SOME DISTRIBUTIONS OF THE LATENT ROOTS OF A COMPLEX WISHART MATRIX VARIATE

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1. Introduction

Test statistics have been proposed in multivariate analysis concerned with the latent roots of certain matrix variates. In this paper we shall consider the distributions related to the latent roots of an $m\times m$ complex Wishart matrix A proportional to sample covariance matrix on the basis of observations drawn from a complex multivariate normal population. In Section 3 we shall derive the distributions of the maximum latent root λ_1 and the minimum latent root λ_m of A and in Section 4 that of the range R defined such that $R=\lambda_1(1+\lambda_1)^{-1}-\lambda_m(1+\lambda_m)^{-1}$ and in Section 5 those of the trace A and the trace $A(I_m+A)^{-1}$. All the representation in the three cases stated above are power series forms.

2. Evaluation of integrals

Here we shall describe summarily the useful integrals in the following sections.

THEOREM 1. *Let A be a positive definite Hermitian matrix of $m\times m$ and let B be any Hermitian matrix of $m\times m$, then*

$$(1)\quad \int_{I_m>A>xI_m} |A|^{n-m}\tilde{C}_\kappa(AB)dA$$
$$=\frac{(\tilde{\Gamma}_m(m))^2}{\tilde{\Gamma}_m(2m)}\tilde{C}_\kappa(B)\sum_{l=0}^{mn-m^2+k}(-1)^l(1-x)^{m^2+l}\sum_\nu\binom{\kappa_1}{\nu}\frac{[m]_\nu}{[2m]_\nu},$$

where n is an integer such that $n\geqq m$ and x is a nonnegative number and $\tilde{C}_\kappa(C)$ is a zonal polynomial of a Hermitian matrix C which is a symmetric function of the latent roots of C corresponding to the partition $\kappa=(k_1, k_2,\cdots, k_m)$, $k_1\geqq k_2\geqq\cdots\geqq k_m$, of an integer k and the partition κ_1 is denoted by $\kappa_1=(k_1+n-m, k_2+n-m,\cdots, k_m+n-m)$ and the constants $\binom{\kappa}{\nu}$'s are one's in the equality

$$(2)\qquad \frac{\tilde{C}_\kappa(I_m-A)}{\tilde{C}_\kappa(I_m)}=\sum_{l=0}^{k}\sum_{\nu}(-1)^l\binom{\kappa}{\nu}\frac{\tilde{C}_\nu(A)}{\tilde{C}_\nu(I_m)}$$

and multivariate gamma functions are

$$\tilde{\Gamma}_m(a)=\pi^{m(m-1)/2}\prod_{i=1}^{m}\Gamma(a-i+1)$$

and generalized binomial coefficients are

$$[a]_\kappa=\prod_{i=1}^{m}(a-i+1)_{k_i}\,,\qquad (a)_n=a(a+1)\cdots(a+n-1),\ (a)_0=1\,.$$

PROOF. We shall put the left-hand side of (1) to be $f(B)$. Then $f(B)$ is invariant under the transformation such that $B\to UBU^*$ where U is a unitary matrix normalized to make its volume unity. Hence replacing B by UBU^* and integrating with respect to U over the unitary group $U(m)$, it follows that

$$(3)\qquad f(B)=\frac{\tilde{C}_\kappa(B)}{\tilde{C}_\kappa(I_m)}f(I_m)\,.$$

On the other hand, we shall set $B=I_m$ in the left-hand side of (1) and perform the transformation $A=I_m-C$. Then noting that

$$(4)\qquad |D|^n\tilde{C}_\kappa(D)=\frac{\tilde{C}_\kappa(I_m)}{\tilde{C}_{\kappa_2}(I_m)}\tilde{C}_{\kappa_2}(D)$$

where n is arbitrary nonnegative integer and $\kappa_2=(k_1+n, k_2+n,\cdots, k_m+n)$, we get

$$(5)\qquad f(I_m)=\frac{\tilde{C}_\kappa(I_m)}{\tilde{C}_{\kappa_1}(I_m)}\int_{(1-x)I_m>C>0}\tilde{C}_{\kappa_1}(I_m-C)dC\,.$$

Substituting (2) in the above expression and integrating term by term with the help of (8) in Sugiyama [9] that

$$(6)\qquad \int_{xI>A>0}|A|^{t-m}\tilde{C}_\kappa(AB)dA=\frac{\tilde{\Gamma}_m(m)\tilde{\Gamma}_m(t)}{\tilde{\Gamma}_m(t+m)}\frac{[t]_\kappa}{[t+m]_\kappa}\tilde{C}_\kappa(B)x^{mt+k}\,,$$

we have

$$(7)\qquad f(I_m)=\frac{(\tilde{\Gamma}_m(m))^2}{\tilde{\Gamma}_m(2m)}\tilde{C}_\kappa(I_m)\sum_{l=0}^{mn-m^2+k}(-1)^l(1-x)^{m^2+l}\sum_\nu\binom{\kappa_1}{\nu}\frac{[m]_\nu}{[2m]_\nu}\,.$$

Hence from (3) and (7) we finish the proof.

THEOREM 2. *Let* $\Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_m)$, *then*

$$(8)\qquad \int_{\lambda_1>\lambda_2>\cdots>\lambda_m>\lambda_1-t}|\Lambda|^{n-m}\tilde{C}_\kappa(\Lambda)\prod_{i<j}(\lambda_i-\lambda_j)^2\prod_{i=2}^{m}d\lambda_i$$

$$=\frac{(\tilde{\Gamma}_m(m))^2\tilde{\Gamma}_m(m+2)\Gamma(2m+1)}{\pi^{m(m-1)}\tilde{\Gamma}_m(2m+1)(\Gamma(m))^2\Gamma(m+2)}\tilde{C}_\kappa(I_m)\lambda_1^{mn+k-1}$$
$$\cdot\sum_{l=0}^{mn-m^2+k}(-1)^l(t/\lambda_1)^{m^2+l-1}\sum_\nu\binom{\kappa_1}{\nu}\frac{[m+1]_\nu}{[2m]_\nu}\frac{\tilde{C}_\nu(\mathring{I}_{m-1})}{\tilde{C}_\nu(I_m)},$$
$$\lambda_1\geqq t\geqq 0,$$

where $\mathring{I}_{m-1}=\text{diag}(0, 1, \cdots, 1)$ *and the other notations are the same as in Theorem* 1.

PROOF. Put the left-hand side of (8) to be $f(\lambda_1, t)$. Then, transforming such that $\lambda_i=\lambda_1(1-t_i)$, $i=2, 3, \cdots, m$, we get using (4)

$$(9)\qquad f(\lambda_1, t)=\frac{\tilde{C}_\kappa(I_m)}{\tilde{C}_{\kappa_1}(I_m)}\lambda_1^{mn+k-1}\int_{t/\lambda_1>t_m>t_{m-1}>\cdots>t_2>0}|\Lambda_t|^2\tilde{C}_{\kappa_1}(I_m-\mathring{\Lambda}_t)$$
$$\cdot\prod_{i<j}(t_i-t_j)^2\prod_{i=2}^m dt_i$$

where $\Lambda_t=\text{diag}(t_2, \cdots, t_m)$ and $\mathring{\Lambda}_t=\text{diag}(0, t_2, \cdots, t_m)$. Using the results that

$$\tilde{C}_\kappa(\mathring{\Lambda}_t)=0 \qquad \text{for } k_m\neq 0$$
$$\frac{\tilde{C}_\kappa(\mathring{\Lambda}_t)}{\tilde{C}_\kappa(\mathring{I}_{m-1})}=\frac{\tilde{C}_{\kappa'}(\Lambda_t)}{\tilde{C}_{\kappa'}(I_{m-1})} \qquad \text{for } k_m=0$$

where $\kappa'=(k_1, k_2, \cdots, k_{m-1})$ in the series (2) for $\tilde{C}_\kappa(I_m-\mathring{\Lambda}_t)/\tilde{C}_\kappa(I_m)$ and substituting it in the integrand of (9) and further integrating term by term with respect to t_i's, $i=2, 3, \cdots, m$, by the application of (6), we get

$$(10)\qquad f(\lambda_1, t)=\frac{(\tilde{\Gamma}_{m-1}(m-1))^2\tilde{\Gamma}_{m-1}(m+1)}{\pi^{(m-1)(m-2)}\tilde{\Gamma}_{m-1}(2m)}\tilde{C}_\kappa(I_m)\lambda_1^{mn+k-1}$$
$$\cdot\sum_{l=0}^{mn-m^2+k}(-1)^l(t/\lambda_1)^{m^2+l-1}\sum_\nu\binom{\kappa_1}{\nu}\frac{[m+1]_{\nu'}}{[2m]_{\nu'}}\frac{\tilde{C}_\nu(\mathring{I}_{m-1})}{\tilde{C}_\nu(I_m)}.$$

Furthermore in consideration that

$$\tilde{\Gamma}_{m-1}(a)=\frac{\tilde{\Gamma}_m(a+1)}{\pi^{m-1}\Gamma(a+1)}$$

and

$$[a]_\nu=[a]_{\nu'} \qquad \text{for } l_m=0$$

where $\nu=(l_1, l_2, \cdots, l_m)$, $l_1\geqq l_2\geqq\cdots\geqq l_m$, (10) is reformulated such that

$$(11)\qquad f(\lambda_1, t)=\frac{(\tilde{\Gamma}_m(m))^2\tilde{\Gamma}_m(m+2)\Gamma(2m+1)}{\pi^{m(m-1)}\tilde{\Gamma}_m(2m+1)(\Gamma(m))^2\Gamma(m+2)}\tilde{C}_\kappa(I_m)\lambda_1^{mn+k-1}$$

$$\cdot \sum_{l=0}^{mn-m^2+k} (-1)^l (t/\lambda_1)^{m^2+l-1} \sum_{\nu} \binom{\kappa_1}{\nu} \frac{[m+1]_\nu}{[2m]_\nu} \frac{\tilde{C}_\nu(\mathring{I}_{m-1})}{\tilde{C}_\nu(I_m)} ,$$

which is the desired result.

3. The distributions of the maximum latent root and the minimum latent root

Let A be a positive definite Hermitian matrix distributed as $\tilde{W}_m(n, \Sigma)$, that is, having the density function such as

$$(12) \qquad \frac{1}{\tilde{\Gamma}_m(n)|\Sigma|^n} \operatorname{etr}(-\Sigma^{-1}A)|A|^{n-m} , \qquad ((5.0) \text{ of Goodman [3]}) .$$

Then the density function of the latent roots λ_i's, $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_m \geqq 0$ of A is given by

$$(13) \qquad \frac{\pi^{m(m-1)}}{\tilde{\Gamma}_m(m)\tilde{\Gamma}_m(n)|\Sigma|^n} |\Lambda|^{n-m} \prod_{i<j} (\lambda_i - \lambda_j)^2 \int_{U(m)} \operatorname{etr}(-\Sigma^{-1}U\Lambda U^*)dU$$

where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m)$. Hence making the transformation such that $\lambda_i = b_i(1-b_i)^{-1}$, $i=1, 2, \cdots, m$, we get the density function of b_i's as follows

$$(14) \qquad \frac{\pi^{m(m-1)}}{\tilde{\Gamma}_m(m)\tilde{\Gamma}_m(n)|\Sigma|^n} |\Lambda_b|^{n-m} \prod_{i<j} (b_i - b_j)^2 \sum_k \sum_\kappa \frac{\tilde{L}_\kappa^n(\Sigma^{-1})\tilde{C}_\kappa(\Lambda_b)}{k!\tilde{C}_\kappa(I_m)}$$

where $\Lambda_b = \operatorname{diag}(b_1, b_2, \cdots, b_m)$ by using the generating function for generalized Laguerre polynomials in complex variate case, that is, (16) of Hayakawa [4]

$$(15) \qquad |I-Z|^{-r-m} \int_{U(m)} \operatorname{etr}(-SUZ(I-Z)^{-1}U^*)dU = \sum_k \sum_\kappa \frac{\tilde{L}_\kappa^r(S)\tilde{C}_\kappa(Z)}{k!\tilde{C}_\kappa(I_m)} , \qquad ||Z|| < 1 .$$

Hence with the use of (6) in (14) the c.d.f. of the maximum latent root b_1 is obtained in the following way

$$(16) \qquad \Pr(b_1 \leqq x) = \frac{\tilde{\Gamma}_m(m)}{\tilde{\Gamma}_m(m+n)|\Sigma|^n} \sum_k \sum_\kappa \frac{[n]_\kappa}{k![m+n]_\kappa} \tilde{L}_\kappa^n(\Sigma^{-1}) x^{mn+k} .$$

Secondly making the use of (1) in (14), the c.d.f. of the minimum latent root b_m is gotten as follows

$$(17) \qquad \Pr(b_m \leqq x) = 1 - \frac{(\tilde{\Gamma}_m(m))^2}{\tilde{\Gamma}_m(2m)\tilde{\Gamma}_m(n)|\Sigma|^n} \sum_k \sum_\kappa \frac{\tilde{L}_\kappa^n(\Sigma^{-1})}{k!} \cdot \sum_{l=0}^{mn-m^2+k} (-1)^l (1-x)^{m^2+l} \sum_\nu \binom{\kappa_1}{\nu} \frac{[m]_\nu}{[2m]_\nu} .$$

By summarizing the above discussed results, we have the following theorem.

THEOREM 3. *Let A be distributed as* (12), *then the c.d.f.'s of the maximum latent root b_1 and the minimum latent root b_m of $A(I_m+A)^{-1}$ are given by* (16) *and* (17) *respectively.*

Here it should be noticed that the c.d.f. of the maximum latent root is immediately obtained from (16) and that of the minimum latent root of A from (17) because $\Pr(\lambda_i \leqq x) = \Pr(b_i \leqq x(1+x)^{-1})$, $i=1, 2, \cdots, m$.

When we set $x=1$ in (16), the following corollary is easily shown.

COROLLARY 1.

$$\sum_k \sum_\kappa \frac{[n]_\kappa \tilde{L}_\kappa^n(\Sigma^{-1})}{k![m+n]_\kappa} = \frac{\tilde{\Gamma}_m(m+n)}{\tilde{\Gamma}_m(m)} |\Sigma|^n . \tag{18}$$

4. The distribution of R

We shall define the range R such that $R=\lambda_1(1+\lambda_1)^{-1}-\lambda_m(1+\lambda_m)^{-1}$ where λ_i's, $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_m \geqq 0$, are the latent roots of A distributed according to (12). Then the c.d.f. of R is obtained by calculating

$$\Pr(R \leqq t) = \int_0^1 \Pr(b_1 \geqq b_2 \geqq \cdots \geqq b_m \geqq b_1 - t \,|\, b_1) dF(b_1) . \tag{19}$$

By the way, for $t \geqq b_1 \geqq 0$, $\Pr(b_1 \geqq b_2 \geqq \cdots \geqq b_m \geqq b_1 - t \,|\, b_1) = 1$ and for $b_1 \geqq t \geqq 0$, we get with the help of (8) in (14)

$$\begin{aligned} &\Pr(b_1 \geqq b_2 \geqq \cdots \geqq b_m \geqq b_1 - t \,|\, b_1) F'(b_1) \\ &= \frac{\tilde{\Gamma}_m(m)\tilde{\Gamma}_m(m+2)\Gamma(2m+1)}{\tilde{\Gamma}_m(2m+1)\tilde{\Gamma}_m(n)(\Gamma(m))^2\Gamma(m+2)} |\Sigma|^{-n} \sum_k \sum_\kappa \frac{\tilde{L}_\kappa^n(\Sigma^{-1})}{k!} b_1^{mn+k-1} \\ &\quad \cdot \sum_{l=0}^{mn-m^2+k} (-1)^l (t/b_1)^{m^2+l-1} \sum_\nu \binom{\kappa_1}{\nu} \frac{[m+1]_\nu}{[2m]_\nu} \frac{\tilde{C}_\nu(\mathring{I}_{m-1})}{\tilde{C}_\nu(I_m)} . \end{aligned} \tag{20}$$

Therefore, adding (16) with x replaced by t and the integrated result of (20) with respect to b_1 over $1 \geqq b_1 \geqq t$, we have the following theorem.

THEOREM 4. *Let A be distributed as* (12), *then the c.d.f. of $R=\lambda_1(1+\lambda_1)^{-1}-\lambda_m(1+\lambda_m)^{-1}$ where λ_1 and λ_m are the maximum latent root and the minimum latent root of A respectively is given by*

$$\begin{aligned} \Pr(R \leqq t) = \frac{\tilde{\Gamma}_m(m)}{\tilde{\Gamma}_m(n)|\Sigma|^n} \sum_k \sum_\kappa \frac{\tilde{L}_\kappa^n(\Sigma^{-1})}{k!} \Bigg\{ &\frac{\tilde{\Gamma}_m(n)}{\tilde{\Gamma}_m(m+n)} \frac{[n]_\kappa}{[m+n]_\kappa} t^{mn+k} \\ &+ \frac{\tilde{\Gamma}_m(m+2)\Gamma(2m+1)}{\tilde{\Gamma}_m(2m+1)(\Gamma(m))^2\Gamma(m+2)} \sum_{l=0}^{mn-m^2+k} (-1)^l (t^{m^2+l-1} - t^{mn+k}) \end{aligned} \tag{21}$$

$$\cdot (mn-m^2+k-l+1)^{-1} \sum_\nu \binom{\kappa_1}{\nu} \frac{[m+1]_\nu}{[2m]_\nu} \frac{\tilde{C}_\nu(\mathring{I}_{m-1})}{\tilde{C}_\nu(I_m)} \Big\} .$$

5. The distributions of $\operatorname{tr} A$ and $\operatorname{tr} A(I_m+A)^{-1}$

By inverting the Laplace transform of the density function of $\operatorname{tr} A$, that is, inverting $\mathrm{E}(\operatorname{etr}(-tA))$, we have the p.d.f. of $\operatorname{tr} A$ in the following way.

THEOREM 5. *Let A be distributed as* (12), *then the p.d.f. of* $T=\operatorname{tr} A$ *is given by*

$$\frac{1}{\Gamma(mn)|\Sigma|^n} T^{mn-1} \sum_k \frac{T^k}{k!(mn)_k} \sum_\kappa [n]_\kappa \tilde{C}_\kappa(-\Sigma^{-1}) . \tag{22}$$

On the other hand, the density function of T is also obtained by integrating (13) with respect to λ_i's, $i=2, 3, \cdots, m$, over the hyperplane $T=\sum_{i=1}^m \lambda_i$ with fixed T. Therefore comparing the coefficients of $C_\kappa(-\Sigma^{-1})$ between the representations of the density function of $\operatorname{tr} A$ obtained by the two methods mentioned above, we get the following fact, which is the extension of (2.1) in Khatri [9] to complex variate case.

COROLLARY 2.

$$\int_{\mathcal{D}} |\Lambda|^{n-m} \tilde{C}_\kappa(\Lambda) \prod_{i<j} (\lambda_i-\lambda_j)^2 \prod_{i=2}^m d\lambda_i = \frac{\tilde{\Gamma}_m(m)\tilde{\Gamma}_m(n)[n]_\kappa}{\Gamma(mn)(mn)_k \pi^{m(m-1)}} \tilde{C}_\kappa(I_m) T^{mn+k-1} , \tag{23}$$

where

$$\mathcal{D}=\{\lambda_1=T-(\lambda_2+\lambda_3+\cdots+\lambda_m)>\lambda_2>\lambda_3>\cdots>\lambda_m>0\} .$$

With the direct use of (23) in (14), the density function of $\operatorname{tr} A(I_m+A)^{-1}$ is gotten as follows:

THEOREM 6. *Let A be distributed as* (12). *Then the p.d.f. of* $F=\operatorname{tr} A(I_m+A)^{-1}$ *is given by*

$$\frac{1}{\Gamma(mn)|\Sigma|^n} F^{mn-1} \sum_k \frac{F^k}{k!(mn)_k} \sum_\kappa [n]_\kappa \tilde{L}_\kappa^n(\Sigma^{-1}) . \tag{24}$$

Considering that $(\operatorname{tr} A)^k=\sum_\kappa \tilde{C}_\kappa(A)$, we shall multiply (12) by $(\operatorname{tr} A)^k$ and integrate with respect to A over $A=A^*>0$ with the use of (86) of James [6] that

$$\int_{A=A^*>0} \operatorname{etr}(-AB)|A|^{t-m}\tilde{C}_\kappa(AC)dA=\tilde{\Gamma}_m(t)[t]_\kappa|B|^{-t}\tilde{C}_\kappa(B^{-1}C) . \tag{25}$$

Then we get the kth moment of tr A as follows:

COROLLARY 3.

$$\mathrm{E}\,((\mathrm{tr}\,A)^k)=\sum_{\kappa}[n]_{\kappa}\tilde{C}_{\kappa}(\Sigma)\,. \tag{26}$$

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