DECOMPOSITION OF INFINITELY DIVISIBLE CHARACTERISTIC FUNCTIONS WITH ABSOLUTELY CONTINUOUS POISSON SPECTRAL MEASURE

SHIGERU MASE

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Abstract

We shall consider the problem of characterizing infinitely divisible characteristic functions which have only infinitely divisible factors. Infinitely divisible characteristic functions treated in this paper are those which have absolutely continuous Poisson spectral measures and have no Gaussian component in their Lévy canonical representations. It will be shown that Ostrovskii's sufficient condition is also necessary in this case.

1. Introduction

Arithmetic of characteristic functions (abbrev. ch.f.'s) of probability measures concerns possible decompositions $f(t)=g(t)h(t)$ of a ch.f. $f(t)$ into the product of another ch.f.'s $g(t)$ and $h(t)$. $g(t)$ and $h(t)$ are called factors of $f(t)$. Two extremal classes are important in this connection:

(a) The indecomposable ch.f.'s, i.e. those for which one of two factors is always degenerated (i.e., of the form e^{iat}).

(b) The ch.f.'s without indecomposable factors, i.e. those which have no indecomposable factor. This class is usually referred to as the class I_0.

It is well-known that the ch.f.'s of the class I_0 are necessarily infinitely divisible (see, Lukacs [4], Chapter 6). The characterization of the class I_0 has been the central problem of the arithmetic of ch.f.'s. In this paper we shall consider a necessary and sufficient condition for the membership of the class I_0 for a certain class of infinitely divisible ch.f.'s.

Consider the Lévy canonical representation of an infinitely divisible ch.f. $f(t)$ without Gaussian component;

$$(1) \qquad \log f(t)=iat+\int_{-\infty}^{-0} K(t, x)dM(x)+\int_{+0}^{+\infty} K(t, x)dN(x) ,$$

where the kernel function $K(t, x)=e^{itx}-1-itx/(1+x^2)$. We shall be especially interested in those which have absolutely continuous Poisson spectral measures $dM(x)$ and $dN(x)$;

$$\text{(2)} \qquad \log f(t)=iat+\int_{-\infty}^{+\infty} K(t, x)\alpha(x)dx ,$$

where a is a real constant and the spectral density function $\alpha(x)$ is a nonnegative function satisfying conditions

$$\int_{|x|\geq 1} \alpha(x)dx<+\infty \qquad \text{and} \qquad \int_{|x|\leq 1} x^2\alpha(x)dx<+\infty .$$

Cramér [1] was the first who obtained a necessary condition that $f(t)$ of the form (2) belongs to the class I_0. Shimizu [6] then extended Cramér's result. Especially he considered also a case in which spectral measures are not absolutely continuous. Their results are further extended by Cuppens [2] to the following form:

THEOREM (Cuppens). *Let a ch.f. $f(t)$ of the form* (2) *belong to the class I_0. Suppose that $\alpha(x)$ is continuous a.e. on the real axis. Then there is a positive constant c such that either $\alpha(x)=0$ a.e. outside the interval* $[c, 2c]$ *or $\alpha(x)=0$ a.e. outside the interval* $[-2c, -c]$.

On the other hand, Ostrovskii [5] proved the following sufficient condition that a ch.f. $f(t)$ of the form (1) belongs to the class I_0:

THEOREM (Ostrovskii). *Let a ch.f. $f(t)$ be of the form* (1). *Suppose that there is a positive constant c such that either $dN(x)=0$ outside the interval* $[c, 2c]$ *and $dM(x)\equiv 0$ or $dM(x)=0$ outside the interval* $[-2c, -c]$ *and $dN(x)\equiv 0$. Then $f(t)$ belongs to the class I_0.*

The purpose of the present paper is that the Cuppens' necessary condition is valid without any restriction on the spectral density function $\alpha(x)$. Our method will be a refinement of the one used by Cuppens [2].

Therefore, taking note of Ostrovskii's sufficient condition, we shall solve completely the problem of characterizing the class I_0 for the class of infinitely divisible ch.f.'s with absolutely continuous Poisson spectral measures and without Gaussian component.

2. Preliminary lemmas

We define the n-fold vectorial sum $(n)I$ of a set I recurrently by $(n+1)I\equiv(n)I(+)I$, where the symbol $(+)$ means the vectorial summation. If the set I is of the form $(a, b)\cup(c, d)$, then $(n)I=\bigcup_{k=0}^{n}(ka+$

$(n-k)c, kb+(n-k)d)$. This is easily seen by induction.

For an integrable function $g(x)=g_1(x)$, we define the iterated convolutions $g_n(x)$ of $g_1(x)$ with itself by the recurrence formula $g_n(x)\equiv \int_{-\infty}^{+\infty} g_{n-1}(x-y)g_1(y)dy,\ n=2, 3, \cdots$. We shall summarize some properties of $g_n(x)$ in the next lemma. Proof is easy, so it will be omitted.

LEMMA 1. *Suppose that $g_1(x)$ is bounded ($\|g_1\|=B$) and vanishes outside an open set of finite length c. Then, for $n=1, 2, \cdots$,*

(a) *$g_n(x)$ is continuous ($n\geqq 2$),*

(b) *$\|g_n\|\leqq c^{n-1}B^n$,*

(c) *$g_n(x)=0$ outside $(n)I$.*

Further, if we assume that $g_1(x)$ is positive on I, we have

(d) *$g_n(x)>0$ on $(n)I$.*

Remark. $\|g\|$ means the maximal norm $\max_{-\infty<x<+\infty} |g(x)|$.

Next we shall consider the iterated convolutions of the spectral density $\alpha(x)$. The next lemma is basic for the rest of the paper.

LEMMA 2. *Let $\alpha(x)=\alpha_1(x)$ be a nonnegative bounded function vanishing outside an interval (d, c) and $\alpha_n(x)$ its iterated convolutions. Assume that, for any positive number ε, both the sets $\{x\colon c-\varepsilon<x<c,\ \alpha(x)>0\}$ and $\{x\colon d<x<d+\varepsilon,\ \alpha(x)>0\}$ have positive Lebesgue measures. Then, for any fixed positive number h, $\alpha_n(x)$ is positive on the interval $[nd+h, nc-h]$ for all sufficiently large n.*

PROOF. Fix any $h>0$. Since $\alpha_2(x)$ is continuous, there are four numbers d_1, d_2, c_1, c_2 ($2d<d_1<d_2<c_1<c_2<2c,\ h>2c-c_2,\ h>d_1-2d$) such that $\alpha_2(x)>0$ on the set $(c_1, c_2)\cup(d_1, d_2)$. Therefore $\alpha_4(x)$ is positive on $(2c_1, 2c_2)$. On the other hand, there is an increasing sequence $\{x_n\}$ tending to $2c$ such that $2c-x_1<c_2-c_1$ and $\alpha_2(x_n)>0,\ n=1, 2, \cdots$. So $\alpha_4(x)$ is also positive on the set $\bigcup_{n=1}^{\infty} (c_1+x_n, c_2+x_n)$. Calculating distances between adjoining two intervals in this set, we see that $\alpha_4(x)$ is positive on (c_1+x_1, c_2+2c).

Repeating the same reasoning, we can show generally that $\alpha_{2n}(x)$ is positive on the set $\bigcup_{k=1}^{n} (kc_1+(n-k)x_1, kc_2+2(n-k)c)$. Again calculating distances between adjoining two intervals in this set, we can conclude that, for all sufficiently large n, $\alpha_{2n}(x)$ is positive on the interval $(nc_1, c_2+2(n-1)c)$. Hence, of course, $\alpha_{2n}(x)$ is positive on $(nc_1, 2nc-h]$.

Similarly we can show that, for all sufficiently large n, $\alpha_{2n}(x)$ is positive on $[2nd+h, nd_2)$.

On the other hand, $\alpha_{2n}(x)$ is positive on the set $(c_1, c_2)\cup(d_1, d_2)$, so

$\alpha_{2n}(x)$ is positive on the set $\bigcup_{k=1}^{n}(kd_1+(n-k)c_1,\, kd_2+(n-k)c_2)$. Again calculating distances between adjoining two intervals in this set, we can conclude that, for all sufficiently large n, $\alpha_{2n}(x)$ is positive on (nd_1, nc_2).

Adding the preceding results, we have proved the assertion for all sufficiently large even integers. Taking note of the condition about $\alpha(x)$, we can immediately show that the assertion is true for all sufficiently large n.

The last parts of proofs of Theorems 1 and 2 are the same. So we shall summarize it in the next lemma.

LEMMA 3. *Let $f(t)$ be the infinitely divisible ch.f. defined by* (2). *Suppose that there exists a bounded function $\beta(x)=\beta_1(x)$ with the properties;*

(a) *$\beta(x)$ vanishes outside a compact set,*
(b) *$\beta(x)<0$ with positive Lebesgue measure,*
(c) *$\alpha(x)-\beta(x)\geqq 0$ everywhere,*
(d) *$\sum_{n=1}^{\infty}\beta_n(x)/n!\geqq 0$ everywhere.*

Then $f(t)$ has an indecomposable factor.

PROOF. Note that the series in (d) is absolutely convergent by Lemma 1.

Define the functions $\phi(t)$ and $\psi(t)$ by formulas $\log\phi(t)=iat+\int_{-\infty}^{+\infty}K(t,x)\cdot[\alpha(x)-\beta(x)]dx$, $\log\psi(t)=\int_{-\infty}^{+\infty}K(t,x)\beta(x)dx$. Then $f(t)=\phi(t)\psi(t)$ and $\phi(t)$ is obviously an infinitely divisible ch.f. We shall show that $\psi(t)$ is also a ch.f. If so, $\psi(t)$ cannot be infinitely divisible (see, Linnik [3], Chapter 6) and, by a theorem due to Khintchin (see, Lukacs [4], Chapter 6), $\psi(t)$ has an indecomposable factor. Hence $f(t)$ itself has an indecomposable factor.

Set $b=\int_{-\infty}^{+\infty}\beta(x)dx$ and $c=\int_{-\infty}^{+\infty}x(1+x^2)^{-1}\beta(x)dx$. Then

$$\begin{aligned}\psi(t)&=\exp\left\{\int_{-\infty}^{+\infty}K(t,x)\beta(x)dx\right\}\\&=e^{-ict}e^{-b}\left\{1+\sum_{n=1}^{\infty}\left[\int_{-\infty}^{+\infty}e^{itx}\beta(x)dx\right]^n\Big/n!\right\}\\&=e^{-ict}e^{-b}\int_{-\infty}^{+\infty}e^{itx}\left[d\varepsilon(x)+\sum_{n=1}^{\infty}\beta_n(x)/n!dx\right],\end{aligned}$$

where $d\varepsilon(x)$ is the unit distribution at $x=0$. From (d), we see that $\psi(t)$ is a Fourier transform of a positive measure. Noting $\psi(0)=1$, we conclude that $\psi(t)$ is a ch.f.

3. The case when a both-sided Poisson spectrum exists

THEOREM 1. *Let $f(t)$ be the infinitely divisible ch.f. defined by* (2). *Suppose that both the sets $\{x\colon x>0, \alpha(x)>0\}$ and $\{x\colon x<0, \alpha(x)>0\}$ have positive Lebesgue measures. Then $f(t)$ has an indecomposable factor.*

PROOF. There are two numbers c, d $(d<0<c)$ such that, for any $\varepsilon>0$, both the sets $\{x\colon c-\varepsilon<x<c, \alpha(x)>0\}$ and $\{x\colon d<x<d+\varepsilon, \alpha(x)>0\}$ have positive Lebesgue measures. Fix a positive number $h>(c-d)/2$. Define the function $\alpha_1(x)$ by

$$\alpha_1(x)=\begin{cases} 1 & \text{if } x\in(d, d+h)\cup(c-h, c) \text{ and } \alpha(x)>1\,, \\ \alpha(x) & \text{if } x\in(d, d+h)\cup(c-h, c) \text{ and } \alpha(x)\leqq 1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and its iterated convolutions $\alpha_n(x)$. Set the intervals $I_n=[nd+h, nc-h]$, $n=1, 2, \cdots$. By Lemma 2, for sufficiently large n, say from m onwards, $\alpha_n(x)$ is positive on I_n. Note also that, since $d<0$, $I_n\subset I_{n+1}$ for all n.

For a given small number $\varepsilon>0$, define the function $\beta(x)=\beta_1(x)$ by

$$\beta(x)=\begin{cases} \alpha_1(x) & \text{if } x\in(d, d+h]\cup[c-h, c)\,, \\ -\varepsilon & \text{if } x\in(d+h, c-h)\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and its iterated convolutions $\beta_n(x)$. We shall show that $\beta(x)$ satisfies the assumptions of Lemma 3. Only (d) of Lemma 3 should be verified.

First we have, for $n=1, 2, \cdots$,

$$\|\alpha_n(x)-\beta_n(x)\|\leqq\varepsilon n(c-d)^{n-1}\,. \tag{3}$$

This is easily seen by induction from the inequality

$$\begin{aligned}|\alpha_n(x)-\beta_n(x)| &\leqq \left|\int_d^c \beta_{n-1}(x-y)[\beta_1(y)-\alpha_1(y)]dy\right| \\ &\quad+\left|\int_d^c [\beta_{n-1}(x-y)-\alpha_{n-1}(x-y)]\alpha_1(y)dy\right| \\ &\leqq \varepsilon(c-d)^{n-1}+(c-d)\|\alpha_{n-1}-\beta_{n-1}\|\,.\end{aligned}$$

We have also, for $n=1, 2, \cdots$,

$$\alpha_n(x)=\beta_n(x) \qquad \text{outside } I_n\,. \tag{4}$$

This can be proved by induction. It is true for $n=1$. Suppose that it is also true for $1, 2, \cdots, n$. Let $x<(n+1)d+h$. Then

$$\beta_{n+1}(x)=\int_d^{d+h}\beta_n(x-y)\beta_1(y)dy+\int_{d+h}^{c}\beta_n(x-y)\beta_1(y)dy\ .$$

In the first term of the right-hand side of this equality, $x-y<nd+h$. Hence, by induction hypothesis, $\alpha_n(x-y)=\beta_n(x-y)$ and $\alpha_1(y)=\beta_1(y)$. In the second term, $x-y<nd$, so $\alpha_n(x-y)=\beta_n(x-y)=0$. Therefore $\beta_{n+1}(x)=\alpha_{n+1}(x)$. The case $x>(n+1)c-h$ can be treated similarly.

Set the constant $\xi=\min\{\alpha_n(x)\colon x\in I_n,\ m\leqq n<2m\}$. ξ is positive. For $m\leqq n<2m$, we have

$$\beta_n(x)\geqq\alpha_n(x)-||\alpha_n-\beta_n||\geqq\xi-||\alpha_n-\beta_n|| \qquad \text{if } x\in I_n\ .$$

Using (3), we can conclude that, for sufficiently small $\varepsilon>0$, $\beta_n(x)$ is positive on I_n for $m\leqq n<2m$. Outside I_n, by (4), $\beta_n(x)=\alpha_n(x)\geqq0$. Therefore, if we choose $\varepsilon>0$ sufficiently small, $\beta_n(x)\geqq0$ everywhere for $m\leqq n<2m$. Furthermore $\beta_n(x)\geqq0$ everywhere for $n\geqq m$. For example,

$$\beta_{2m}(x)=\int_{-\infty}^{+\infty}\beta_m(x-y)\beta_m(y)dy\geqq0\ .$$

Let x be a point of I_m, then

$$\sum_{n=1}^{m}\beta_n(x)/n!\geqq\sum_{n=1}^{m}\alpha_n(x)/n!-\sum_{n=1}^{m}||\alpha_n-\beta_n||/n!$$
$$\geqq\xi/m!-\sum_{n=1}^{m}||\alpha_n-\beta_n||/n!\ .$$

From (3), the right-hand side of this inequality can be made positive if we choose $\varepsilon>0$ sufficiently small. Since $I_m\supset I_n$, $\beta_n(x)=\alpha_n(x)\geqq0$ outside I_m for $n=1,2,\cdots,m$. Therefore,

$$\sum_{n=1}^{m}\beta_n(x)/n!\geqq0 \qquad \text{everywhere}\ . \tag{5}$$

From (5) and positivity of $\beta_n(x)$, $n\geqq m$, we have, for sufficiently small $\varepsilon>0$,

$$\sum_{n=1}^{\infty}\beta_n(x)/n!\geqq0 \qquad \text{everywhere}\ .$$

This is just (d) of Lemma 3 and Theorem 1 has proved.

4. The case when a one-sided Poisson spectrum exists

THEOREM 2. *Let $f(t)$ be the infinitely divisible ch.f. defined by* (2) *which is not degenerated. Suppose that $\alpha(x)=0$ a.e. for $x<0$ and that there is no positive number c such that $\alpha(x)=0$ a.e. outside* $[c, 2c]$. *Then $f(t)$ has an indecomposable factor.*

If we suppose that $\alpha(x)=0$ a.e. for $x>0$ and that there is no posi-

tive number c such that $\alpha(x)=0$ a.e. outside $[-2c, -c]$, then the same conclusion holds.

We shall prove the first case only. The second can be proved similarly. We need the next lemma.

LEMMA 4. *Under the assumption of Theorem 2, there is a positive number c such that both the sets $A(c)\equiv\{x:\ 0<x<c, \alpha(x)>0\}$ and $B(c)\equiv\{x:\ 2c<x, \alpha(x)>0\}$ have positive Lebesgue measures.*

PROOF OF LEMMA 4. Assume the contrary. Then, for any positive c, either $|A(c)|=0$ or $|B(c)|=0$, where $|E|$ means the Lebesgue measure of a set E. For sufficiently large $c>0$, $|A(c)|>0$. Let c_0 be the infimum of the values c for which $|A(c)|>0$. Of course, c_0 is positive. For $c>c_0$, $|A(c)|>0$ and $|B(c)|=0$, and, for $c<c_0$, $|A(c)|=0$ and $|B(c)|>0$. Hence $|A(c_0)|=|B(c_0)|=0$. This means that $\alpha(x)=0$ a.e. outside $[c_0, 2c_0]$. This contradicts the assumption.

PROOF OF THEOREM 2. From Lemma 4, there are three positive numbers b, c, d $(c<b<2c<2b<d)$ such that, for every $\varepsilon>0$, both the sets $\{x:\ c<x<c+\varepsilon, \alpha(x)>0\}$ and $\{x:\ d\ \ \varepsilon<x<d, \alpha(x)>0\}$ have positive Lebesgue measures.

Define the function $\alpha_1(x)$ by

$$\alpha_1(x)=\begin{cases} 1 & \text{if } x\in(c, b)\cup(2b, d) \text{ and } \alpha(x)>1 \\ \alpha(x) & \text{if } x\in(c, b)\cup(2b, d) \text{ and } \alpha(x)\leqq 1 \\ 0 & \text{otherwise}, \end{cases}$$

and its iterated convolutions $\alpha_n(x)$.

Fix some positive number $h<\min(b-c, d-2b)$ and set the intervals $I_n\equiv[nc+h, nd-h]$, $n=1, 2, \cdots$. Then, by Lemma 2, for sufficiently large n, say from m onwards, $\alpha_n(x)$ is positive on I_n. In the present case I_n is not contained in I_{n+1}. Therefore we cannot use the argument in the proof of Theorem 1 and must modify it more subtly as the following.

For some given small number $\varepsilon>0$, define the function $\beta(x)=\beta_1(x)$ by

$$\beta(x)=\begin{cases} \alpha_1(x) & \text{if } x\in(c, b)\cup[2b, d) \\ -\varepsilon\alpha_2(x) & \text{if } x\in(2c, 2b) \\ 0 & \text{otherwise}, \end{cases}$$

and its iterated convolutions $\beta_n(x)$. We shall show that $\beta(x)$ satisfies the assumption of Lemma 3. Again we must prove only (d) of Lemma 3.

As (3) and (4) in the proof of Theorem 1, we can show that, for $n=1, 2, \cdots$,

(6) $$\|\alpha_n-\beta_n\| \leqq \varepsilon n(d-c)^n$$

(7) $$\alpha_n(x)=\beta_n(x) \qquad \text{outside } I_n .$$

Set the constant $\xi \equiv \min\{\alpha_n(x)\colon x \in I_n,\ m \leqq n < 2m\}$. According to Lemma 2, ξ is positive. For $m \leqq n < 2m$, we have

$$\beta_n(x) \geqq \alpha_n(x) - \|\alpha_n-\beta_n\| \geqq \xi - \|\alpha_n-\beta_n\| \qquad \text{if } x \in I_n .$$

Using (6), we can conclude that, for sufficiently small $\varepsilon>0$, $\beta_n(x)$ is positive on I_n for $m \leqq n < 2m$. Outside I_n, by (7), $\beta_n(x)=\alpha_n(x) \geqq 0$. Therefore, if we choose sufficiently small $\varepsilon>0$, $\beta_n(x) \geqq 0$ everywhere for $m \leqq n < 2m$. Moreover

(8) $$\beta_n(x) \geqq 0 \qquad \text{everywhere for } n \geqq m .$$

Next, define the functions $\rho_1(x)$ and $\eta_1(x)$ by

$$\rho_1(x)=\begin{cases} \alpha_1(x) & \text{if } x \in (c, b) \\ 0 & \text{otherwise ,} \end{cases}$$

$$\eta_1(x)=\begin{cases} \alpha_1(x) & \text{if } x \in [2b, d) \\ 0 & \text{otherwise ,} \end{cases}$$

and their iterated convolutions $\rho_n(x)$, $\eta_n(x)$. It is easily seen that $\rho_2(x)$ is the restriction of $\alpha_2(x)$ to the interval $(2c, 2b)$, so $\beta_1(x)=\rho_1(x)+\eta_1(x)-\varepsilon\rho_2(x)$. Making n-fold convolution, we get

$$\begin{aligned} \beta_n(x) &= \sum_{\substack{i+j+k=n \\ i,j,k\geqq 0}} N_n(i, j, k)(-1)^k \varepsilon^k \rho_{i+2k} * \eta_j(x) \\ &= \sum_{\substack{i+j+2k=n \\ i,j,k\geqq 0}} N_n(i, j, 2k)\varepsilon^{2k} \rho_{i+4k} * \eta_j(x) \\ &\quad - \sum_{\substack{i+j+2k+1=n \\ i,j,k\geqq 0}} N_n(i, j, 2k+1)\varepsilon^{2k+1} \rho_{i+4k+2} * \eta_j(x) \\ &\equiv \beta_n^+(x) - \beta_n^-(x) , \end{aligned}$$

where the symbol $*$ means the operation of convolution and the numbers $N_n(i, j, k) \equiv n!/(i!j!k!)$. Calculating further,

$$\begin{aligned} &\beta_{n+1}^+(x)/2(n+1) - \beta_n^-(x) \\ &\quad \geqq \sum_{\substack{i+j+2k=n \\ i,j,k\geqq 0}} \frac{1}{2} N_n(i+1, j, 2k)\varepsilon^{2k} \rho_{i+4k+1} * \eta_j(x) \\ &\quad\quad - \sum_{\substack{i+j+2k+1=n \\ i,j,k\geqq 0}} N_n(i, j, 2k+1)\varepsilon^{2k+1} \rho_{i+4k+2} * \eta_j(x) \end{aligned}$$

$$\geqq \sum_{\substack{i+j+2k+1=n \\ i,j,k\geqq 0}} \varepsilon^{2k}\left[\frac{1}{2}N_n(i+2,\, j,\, 2k)-\varepsilon N_n(i,\, j,\, 2k+1)\right]\rho_{i+4k+2} * \eta_j(x) .$$

Therefore, if we choose sufficiently small $\varepsilon>0$, we have, for $1\leqq n<m$,

$$\beta^+_{n+1}(x)/2(n+1)-\beta^-_n(x)\geqq 0 \qquad \text{everywhere} .$$

For such $\varepsilon>0$, if $x\in I_m$,

$$\begin{aligned}
&\sum_{n=1}^{m}\beta_n(x)/n! \\
&\quad \geqq \beta^+_1(x)+\sum_{n=1}^{m-1}\{2^{-1}[(n+1)!]^{-1}\beta^+_{n+1}(x)-(n!)^{-1}\beta^-_n(x)\} \\
&\qquad +2^{-1}(m!)^{-1}\beta_m(x)-2^{-1}(m!)^{-1}\beta^-_m(x) \\
&\quad \geqq 2^{-1}(m!)^{-1}[\beta_m(x)-\beta^-_m(x)] \\
&\quad \geqq 2^{-1}(m!)^{-1}[\alpha_m(x)-||\alpha_m-\beta_m||-||\beta^-_m||] \\
&\quad \geqq 2^{-1}(m!)^{-1}[\xi-||\alpha_m-\beta_m||-||\beta^-_m||] .
\end{aligned}$$

According to the definition of $\beta^-_m(x)$, it is easy to see that $||\beta^-_m||$ can be made arbitrarily small with ε and that $\beta^-_m(x)=0$ outside I_m. Hence, for sufficiently small $\varepsilon>0$,

$$\sum_{n=1}^{m}\beta_n(x)/n!\geqq 0 \qquad \text{for } x\in I_m .$$

On the other hand, if $x\notin I_m$, then

$$\sum_{n=1}^{m}\beta_n(x)/n!\geqq 2^{-1}(m!)^{-1}[\beta_m(x)-\beta^-_m(x)]=2^{-1}(m!)^{-1}\alpha_m(x)\geqq 0 .$$

Consequently

$$\sum_{n=1}^{m}\beta_n(x)/n!\geqq 0 \qquad \text{everywhere} . \tag{9}$$

Finally, adding (8) and (9), we have

$$\sum_{n=1}^{\infty}\beta_n(x)/n!\geqq 0 \qquad \text{everywhere} .$$

This completes the proof of Theorem 2.

5. Main theorems

Combining Theorem 1 and Theorem 2, we immediately get the following main theorems.

THEOREM 3. *Let $f(t)$ be the infinitely divisible ch.f. defined by* (2). *A necessary and sufficient condition that $f(t)$ has no indecomposable factor* *is that there is a positive constant c such that either* $\alpha(x)$ *vanishes a.e. outside the interval* $[c, 2c]$ *or* $\alpha(x)$ *vanishes a.e. outside the interval* $[-2c, -c]$.

THEOREM 4. *Let* $f(t)$ *be the infinitely divisible ch.f. defined by* (1) *and* $\alpha(x)$ *be the density function of the absolutely continuous part of spectral measures* $dM(x)$ *and* $dN(x)$. *Suppose that there is no positive constant c such that either* $\alpha(x)=0$ *a.e. outside* $[c, 2c]$ *or* $\alpha(x)=0$ *outside* $[-2c, -c]$. *Then* $f(t)$ *has an indecomposable factor.*

Remark. The problem of characterizing the class I_0 has not been solved completely at present. If an infinitely divisible ch.f. has no Gaussian component, Theorem 4 tells us that the absolutely continuous part of its Poisson spectral measure in the Lévy canonical representation must be supported by an interval of special form. But, as to the singular and discrete part of the Poisson spectral measure, it is known that Ostrovskii's sufficient condition is not always necessary. It will be interesting to study what restrictions on the singular and the discrete component exist if a ch.f. of the class I_0 has a nonvanishing absolutely continuous Poisson spectral measure in its Lévy canonical representation.

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