Srivastava [3] has established a connection between a balanced fractional 2^m factorial design of resolution V and a balanced array of strength 4 with index set $\{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4\}$. Furthermore Srivastava and Chopra [5] have given explicitly the characteristic polynomial of the information matrix of this design, which is useful for comparing the fraction by the popular criteria (such as the deteminant, the trace and the largest root of the covariance matrix (see Srivastava and Chopra [4]).

Those investigations, however, have been restricted only to the effects up to two-factor interactions. With the development of a means of computation, it has been possible to deal with many factors simultaneously. Among these factors, there may be similar ones. This implies that we can not ignore three-factor or more interactions, that is, we need to consider designs of resolution VII or higher resolution.

Yamamoto, Shirakura and Kuwada [6], in general, established that a necessary and sufficient condition for a fractional 2^m factorial design T of resolution $2l+1$ to be balanced is that T is a balanced array of strength

$2l$ with index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$ provided the information matrix M_T is non-singular. Furthermore, Yamamoto, Shirakura and Kuwada [7] gave explicitly the characteristic polynomial of the information matrix M_T of a balanced fractional 2^m factorial design of resolution $2l+1$, using the decomposition of the triangular MDPB (multi-dimensional partially balanced) association algebra into its two-sided ideals.

The purpose of this paper is to show that the characteristic roots of the information matrix of a balanced fractional 2^m factorial design T of resolution $2l+1$ are the same as those of its complementary design $\bar{T}$ (which is obtained from T by an interchange of 0 and 1). It is useful for finding optimal balanced designs with respect to the criteria such as $|M_T^{-1}|$, $\mathrm{tr}\, M_T^{-1}$ and $\mathrm{ch}_{\max} M_T^{-1}$. Another purpose is to give explicitly some necessary conditions for the existence of a balanced array such that $\nu_l \left(=1+m+\binom{m}{2}+\cdots+\binom{m}{l}\right)$ effects are estimable.

2. Preliminaries

Consider a 2^m factorial experiment with m factors each at two levels. An assembly or treatment combination will be represented by $(j_1, j_2; \cdots, j_m)$, where j_k, the level of kth factor, equals 0 or 1. We shall consider the situation where $(l+1)$-factor or more interactions are assumed negligible for a given integer l $(1 \leqq l \leqq m/2)$. Then the vector of unknown parameters is given by $\boldsymbol{\theta}' = (\theta_\phi; \theta_1, \theta_2, \cdots, \theta_m; \theta_{12}, \theta_{13}, \cdots, \theta_{m-1\,m}; \cdots; \theta_{12\cdots l}, \cdots, \theta_{m-l+1\cdots m})$, $(1 \times \nu_l)$, where θ_ϕ, θ_t and in general $\theta_{t_1 t_2 \cdots t_k}$ denote the general mean, the main effect of tth factor, and the k-factor interaction of corresponding factors, respectively.

Let T be a fraction with n assemblies, then T can be expressed as a (0, 1) matrix of size $n \times m$ whose rows denote assemblies. The normal equation for estimating $\boldsymbol{\theta}$ based on T is given by $M_T \boldsymbol{\theta} = E_T' \boldsymbol{y}(T)$ where $E_T(n \times \nu_l)$ is the design matrix, $\boldsymbol{y}(T)$ is the observation vector and M_T $(=E_T' E_T)$ is the information matrix (see, e.g., Yamamoto, Shirakura and Kuwada [6]). If an information matrix M_T is non-singular and M_T^{-1} is invariant under any permutation of m factors, then T is called a balanced design of resolution $2l+1$.

Let $\varepsilon(t_1, \cdots, t_u; t_1', \cdots, t_v')$ be the element of M_T whose row and column correspond to $\theta_{t_1 \cdots t_u}$ and $\theta_{t_1' \cdots t_v'}$ of $\boldsymbol{\theta}$, respectively. Let $x \ominus y$ and $|x|$ denote the symmetric difference of sets x and y (i.e., $x \ominus y = (x \cup y) - (x \cap y)$) and the cardinality of x, respectively. Then it has been shown in Yamamoto, Shirakura and Kuwada [6] that a necessary and sufficient condition for T to be a balanced array (B-array) of strength $2l$ is that the information matrix M_T has at most $2l+1$ distinct elements γ_i $(i = 0, 1, \cdots, 2l)$ as indicated below:

$$\gamma_i=\varepsilon(t_1,\cdots,t_u;t'_1,\cdots,t'_v) \qquad \text{if } |\{t_1,\cdots,t_u\}\ominus\{t'_1,\cdots,t'_v\}|=i\,,$$

and that a connection between the elements γ_i of M_T and the indices μ_j of the B-array T is given by

$$\text{(2.1)} \qquad \begin{aligned}\gamma_i&=\sum_{j=0}^{2l}\sum_{p=0}^{i}(-1)^p\binom{i}{p}\binom{2l-i}{j-i+p}\mu_j\,,\\ \mu_i&=\frac{1}{2^{2l}}\sum_{j=0}^{2l}\sum_{p=0}^{j}(-1)^p\binom{i}{j-p}\binom{2l-i}{p}\gamma_j\,.\end{aligned}$$

Suppose $\binom{a}{b}=0$ if and only if $b>a\geqq 0$ or $b<0$ throughout this paper. We define a B-array of strength t. A (0, 1) matrix T of size $n\times m$ is said to be a B-array of strength t, size n, m constraints, 2 levels and index set $\{\mu_0^{(t)},\mu_1^{(t)},\cdots,\mu_t^{(t)}\}$ if every $n\times t$ submatrix T_0 of T is such that every (0, 1) vector with weight (or number of non-zero components) j occurs exactly $\mu_j^{(t)}$ times $(j=0,1,\cdots,t)$ as a row of T_0. Then it is easily seen that $n=\sum_{j=0}^{t}\binom{t}{j}\mu_j^{(t)}$. For simplicity, we write $\mu_i=\mu_i^{(2l)}$ $(i=0, 1,\cdots,2l)$ throughout this paper. Since a *B*-array T of strength $2l$ with indices μ_i is also of strength k $(k\leqq 2l)$ with indices

$$\text{(2.2)} \qquad \mu_j^{(k)}=\sum_{i=0}^{2l}\binom{2l-k}{i-j}\mu_i \qquad \text{for } j=0,1,\cdots,k\,,$$

a connection between the γ_i and $\mu_i^{(k)}$ is given as follows: For $i=0,1,\cdots,k$,

$$\text{(2.3)} \qquad \begin{aligned}\gamma_i&=\sum_{j=0}^{k}\sum_{p=0}^{i}(-1)^p\binom{i}{p}\binom{k-i}{j-i+p}\mu_j^{(k)}\,,\\ \mu_i^{(k)}&=\frac{1}{2^{k}}\sum_{j=0}^{k}\sum_{p=0}^{j}(-1)^p\binom{i}{j-p}\binom{k-i}{p}\gamma_j\,.\end{aligned}$$

3. Complementary balanced arrays

For a B-array T of strength $2l$, m constraints and index set $\{\mu_0, \mu_1,\cdots,\mu_{2l}\}$, we consider

$$\text{(3.1)} \qquad \kappa_\beta^{(i,j)}=\sum_{\alpha=0}^{\beta+i}\gamma_{j-i+2\alpha}z_{\beta\alpha}^{(\beta+i,\beta+j)} \qquad \text{for } 0\leqq i\leqq j\leqq l-\beta\,,$$

where

$$z_{\beta\alpha}^{(u,v)}=\sum_{b=0}^{\alpha}(-1)^{\alpha-b}\frac{\binom{u-\beta}{b}\binom{u-b}{u-\alpha}\binom{m-u-\beta+b}{b}}{\binom{v-u+b}{b}}\sqrt{\binom{m-u-\beta}{v-u}\binom{v-\beta}{v-u}}\,.$$

Let K_β be the $(l-\beta+1)\times(l-\beta+1)$ matrix such that for each $\beta=0, 1, \cdots, l$,

$$(3.2) \qquad K_\beta=\begin{bmatrix} \kappa_\beta^{(0,0)}, & \kappa_\beta^{(0,1)}, & \cdots, & \kappa_\beta^{(0,l-\beta)} \\ & \kappa_\beta^{(1,1)} & & \vdots \\ & \text{Sym.} & \ddots & \vdots \\ & & & \kappa_\beta^{(l-\beta,l-\beta)} \end{bmatrix}.$$

The following theorem has been given by Yamamoto, Shirakura and Kuwada [7].

THEOREM 3.1. *The characteristic polynomial $\phi_T(\lambda)$ of the information matrix M_T of a balanced fractional 2^m factorial design T or a balanced array of strength $2l$ with index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$ is given by*

$$\phi_T(\lambda)=|M_T-\lambda I_{\nu_l}|=\prod_{\beta=0}^{l}|K_\beta-\lambda I_{l-\beta+1}|^{\phi_\beta},$$

where $\phi_\beta=\binom{m}{\beta}-\binom{m}{\beta-1}$.

Let $\bar{T}$ be the array obtained by interchanging symbols 0 and 1 in T. Then it is easily shown that the array $\bar{T}$ is also the B-array of strength $2l$ with indices $\bar{\mu}_i=\mu_{2l-i}$, $(i=0, 1, \cdots, 2l)$.

DEFINITION 3.1. The array $\bar{T}$ is called the complementary balanced array of a B-array T.

We shall consider the relations between a B-array T and its complementary B-array $\bar{T}$. Let $\bar{\gamma}_i$ and $\bar{\kappa}_\beta^{(i,j)}$ be those values of $\bar{T}$ corresponding to γ_i and $\kappa_\beta^{(i,j)}$ of T, respectively. Then we have

LEMMA 3.1. *For a B-array T and its complementary B-array $\bar{T}$, it holds that*

$$\bar{\gamma}_i=(-1)^i\gamma_i \qquad \textit{for all } i=0, 1, \cdots, 2l,$$

$$\bar{\kappa}_\beta^{(i,j)}=(-1)^{i+j}\kappa_\beta^{(i,j)} \qquad \textit{for all } 0\leqq i\leqq j\leqq l-\beta.$$

PROOF. From (2.1) and (3.1), we have

$$\bar{\gamma}_i=\sum_{j=0}^{2l}\sum_{p=0}^{i}(-1)^p\binom{i}{p}\binom{2l-i}{j-i+p}\bar{\mu}_j$$

$$\bar{\kappa}_\beta^{(i,j)}=\sum_{\alpha=0}^{\beta+i}\bar{\gamma}_{j-i+2\alpha}z_{\beta\alpha}^{(\beta+i,\beta+j)}.$$

From Definition 3.1, we have

$$\begin{aligned}\bar{\gamma}_i&=\sum_{j=0}^{2l}\sum_{p=0}^{i}(-1)^p\binom{i}{p}\binom{2l-i}{j-i+p}\mu_{2l-j}\\&=\sum_{k=0}^{2l}\sum_{q=0}^{i}(-1)^{i-q}\binom{i}{i-q}\binom{2l-i}{2l-k-q}\mu_k\\&=(-1)^i\sum_{k=0}^{2l}\sum_{q=0}^{i}(-1)^q\binom{i}{q}\binom{2l-i}{k-i+q}\mu_k\\&=(-1)^i\gamma_i\ .\end{aligned}$$

Hence we have

$$\bar{\kappa}_\beta^{(i,j)}=\sum_{\alpha=0}^{\beta+i}(-1)^{j-i+2\alpha}\gamma_{j-i+2\alpha}z_{\beta\alpha}^{(\beta+i,\beta+j)}=(-1)^{i+j}\kappa_\beta^{(i,j)}\ .$$

THEOREM 3.2. *For a B-array T and its complementary B-array,* $\bar{T}$,

$$\phi_{\bar{T}}(\lambda)=\phi_T(\lambda)\ ,$$

where $\phi_{\bar{T}}(\lambda)$ *is the characteristic polynomial of the information matrix* $M_{\bar{T}}$ *of* $\bar{T}$.

PROOF. From Theorem 3.1, we have

$$\phi_{\bar{T}}(\lambda)=\prod_{\beta=0}^{l}|\bar{K}_\beta-\lambda I_{l-\beta+1}|^{\phi_\beta}\ ,$$

where $\bar{K}_\beta$ is the $(l-\beta+1)\times(l-\beta+1)$ symmetric matrix whose (i, j) element is $\bar{\kappa}_\beta^{(i,j)}$. From Lemma 3.1, we have

$$\bar{K}_\beta=D_{l-\beta+1}K_\beta D_{l-\beta+1}\ ,$$

where $D_{l-\beta+1}$ is the $(l-\beta+1)\times(l-\beta+1)$ diagonal matrix whose (i, i) element is $(-1)^i$ for $i=0, 1, \cdots, l-\beta$. Hence

$$\begin{aligned}|\bar{K}_\beta-\lambda I_{l-\beta+1}|&=|D_{l-\beta+1}K_\beta D_{l-\beta+1}-\lambda I_{l-\beta+1}|\\&=|D_{l-\beta+1}(K_\beta-I_{l-\beta+1})D_{l-\beta+1}|\\&=|K_\beta-\lambda I_{l-\beta+1}|\ .\end{aligned}$$

Thus we have $\phi_{\bar{T}}(\lambda)=\phi_T(\lambda)$.

The theorem is useful for finding optimal balanced designs with respect to criteria (e.g., the trace, determinant and largest root criteria) which are based on the characteristic roots of M_T. For example, consider the selection of optimal balanced designs of resolution VII (i.e., $l=3$) with respect to the trace criterion. For a given pair (m, n), there are, in general, a large number of B-arrays of strength 6 whose indices satisfy $n=\mu_0+\mu_6+6(\mu_1+\mu_5)+15(\mu_2+\mu_4)+20\mu_3$. Among these, we must select a design which minimizes $\mathrm{tr}\, M_T^{-1}$. However Theorem 3.2 means that among ones such that (a) $\mu_2>\mu_4$, if $\mu_2\neq\mu_4$, (b) $\mu_1>\mu_5$, if $\mu_2=\mu_4$ and

$\mu_1 \neq \mu_5$ or (c) $\mu_0 \geqq \mu_6$, if $\mu_2 = \mu_4$ and $\mu_1 = \mu_5$ (with given (m, n), of course), we can select such a design.

4. Some existence conditions of balanced designs

Each matrix K_β given in (3.2), obviously, is a function of the m and μ_i $(i=0, 1, \cdots, 2l)$ of a B-array T. The following two theorems have been given by Yamamoto, Shirakura and Kuwada [7].

THEOREM 4.1. *Let T be a B-array of strength $2l$, m constraints and index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$. Then a necessary condition that T exists is that every matrix K_β is positive semi-definite.*

THEOREM 4.2. *Consider the array T in Theorem 4.1. A necessary and sufficient condition for the parameter vector $\boldsymbol{\theta}$ to be estimable is that every matrix K_β is positive definite.*

In this section, the elements of the matrices K_l, K_{l-1}, K_{l-2} and the diagonal elements $\kappa_{l-3}^{(0,0)}$, $\kappa_{l-3}^{(1,1)}$, $\kappa_{l-3}^{(2,2)}$, $\kappa_{l-3}^{(3,3)}$ of the matrix K_{l-3} will be explicitly expressed in terms of the m and μ_i of a B-array T. For this purpose, we use the following identities (see, e.g., Riordan [2]):

$$(4.1)\qquad \begin{aligned} &\sum_{p=0}^{\omega}(-1)^p\binom{\omega}{p}\binom{\omega}{2\alpha-p}=(-1)^\alpha\binom{\omega}{\alpha} \\ &\sum_{p=0}^{\omega}(-1)^p\binom{\omega}{p}\binom{\omega}{2\alpha+1-p}=0 \\ &\sum_{p=0}^{\omega}\binom{\alpha-\omega}{k-p}\binom{\omega}{p}=\binom{\alpha}{k}\,. \end{aligned}$$

From (2.1)–(2.3), (3.1) and (4.1), we have

$$(4.2)\qquad \kappa_l^{(0,0)}=\sum_{\alpha=0}^{l}(-1)^\alpha\binom{l}{\alpha}\gamma_{2\alpha}=\sum_{j=0}^{2l}\sum_{p=0}^{j}(-1)^p\binom{l}{j-p}\binom{l}{p}\gamma_j=2^{2l}\mu_l\,,$$

$$(4.3)\qquad \kappa_{l-1}^{(0,0)}=2^{2(l-1)}\mu_{l-1}^{(2l-2)}=2^{2l-2}\{(\mu_{l-1}+\mu_{l+1})+2\mu_l\}\,,$$

$$(4.4)\qquad \kappa_{l-2}^{(0,0)}=2^{2l-4}\mu_{l-2}^{(2l-4)}=2^{2l-4}\{(\mu_{l-2}+\mu_{l+2})+4(\mu_{l-1}+\mu_{l+1})+6\mu_l\}\,,$$

$$(4.5)\qquad \kappa_{l-3}^{(0,0)}=2^{2l-6}\{(\mu_{l-3}+\mu_{l+3})+6(\mu_{l-2}+\mu_{l+2})+15(\mu_{l-1}+\mu_{l+1})+20\mu_l\}\,.$$

Furthermore we use the following lemma for the rest.

LEMMA 4.1. *For any non-negative integers j and ω, we have the following combinatorial identity:*

$$\sum_{\alpha=0}^{\omega}\sum_{p=0}^{2\omega}(-1)^{p+\alpha}\binom{2\alpha}{p}\binom{2\omega-2\alpha}{j-2\alpha+p}\binom{\omega}{\alpha}=2^{2\omega}\delta_{j\omega}\,,$$

where $\delta_{j\omega}=0$ or 1 according as $j\neq\omega$ or $j=\omega$.

PROOF. From (2.1), we have another equation, i.e.,

$$\kappa_l^{(0,0)}=\sum_{j=0}^{2l}\sum_{\alpha=0}^{l}\sum_{p=0}^{2l}(-1)^{p+\alpha}\binom{2\alpha}{p}\binom{2l-2\alpha}{j-2\alpha+p}\binom{l}{\alpha}\mu_j .$$

$\kappa_l^{(0,0)}$ is independent of the number m of constraints of a B-array. Particularly consider $m=2l$, then there exists a B-array for any index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$. Hence, comparing the above equation with (4.2), we can derive the identity.

Therefore we have

$$\text{(4.6)}\quad \begin{aligned}\kappa_{l-1}^{(1,1)}&=\sum_{\alpha=0}^{l}(-1)^{\alpha}\binom{l}{\alpha}\gamma_{2\alpha}-\{m-2(l-1)\}\sum_{\alpha=0}^{l}(-1)^{\alpha}\binom{l-1}{\alpha-1}\gamma_{2\alpha}\\
&=2^{2l}\mu_l-(m-2l+2)\sum_{j=0}^{2l}\sum_{\alpha=0}^{l}\sum_{p=0}^{2l}(-1)^{p+\alpha}\left\{\binom{2(\alpha-1)}{p}\right.\\
&\quad\cdot\binom{2(l-1)-2(\alpha-1)}{j-2-2(\alpha-1)+p}+2\binom{2(\alpha-1)}{p-1}\binom{2(l-1)-2(\alpha-1)}{j-1-2(\alpha-1)+p-1}\\
&\quad\left.+\binom{2(\alpha-1)}{p-1}\binom{2(l-1)-2(\alpha-1)}{j-2(\alpha-1)+p-2}\right\}\binom{l-1}{\alpha-1}\mu_j\\
&=2^{2l-2}\{(m-2l+2)(\mu_{l-1}+\mu_{l+1})-2(m-2l)\mu_l\} ,\end{aligned}$$

$$\text{(4.7)}\quad \begin{aligned}\kappa_{l-2}^{(1,1)}&=2^{2l-4}\{(m-2l+4)(\mu_{l-2}^{(2l-2)}+\mu_l^{(2l-2)})-2(m-2l+2)\mu_{l-1}^{(2l-2)}\}\\
&=2^{2l-4}\{(m-2l+4)(\mu_{l-2}+\mu_{l+2})+4(\mu_{l-1}+\mu_{l+1})-2(m-2l)\mu_l\} ,\end{aligned}$$

$$\text{(4.8)}\quad \begin{aligned}\kappa_{l-3}^{(1,1)}&=2^{2l-6}\{(m-2l+6)(\mu_{l-3}+\mu_{l+3})+2(m-2l+8)(\mu_{l-2}+\mu_{l+2})\\
&\quad-(m-2l-10)(\mu_{l-1}+\mu_{l+1})-4(m-2l)\mu_l\} ,\end{aligned}$$

$$\text{(4.9)}\quad \kappa_{l-1}^{(0,1)}=2^{2l-2}\sqrt{m-2l+2}\,(\mu_{l+1}-\mu_{l-1}) ,$$

$$\text{(4.10)}\quad \kappa_{l-2}^{(0,1)}=2^{2l-4}\sqrt{m-2l+4}\,\{(\mu_{l+2}-\mu_{l-2})+2(\mu_{l+1}-\mu_{l-1})\} ,$$

$$\text{(4.11)}\quad \begin{aligned}\kappa_{l-2}^{(2,2)}&=2^{2l-4}\left[\binom{m-2l+4}{2}(\mu_{l-2}+\mu_{l+2})-2(m-2l)(m-2l+3)\right.\\
&\quad\left.\cdot(\mu_{l-1}+\mu_{l+1})+\{3(m-2l)^2+5(m-2l)+4\}\mu_l\right] ,\end{aligned}$$

$$\text{(4.12)}\quad \begin{aligned}\kappa_{l-3}^{(2,2)}&=2^{2l-6}\left[\binom{m-2l+6}{2}(\mu_{l-3}+\mu_{l+3})-(m-2l+5)(m-2l-2)\right.\\
&\quad\cdot(\mu_{l-2}+\mu_{l+2})-\frac{1}{2}\{(m-2l)^2+11(m-2l)-2\}(\mu_{l-1}+\mu_{l+1})\\
&\quad\left.+2\{(m-2l)^2+3(m-2l)+6\}\mu_l\right] ,\end{aligned}$$

$$\text{(4.13)}\quad \kappa_{l-2}^{(0,2)}=2^{2l-4}\sqrt{\binom{m-2l+4}{2}}\{(\mu_{l-2}+\mu_{l+2})-2\mu_l\} ,$$

$$(4.14)\qquad \kappa_{l-2}^{(1,2)}=2^{2l-4}\sqrt{\frac{m-2l+3}{2}}\{(m-2l+4)(\mu_{l+2}-\mu_{l-2})-2(m-2l)\cdot(\mu_{l+1}-\mu_{l-1})\}\ ,$$

$$(4.15)\qquad \kappa_{l-3}^{(3,3)}=2^{2l-6}\left[\binom{m-2l+6}{3}(\mu_{l-3}+\mu_{l+3})-(m-2l)(m-2l+4)(m-2l+5)\cdot(\mu_{l-2}+\mu_{l+2})+\frac{1}{2}(m-2l+4)\{5(m-2l)^2+7(m-2l)+6\}\cdot(\mu_{l-1}+\mu_{l+1})-\frac{2}{3}(m-2l)\{5(m-2l)^2+21(m-2l)+28\}\mu_l\right]\ .$$

From Theorem 4.2 and (4.2)-(4.15), we thus have

THEOREM 4.3. *Let T be a B-array of strength $2l$ ($l\geqq 3$), m constraints and index set $\{\mu_0,\mu_1,\cdots,\mu_{2l}\}$. Then necessary conditions for T to be a balanced fractional 2^m factorial design of resolution $2l+1$ are that the following strict inequalities hold:*

$$(4.16)\qquad \mu_l>0\ ,$$

$$(4.17)\qquad (m-2l+2)(\mu_{l-1}+\mu_{l+1})-2(m-2l)\mu_l>0\ ,$$

$$(4.18)\qquad (m-2l+4)(\mu_{l-2}+\mu_{l+2})+4(\mu_{l-1}+\mu_{l+1})-2(m-2l)\mu_l>0\ ,$$

$$(4.19)\qquad \binom{m-2l+4}{2}(\mu_{l-2}+\mu_{l+2})-2(m-2l)(m-2l+3)(\mu_{l-1}+\mu_{l+1})+\{3(m-2l)^2+5(m-2l)+4\}\mu_l>0\ ,$$

$$(4.20)\qquad (m-2l+6)(\mu_{l-3}+\mu_{l+3})+2(m-2l+8)(\mu_{l-2}+\mu_{l+2})-(m-2l-10)\cdot(\mu_{l-1}+\mu_{l+1})-4(m-2l)\mu_l>0\ ,$$

$$(4.21)\qquad \binom{m-2l+6}{2}(\mu_{l-3}+\mu_{l+3})-(m-2l+5)(m-2l-2)(\mu_{l-2}+\mu_{l+2})-\frac{1}{2}\{(m-2l)^2+11(m-2l)-2\}(\mu_{l-1}+\mu_{l+1})+2\{(m-2l)^2+3(m-2l)+6\}\mu_l>0\ ,$$

$$(4.22)\qquad \binom{m-2l+6}{3}(\mu_{l-3}+\mu_{l+3})-(m-2l)(m-2l+4)(m-2l+5)(\mu_{l-2}+\mu_{l+2})+\frac{1}{2}(m-2l+4)\{5(m-2l)^2+7(m-2l)+6\}(\mu_{l-1}+\mu_{l+1})-\frac{2}{3}(m-2l)\{5(m-2l)^2+21(m-2l)+28\}\mu_l>0\ ,$$

$$(4.23)\qquad (m-2l+2)\mu_{l+1}\mu_{l-1}+(\mu_{l+1}\mu_l+\mu_l\mu_{l-1})-(m-2l)\mu_l^2>0\ ,$$

$$(4.24)\quad (\mu_{l+2}\mu_{l+1}+\mu_{l-1}\mu_{l-2})+(m-2l+6)(\mu_{l+2}\mu_l+\mu_l\mu_{l-2})$$
$$+(2m-4l+9)(\mu_{l+2}\mu_{l-1}+\mu_{l+1}\mu_{l-2})+(m-2l+4)\mu_{l+2}\mu_{l-2}$$
$$-(m-2l)(\mu_{l+1}^2+\mu_{l-1}^2)-2(m-2l-3)(\mu_{l+1}\mu_l+\mu_l\mu_{l-1})$$
$$+2(m-2l+8)\mu_{l+1}\mu_{l-1}-3(m-2l)\mu_l^2>0\ ,$$

$$(4.25)\quad -2(m-2l)(\mu_{l+1}^2\mu_l+\mu_l\mu_{l-1}^2)-4(m-2l)(m-2l+3)$$
$$\cdot(\mu_{l+1}^2\mu_{l-1}+\mu_{l+1}\mu_{l-1}^2)-(m-2l)(m-2l+3)(m-2l+4)$$
$$\cdot(\mu_{l+1}^2\mu_{l-2}+\mu_{l+2}\mu_{l-1}^2)-2(m-2l)(\mu_{l+2}\mu_l^2+\mu_l^2\mu_{l-2})$$
$$+(m-2l)\{3(m-2l)+1\}(\mu_{l+1}\mu_l^2+\mu_l^2\mu_{l-1})$$
$$-(m-2l)^2(m-2l+1)\mu_l^3+2(\mu_{l+2}\mu_{l+1}\mu_l+\mu_l\mu_{l-1}\mu_{l-2})$$
$$+4(m-2l+3)(\mu_{l+2}\mu_{l+1}\mu_{l-1}+\mu_{l+1}\mu_{l-1}\mu_{l-2})$$
$$+(m-2l+3)(m-2l+4)(\mu_{l+2}\mu_{l+1}\mu_{l-2}+\mu_{l+2}\mu_{l-1}\mu_{l-2})$$
$$+2\{2(m-2l)+9\}(\mu_{l+2}\mu_l\mu_{l-1}+\mu_{l+1}\mu_l\mu_{l-2})$$
$$+(m-2l+4)\{(m-2l)^2+5(m-2l)+8\}\mu_{l+2}\mu_l\mu_{l-2}$$
$$+2\{(m-2l)^3+3(m-2l)^2-2(m-2l)+4\}\mu_{l+1}\mu_l\mu_{l-1}>0\ .$$

Finally, we can note the following result from Theorem 4.1:

THEOREM 4.4. *A necessary condition for the existence of a B-array of strength $2l$ ($l\geqq 3$), m constraints and index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$ is that the strict inequalities* (4.16)-(4.25) *are satisfied with inequalities and/or equalities.*

In the same way, we can express other elements of matrices K_β ($\beta\leqq l-3$) in terms of the m and μ_i of a B-array and obtain the results similar to Theorems 4.3 and 4.4. However they are very complicated and not available for practical use.

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