ANALYSIS OF SOME MIXED-MODELS FOR BLOCK AND SPLIT-PLOT DESIGNS*

R. P. BHARGAVA AND K. R. SHAH

(Received Oct. 3, 1972; revised April 1, 1974)

Abstract

In this paper we consider analysis of several models, where the structure of the covariance matrix is intermediate between that of intra-class correlation form and completely arbitrary. The designs considered are incomplete blocks and split-plot. Some of these models arise in studies of growth-curves, learning processes and other areas. Analysis is generally in terms of likelihood ratio tests.

1. Introduction

Conventional analysis of randomised block design uses interaction of treatments with blocks as error variance. This is based on the assumption that every treatment comparison has the same error variance (Scheffé [14]). When different treatment comparisons do not have the same error variance, Scheffé [13] suggests estimation of variances and covariances by a Wishart matrix and using a Hotelling's T^2 statistic for comparing treatment effects. Scheffé thus considered the case of the covariance matrix being an unknown positive definite matrix.

In this paper we consider analysis of various models where the structure for the covariance matrix is intermediate between that of intra-class correlation form (i.e., one with equal variances and equal covariances) and completely arbitrary. These relate to Weiner process, Markoff process (and its generalisations) and others which arise naturally in experiments having split-plot structures. Some of the above covariance matrices have been found suitable in some practical problems, e.g., see (Guttman [10], Anderson [2], Morrison [11], Gabriel [7], Bhargava [4]).

Tests of significance for the hypotheses concerning treatment effects have been obtained using likelihood ratio methods.

Throughout this paper, we shall assume that the observations are

* Supported in part by NRC Grant A. 7272.

normally distributed. We shall discuss the analyses of some incomplete block designs and split-plot designs in the framework of mixed models.

2. Analysis of block designs

It is well known that the standard analysis of a block design remains valid when all the observations have a common variance and the covariance between a pair of observations within a block is the same for all pairs, it being understood that observations in different blocks are independent. In this section we shall consider the analysis for the following structures of the covariance matrix.

(a) *Weiner process*

In each block, observations are taken at time points $t_1<t_2<\cdots<t_p$, and the covariance matrix of these observations is

$$\Sigma=\sigma^2\begin{bmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & \vdots & \cdots & \vdots \\ t_1 & t_2 & \cdots & t_p \end{bmatrix}. \tag{2.1}$$

Let x_{ij} denote the observation in the ith block at time t_j, $i=1, 2, \cdots, b$; $j=1, 2, \cdots, p$. Let $\boldsymbol{x}_i=(x_{i1}, \cdots, x_{ip})'$ and $\boldsymbol{\mu}=(\mu_1, \cdots, \mu_p)'$. Under the mixed model $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_b$ is a random sample from $N_p(\boldsymbol{\mu}, \Sigma)$. Our aim is to estimate μ_j's and to examine hypotheses concerning them.

Let $\boldsymbol{y}_i=(y_{i1}, y_{i2}, \cdots, y_{i\,p-1})'$ where $y_{ij}=x_{i\,j+1}-x_{ij}$. It is easy to see that

$$\mathrm{V}(\boldsymbol{y}_i)=\sigma^2 \,\mathrm{Diag}\,(t_2-t_1, t_3-t_2, \cdots, t_p-t_{p-1}) . \tag{2.2}$$

It follows that $y_{ij}/(t_{j+1}-t_j)^{1/2}$ are all uncorrelated and have the same variance. Standard least squares methods can now be applied to estimate the differences between μ_j's and to examine hypothesis of no differences or of a linear trend (in time) or any other hypothesis of interest.

The above methods can be applied when the blocks are incomplete in the sense that the set of time points in a block is a subset of the set $\{t_1, t_2, \cdots, t_p\}$ but the design is connected, i.e., all differences among μ_j's are estimable. Such an analysis might provide some guidance in the choice of the appropriate design. Of course, for unconnected designs also, an analysis along the same lines can be made to examine hypotheses concerning estimable parametric functions.

(b) *Non-stationary process with independent increments*

Consider observations at time points $t_1<t_2<\cdots<t_p$ (in each block). Suppose the dispersion matrix of these observations is

$$(2.3)\qquad \Sigma=\begin{pmatrix} \theta_1 & \theta_1 & \cdots & \theta_1 \\ \theta_1 & \theta_2 & \cdots & \theta_2 \\ \vdots & \vdots & \cdots & \vdots \\ \theta_1 & \theta_2 & \cdots & \theta_p \end{pmatrix},$$

where $\theta_1, \cdots, \theta_p$ are unknown parameters subject to $0<\theta_1<\theta_2<\cdots<\theta_p$.

Using the same notation and transformation as in (a), we get

$$(2.4)\qquad \mathrm{V}\,(\boldsymbol{y}_i)=\mathrm{Diag}\,(\theta_2-\theta_1, \cdots, \theta_p-\theta_{p-1})\ .$$

It is clear that $\mu_{j+1}-\mu_j$ is estimated by $y._j=1/b\sum_{i=1}^{b} y_{ij}$ (throughout this paper $\cdot$ will indicate average over the suffix replaced) and that the variance of y_j is estimated by $\sum_{i=1}^{b}(y_{ij}-y_j)^2/b(b-1)$. This leads to a '$t$' test for testing for a specific value of $\mu_{j+1}-\mu_j$.

To test $\mu_1=\cdots=\mu_p$, we note that the likelihood ratio test statistic is

$$\lambda=\prod_{j=1}^{p-1} Z_j^{b/2} \qquad \text{where } Z_j\equiv\sum_{i=1}^{b}(y_{ij}-y._j)^2\Big/\sum_{i=1}^{b} y_{ij}^2\ .$$

Under the hypothesis $Z_1, \cdots, Z_{p-1}$ are independent and each is distributed as a beta random variable with parameters $(b-1)/2$ and $1/2$. The asymptotic distribution of $-2\log_e\lambda$ under the hypothesis can be obtained by following the method of Box [8] which is given below. Let M_0 be the observed value of $-2\log_e\lambda$. The significance probability is given by

$$(2.5)\qquad \mathrm{P}\{-2\log\lambda\geqq M_0\}=(1-t)\,\mathrm{P}\{\chi^2_{p-1}\geqq\rho M_0\}+t\,\mathrm{P}\{\chi^2_{p+3}\geqq\rho M_0\} +O(b^{-3})\ ,$$

where

$$\rho=1-3/2b\ ,$$

and

$$(2.6)\qquad t=\frac{(b-1)}{2\rho^2b^2}\{(b-1-\rho b)/2\}\{(b-1)-\rho b-1\}\ .$$

Approximations of higher orders may be obtained by taking more terms in the expansion (see Anderson [1]).

It would often be of interest to examine if μ_j's are linear in time, i.e., $\mu_{j+1}-\mu_j=\alpha(t_{j+1}-t_j)$ for some α. To compute the value of the likelihood ratio test statistic, one computes

$$\lambda(\alpha)=\frac{\prod_{j=1}^{p-1}(\hat{\sigma}_j^2)}{\prod_{j=1}^{p-1}\hat{\sigma}_j^2(\alpha)} \qquad \text{where } \hat{\sigma}_j^2=\sum_{i=1}^{b}(y_{ij}-y._j)^2/b$$

and $\hat{\sigma}_j^2(\alpha)=\sum_i (y_{ij}-\alpha t_{j+1}+\alpha t_j)^2/b$ for a given α. $\lambda\equiv\min_\alpha \lambda(\alpha)$ is the value of the likelihood ratio test statistic. $-2\log_e \lambda$ can be regarded as a χ^2 variate with $p-2$ degrees of freedom when b is large.

Similar methods can be employed when the blocks are not complete but each block contains observations for a contiguous set of time points beginning with the first, i.e., in the case of a staircase design, Graybill [9].

(c) *Markoff type dispersion matrix*

We consider a situation where we have p treatments applied in sequence. The covariance matrix will be said to be of the Markoff type if the regression of the yield of the jth treatment ($j>1$) on the yields of the preceding $j-1$ treatments involves only the yield of the ($j-1$)th treatment.

Let the experiment be performed in b blocks each with all the p treatments. Observations in different blocks will be assumed to be independent. Let x_{ij} denote the yield of the jth treatment in the ith block.

Let $\boldsymbol{x}_i=(x_{i1},\cdots,x_{ip})'$ and $\mathrm{E}\,\boldsymbol{x}_i=\boldsymbol{\mu}=(\mu_1,\cdots,\mu_p)'$. The above description implies that

$$\mathrm{E}\,(x_{ij}\,|\,x_{i\,j-1},\cdots,x_{i1})=\mu_j+\beta_{j-1}^{(j)}(x_{i\,j-1}-\mu_{j-1})$$

for $j=1,2,\cdots,p$, it being understood that $\beta_0^{(1)}=0$. Let

$$\mathrm{V}\,(x_{ij}\,|\,x_{i\,j-1},\cdots,x_{i1})=\sigma_{j(e)}^2\,.$$

We shall now obtain the likelihood ratio test for the hypothesis $\mu_1=\mu_2=\cdots=\mu_p=\mu$, say. Let $L(\Omega)$ and $L(\omega)$ denote the likelihood function under the model and under the hypothesis. Then

$$\max L(\Omega)=e^{-pb/2}(2\pi)^{-pb/2}\left(\prod_1^p \hat{\sigma}_{j(e)}^2\right)^{-b/2},$$

where

$$b\hat{\sigma}_{j(e)}^2=\sum_{i=1}^b (x_{ij}-x_{\cdot j})^2-\frac{\left[\sum_{i=1}^b (x_{ij}-x_{\cdot j})(x_{i\,j-1}-x_{\cdot\,j-1})\right]^2}{\sum_{i=1}^b (x_{i\,j-1}-x_{\cdot\,j-1})^2}\,.$$

Also,

$$L(\omega)=(2\pi)^{-pb/2}\left(\prod_{j=1}^p \sigma_{j(e)}^2\right)^{-b/2}\exp\left\{-\frac{1}{2}\sum_{j=1}^p\sum_{i=1}^b \frac{(x_{ij}-\mu-\beta_{j-1}^{(j)}(x_{ij-1}-\mu))^2}{\sigma_{j(e)}^2}\right\}\,.$$

Let $f(\mu)$ denote maximum of $L(\omega)$ for a fixed value of μ. It is easy to see that

$$f(\mu)=(2\pi e)^{-pb/2}\prod_{j=1}^{p}\{\hat{\sigma}^2_{j(e)}(\mu)\}^{-b/2},$$

where

$$b\hat{\sigma}^2_{j(e)}(\mu)=\sum_i (x_{ij}-\mu)^2-\frac{[\sum_i (x_{ij}-\mu)(x_{ij-1}-\mu)]^2}{\sum_i (x_{ij-1}-\mu)^2}.$$

$\max L(\omega)$ which is maximum of $f(\mu)$ can be obtained by numerical methods. This gives us

$$\lambda=\frac{\max L(\Omega)}{\max L(\omega)}.$$

Under the hypothesis asymptotic distribution of $-2\log_e\lambda$ is χ^2 with $(p-1)$ degrees of freedom.

It is clear that these methods can be applied when the blocks are not complete but are monotonic in the sense that if a block does not contain treatment j, it does not contain treatments $j+1,\cdots,p$.

Similar methods can be developed when the regression of the yield of the jth treatment depends upon the yields of previous treatments $j-1,\cdots,j-s$ $(j>s)$. For $j\leqq s$, it depends on the yields of all the previous treatments*.

3. Treatments with a factorial structure

In this section we shall consider designs where the treatments are combinations of levels of factor A, applied at m levels and factor B, applied at s levels. We shall consider split-plot type situations where levels of A are varied over main plots. Each main plot contains s subplots to which the s levels of B are applied.

Let n_{ij} denote the number of main plots in the ith block to which jth level of A is applied. We shall assume $n_{ij}=0$ or 1. Let y_{ijk} denote the observation in the ith block for the plot containing jth level of A and kth level of B. We shall put

$$\boldsymbol{y}_{ij}=(y_{ij1},y_{ij2},\cdots,y_{ijs})'.$$

Let $\mathrm{V}(\boldsymbol{y}_{ij})=C_j$ when $n_{ij}=1$ and $\mathrm{cov}(\boldsymbol{y}_{ij},\boldsymbol{y}'_{ij})=D_{jj'}$, when $n_{ij}=n_{ij'}=1$.

We shall assume that the observations from different blocks are independent.

Case I. $C_j=C$, $D_{jj'}=D$

Let $sy_{ij\cdot}=\sum_k y_{ijk}$

* Such a model has been termed sth ante-dependence by Gabriel [7]. Gabriel [7] and Bhargava [4] have given methods to examine the adequacy of such a model.

$$V(sy_{ij\cdot})=\alpha\,, \quad \operatorname{cov}(sy_{ij\cdot},\, sy_{ij'\cdot})=\beta \qquad j\neq j'$$

where α is the sum of all elements in C and β is the sum of all elements in D.

It is clear that the main plot totals can be analysed as in any incomplete block design to make inferences concerning main effects of A. Hence, we shall only discuss the tests for the main effects of B and for the interaction AB. In this section H_B will denote the hypothesis of zero main effects of factor B and H_{AB} will denote the hypothesis of no interaction between factors A and B.

I(a). $C_j=C$, $D=E_{ss}$ (E_{mn} denotes an $m\times n$ matrix with each element unity).

Consider an $(s-1)\times s$ matrix P such that

$$PP'=I_{s-1}\,, \qquad P'P=I_s-(1/s)E_{ss}\,.$$

Let $\boldsymbol{u}_{ij}=P\boldsymbol{y}_{ij}$. Further, let

$$E(\boldsymbol{u}_{ij})=\boldsymbol{\mu}_j=(\mu_{j1},\cdots,\mu_{js-1})'$$

and

$$V(\boldsymbol{u}_{ij})=\Sigma_0\,.$$

It is easy to check that $\operatorname{cov}(\boldsymbol{u}_{ij},\boldsymbol{u}_{ij'})=0$ when $j\neq j'$. Evidently, $\boldsymbol{\mu}_j$ is estimated by $\hat{\boldsymbol{\mu}}_j=\sum_{i=1}^{b} n_{ij}\boldsymbol{u}_{ij}/N_j$ where $N_j=\sum_{i=1}^{b} n_{ij}$.

Hypothesis H_B is equivalent to $\sum_{j=1}^{m}\boldsymbol{\mu}_j=\boldsymbol{0}$.

$$V\left(\sum_j \hat{\boldsymbol{\mu}}_j\right)=\left(\sum_j \frac{1}{N_j}\right)\Sigma_0\,.$$

It is clear that $S=\sum_{j=1}^{m}\sum_{i=1}^{b} n_{ij}(\boldsymbol{u}_{ij}-\hat{\boldsymbol{\mu}}_j)(\boldsymbol{u}_{ij}-\hat{\boldsymbol{\mu}}_j)'$ is a Wishart matrix with parameters $(s-1,\Sigma_0)$ and with $\sum_j (N_j-1)=N-m$ d.f. where $N=\sum_{j=1}^{m} N_j$.

It follows that $(\sum_j \hat{\boldsymbol{\mu}}_j)'S^{-1}(\sum_j \hat{\boldsymbol{\mu}}_j)(N-m)/\sum_j 1/N_j$ follows Hotelling's $T^2(s-1)$ distribution with $N-m$ d.f. when H_B is true.

In the special case, $C=\gamma I_s+\delta E_{ss}$, Σ_0 reduces to $\sigma_0^2 I_{s-1}$ where $\sigma_0^2=\gamma$. Here, one uses the F statistic to test H_B.

To obtain this F statistic we note that for each l, $(\sum_j u_{\cdot jl})^2/(\sum_j 1/N_j)\sigma_0^2$ follows χ^2 distribution with one d.f. if H_B is true. It follows that the sum of squares due to H_B is $\sum_l \{(\sum_j u_{\cdot jl})^2/\sum_j 1/N_j\}$. Expressed in terms of original observations this reduces to

$$\left\{\sum_k \left(\sum_j y_{\cdot jk}\right)^2 - \frac{1}{s}\left(\sum_j \sum_k y_{\cdot jk}\right)^2\right\} \Big/ \sum_j \frac{1}{N_j} .$$

Error sum of squares can be verified to be the sum of the so-called "interaction (between blocks and levels of B) sum of squares" computed for different levels of A. This will carry $(s-1)\sum_j (N_j-1)=(s-1)(N-m)$ degrees of freedom.

Next, we consider test of H_{AB} which is equivalent to $\boldsymbol{\mu}_1=\cdots=\boldsymbol{\mu}_m$. In the general case when Σ_0 is arbitrary, this is equivalent to one-way classification analysis of dispersion, (e.g., see Rao [12]).

When $C=\gamma I_s+\delta E_{ss}$ one can use the following F test. To obtain sum of squares due to H_{AB} we note that to test $\mu_{1l}=\mu_{2l}=\cdots=\mu_{ml}$ appropriate sum of squares is $\sum_j N_j u^2_{\cdot jl}-(\sum_j N_j u_{\cdot jl})^2/N$. Adding this over different l and expressing in terms of original observations we get $\sum_j \sum_k N_j y^2_{\cdot jk}-\sum_j sN_j y^2_{\cdot j\cdot}-(1/N)\sum_k(\sum_j N_j y_{\cdot jk})^2+(s/N)(\sum_j N_j y_{\cdot j\cdot})^2$. Of course, the error sum of squares would remain the same as in the test of H_B.

I(b).

$$C=\sigma^2\begin{bmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & \vdots & \cdots & \vdots \\ t_1 & t_2 & \cdots & t_s \end{bmatrix}, \qquad D=\gamma E_{ss}$$

$t_1<t_2<\cdots<t_s$ are known, σ^2 and γ are unknown. Let

$$v_{ijl}=y_{ijl+1}-y_{ijl}, \qquad l=1, 2, \cdots, s-1,$$
$$\boldsymbol{v}_{ij}=(v_{ij1}, \cdots, v_{ijs-1})',$$
$$\mathrm{V}(\boldsymbol{v}_{ij})=\sigma^2\,\mathrm{Diag}\,(t_2-t_1, \cdots, t_s-t_{s-1}) .$$

$\boldsymbol{v}_{ij}$'s are all uncorrelated. Let

$$\mathrm{E}(\boldsymbol{v}_{ij})=\boldsymbol{\beta}_j .$$

To examine the main effects of B we test the hypothesis

$$\sum_{j=1}^{m} \boldsymbol{\beta}_j=\mathbf{0} .$$

Let $N_j v_{\cdot jl}=\sum_{i=1}^{N_j} v_{ijl}$, $j=1, \cdots, m$, $l=1, \cdots, s-1$. Then the s.s. due to main effects of B is

$$SS_B=\frac{\sum_{l=1}^{s-1}\left(\left(\sum_{j=1}^{m} v_{\cdot jl}\right)^2 \Big/ (t_{l+1}-t_l)\right)}{\left(\sum_{j=1}^{m}\frac{1}{N_j}\right)} .$$

H_{AB} is equivalent to

$$\boldsymbol{\beta}_1=\boldsymbol{\beta}_2=\cdots=\boldsymbol{\beta}_m .$$

Then the interaction s.s. reduces to

$$SS_{AB}=\sum_{l=1}^{s-1}\left(\frac{1}{t_{l+1}-t_l}\left(\sum_{j=1}^{m} N_j v_{\cdot jl}^2-\frac{\left(\sum_{j=1}^{m} N_j v_{\cdot jl}\right)^2}{\sum_{j=1}^{m} N_j}\right)\right).$$

The error s.s. based on $(s-1)\sum_{j=1}^{m}(N_j-1)$ d.f. is

$$SS_e=\sum_{j=1}^{m}\sum_{l=1}^{s-1}\sum_{i=1}^{N_j}((v_{ijl}-v_{\cdot jl})^2/(t_{l+1}-t_l)) .$$

I(c).

$$C=\begin{bmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & \vdots & \cdots & \vdots \\ t_1 & t_2 & \cdots & t_p \end{bmatrix}, \qquad D=\rho C .$$

We shall assume here that every block contains each treatment. Let v_{ijl} and $\boldsymbol{v}_{ij}$ be as defined in I(b). It is easy to see that

$$\mathrm{V}(\boldsymbol{v}_{ij})=\sigma^2 \operatorname{Diag}(t_2-t_1,\cdots,t_s-t_{s-1}) ,$$

$$\operatorname{Cov}(\boldsymbol{v}_{ij},\boldsymbol{v}_{ij'})=\rho\sigma^2 \operatorname{Diag}(t_2-t_1,\cdots,t_s-t_{s-1}) , \qquad j\neq j' ,$$

$$\operatorname{Cov}(\boldsymbol{v}_{ij},\boldsymbol{v}_{i'j'})=0 , \qquad i\neq i' ,$$

$$\mathrm{E}(\boldsymbol{v}_{ij})=\boldsymbol{\beta}_j .$$

Let

$$\boldsymbol{x}_i=(x_{i1},\cdots,x_{i\,s-1})'=\sum_j \boldsymbol{v}_{ij} .$$

Then, $\mathrm{V}(\boldsymbol{x}_i)=m\sigma^2(1+\overline{m-1}\rho)\operatorname{Diag}(t_2-t_1,\cdots,t_s-t_{s-1})$. $\sigma^2(1+\overline{m-1}\rho)$ can be looked upon as a single unknown parameter. To examine the main effects of B, one tests $\mathrm{E}(\boldsymbol{x}_i)=0$. s.s. due to the hypothesis turns out to be

$$SS_B=\sum_{l=1}^{s-1}\frac{bx_{\cdot l}^2}{(t_{l+1}-t_l)}$$

where $bx_{\cdot l}=\sum_l x_{il}$. Error s.s. can be shown to be

$$\sum_{l=1}^{s-1}\frac{1}{(t_{l+1}-t_l)}\sum_{i=1}^{b}(x_{il}-x_{\cdot l})^2 .$$

This will be based on $(s-1)(b-1)$ d.f. and is independent of SS_B. Interaction AB may be examined by testing $\boldsymbol{\beta}_1=\boldsymbol{\beta}_2=\cdots=\boldsymbol{\beta}_m$.

It is easy to verify that for fixed i and l, orthogonal comparisons involving v_{ijl} are uncorrelated, each normalised comparison having variance $(t_{l+1}-t_l)(1-\rho)\sigma^2$. From this, one can show that

$$SS_{AB}=\sum_{l=1}^{s-1}\frac{b}{t_{l+1}-t_l}\sum_{j=1}^{m}(v_{\cdot jl}-v_{\cdot\cdot l})^2$$

with $(m-1)(s-1)$ d.f. where $bv_{\cdot jl}=\sum_i v_{ijl}$ and $bmv_{\cdot\cdot l}=\sum_i\sum_j v_{ijl}$. Let $mv_{i\cdot l}=\sum_j v_{ijl}$. Error s.s. to test interaction AB reduces to

$$\sum_{l=1}^{s-1}\frac{1}{(t_{l+1}-t_l)}\sum_i\sum_j(v_{ijl}-v_{i\cdot l}-v_{\cdot jl}+v_{\cdot\cdot l})^2 .$$

This will be based on $(s-1)(m-1)(b-1)$ d.f. and is independent of SS_{AB}.

Case II. $C_j=\alpha_j I_s+\beta_j E_{ss}$ *and* $D_{jj'}=p_{jj'}E_{ss}$

We shall analyse this case on by when the design in the levels of A is a staircase design. Of course, this includes the complete design.

For the main-plot totals we have completely arbitrary covariance matrix. Levels of A can be compared as in Bhargava [5].

For the sub-plot analysis we consider vector $\boldsymbol{u}_{ij}$ defined as in I(a). Clearly, $\mathrm{V}(\boldsymbol{u}_{ij})=\sigma_j^2 I_{s-1}$ where $\sigma_j^2=\alpha_j$.

$$\mathrm{Cov}(\boldsymbol{u}_{ij},\boldsymbol{u}_{i'j'})=0 \qquad \text{for } j\neq j' .$$

Again $\hat{\boldsymbol{\mu}}_j=(1/N_j)\sum_i \boldsymbol{u}_{ij}$. H_B is equivalent to $\sum_j \boldsymbol{\mu}_j=0$. We note that $\mathrm{V}(\sum\hat{\boldsymbol{\mu}}_j)=(\sum_j(1/N_j)\sigma_j^2)I_s$. To estimate $\sum(1/N_j)\sigma_j^2$ we construct $w_{il}=\sum_j(1/\sqrt{N_j})u_{ijl}$. Clearly, $\mathrm{V}(w_{il})=\sum(1/N_j)\sigma_j^2$ and w_{il} are all uncorrelated. This implies that if $\bar{w}_l$ denote means of w_{il} from the N_m complete blocks, $\sum_{l=1}^{s-1}\sum_{i=1}^{N_m}(w_{il}-\bar{w}_l)^2/(\sum_j(1/N_j)\sigma_j^2)$ is a χ^2 with $(s-1)(N_m-1)$ degrees of freedom and is distributed independently of $\hat{\boldsymbol{\mu}}_j$. This gives us an F statistic with $(s-1)$ and $(s-1)(N_m-1)$ d.f. for test of H_B.

H_{AB} is equivalent to $\boldsymbol{\mu}_1=\boldsymbol{\mu}_2=\cdots=\boldsymbol{\mu}_m$. We note that any particular component of AB can be tested by a procedure exactly similar to the one above.

The problem of testing the entire AB interaction involves comparing mean vectors of m populations where within a population all the $(s-1)$ variables are independent and have the same variance. For any fixed common mean vector $\boldsymbol{\mu}$ the maximum of the likelihood can be easily obtained. For small values of s, maximum of this w.r.t. $\boldsymbol{\mu}$ can be obtained by numerical methods. This provides maximum of the likelihood under H_{AB}. Since the maximum of the likelihood under the model can be obtained easily, likelihood ratio test statistic can be calculated.

When the main plot design is complete the following approach can also be used.

Let $\boldsymbol{\beta}_l=(\beta_{1l},\cdots,\beta_{m-1\,l})'$; $l=1,2,\cdots,s-1$ be independent contrasts in $\mu_{1l},\cdots,\mu_{ml}$ such that the coefficients do not depend upon l. Clearly, H_{AB} is equivalent to $\boldsymbol{\beta}_1=\boldsymbol{\beta}_2=\cdots=\boldsymbol{\beta}_{s-1}=\mathbf{0}$.

Let $\mathrm{V}(\hat{\boldsymbol{\beta}}_l)=\Sigma$. The data provides a Wishart matrix S with $(b-1)\cdot(s-1)$ degrees of freedom such that $\mathrm{E}(S)=(b-1)(s-1)\Sigma$. Further, S and $\hat{\boldsymbol{\beta}}_l$ are independently distributed. This leads to $|S|\Big/\left|S+\sum_{l=1}^{s-1}\hat{\boldsymbol{\beta}}_l\hat{\boldsymbol{\beta}}_l'\right|$ as a test statistic for H_{AB}. Under H_{AB}, the distribution of this will be that of the product of $(m-1)$ independent Beta variables, the ith Beta having parameters $((s-1)/2, ((b-1)(s-1)-(i-1))/2)$.

In this method we are estimating the covariance matrix of order $(m-1)$ ignoring the fact that all the elements of this matrix are linear functions of m parameters. Anderson [2] and Bhargava [4] have adopted such methods in similar situations.

ONTARIO INSTITUTE FOR STUDIES IN EDUCATION
UNIVERSITY OF TORONTO AND UNIVERSITY OF WATERLOO

REFERENCES

[1] Anderson, T. W. (1958). *An Introduction to Multivariate Analysis*, John Wiley & Sons, Inc., New York.

[2] Anderson, T. W. (1960). Some stochastic process models for intelligence test scores, In K. J. Arrow, et al. ed., *Mathematical Methods in the Social Sciences*, Stanford University Press, Stanford, California, 205-220.

[3] Anderson, T. W. (1963). A test for equality of means when covariance matrices are unequal, *Ann. Math. Statist.*, **34**, 671-672.

[4] Bhargava, R. (1962). Multivariate tests of hypotheses with incomplete data, *Technical Report*, No. 3, Applied Mathematics and Statistics Laboratories, Stanford University.

[5] Bhargava, R. P. (1967). Analysis of variance for Scheffé's mixed model for stair-case design, *Sankhya*, A, **29**, 391-398.

[6] Bhargava, R. P. (1971). A test for equality of means when covariance matrices are unequal, *Calcutta Statist. Ass. Bull.*, **20**, 153-156.

[7] Gabriel, K. R. (1962). Ante dependence analysis of an ordered set of variables, *Ann. Math. Statist.*, **33**, 201-212.

[8] Box, G. E. P. (1949). A general distribution theory for a class of likelihood ratio criteria, *Biometrika*, **36**, 317-346.

[9] Graybill, Franklin and Pruitt, William E. (1957). The stair-case design: Theory, *Ann. Math. Statist.*, **29**, 523-533.

[10] Guttman, L. (1954). A new approach to factor analysis: the Radex (1954), In P. F. Lazarsfeld ed., *Mathematical Thinking in the Social Sciences*, The Free Press, Glencoe, Illinois, 258-348.

[11] Morrison, D. F. (1970). The optimal spacing of repeated measurements, *Biometrics*, **26**, 281-290.

[12] Rao, C. Radhakrishna (1965). *Linear Statistical Inference and Its Applications*, John Wiley & Sons, Inc., New York.

[13] Scheffé, H. (1956). A 'mixed model' for the analysis of variance, *Ann. Math. Statist.*, **27**, 23-36.

[14] Scheffé, H. (1959). *The Analysis of Variance*, John Wiley & Sons, Inc., New York.