SOME PROBLEMS OF UNBIASED SEQUENTIAL BINOMIAL ESTIMATION

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(Received August 29, 1972)

1. Introduction

Much yet remains as undone regarding some problems of unbiased sequential binomial estimation, though there is no denying the fact that the technique of sequential analysis is receiving increasing attention of the statistical workers in different parts of the world. Restricting our attention to the problem of unbiased sequential binomial estimation of $1/p$, this paper aims at putting together the few results derived so far, pointing out the inexactness of some of these results, adding some further results in this direction and emphasizing the need for more thinking on such problems.

Basically a sequential estimation problem is a difficult one, because with each in-coming observation the sample space is changed and the distribution problem becomes complicated. Still the problem of sequential estimation of the binomial proportion p has been taken up by various authors and considerable work has been done up till now [1], [2], [3], [5], [6], [7], [8].

So far as the problem of unbiased estimation (ue) of $1/p$ is concerned, the relevant work of DeGroot [2] puts forward a remarkable solution. He introduces the notion of *efficient* sampling plans. A (sequential) sampling plan S together with an unbiased estimator (ue) f (of a function $g(p)$ of the binomial proportion p) are said to be jointly efficient if the sampling variance of f attains its relevant lower bound (which certainly depends on $g(p)$ and the particular plan S). Concerning ue of $1/p$, he has shown that the only efficient binomial sequential sampling plans are the 'inverse binomial' sampling plans

$$[S(c):\ B(c)=\{\gamma:\ Y(\gamma)=c\};\ c=1, 2, \cdots]^{*}$$

and the corresponding variance-bounds are

* To be precise, the boundary points of $S(c)$ are the sequence of points $(0, c), (1, c), (2, c), \cdots$ i.e., the points γ on the line '$Y=c$'.

$$\mathrm{Var}\,(f\,|\,p, S(c)) = q/cp^2\,; \qquad c=1, 2, \cdots .$$

In general, however, the problem of existence of an ue of $1/p$ for any arbitrary sequential binomial sampling plan has not yet been tackled satisfactorily. The investigation relating to the characterization of the class of sampling plans providing ue's of $1/p$ has got some independent interest. Recently, Gupta [4] furnished some interesting results on ue of $1/p$. He laid down a set of necessary conditions on a sampling plan for providing an ue of $1/p$ and also a very powerful sufficient condition on a sampling plan which enables one to practically construct an ue of $1/p$ (and on the analogous line of arguements, also of $1/q$, $1/pq$ and p). Unfortunately, however, most of the stated necessary conditions turn out to be invalid (vide Section 3, Subsection 3.1). As a matter of fact, it will be seen that of the stated ones of Gupta, only the unboundedness of the plan will hold as a necessary condition.

Our main concern in this paper, apart from pointing out the invalidity of most of the stated necessary conditions of Gupta, is

i) to provide one necessary condition for ue of $1/p$

ii) to formulate a sufficient condition (equivalent to that of Gupta) for ue of $1/p$

iii) to write down a set of rules to examine whether this sufficient condition is satisfied by a plan

iv) to give support to our own conjecture that Gupta's sufficient condition is necessary as well.

Analogous necessary and sufficient conditions for ue's of $1/q$ and $1/pq$ have also been derived but these are not reported here.

2. Nomenclature

We shall use the notation and terminology of [4]. For the sake of completeness, we discuss them briefly. Other definitions and notations will be incorporated in proper places.

The word *point* will refer to points in the XY-plane with positive integral co-ordinates. A *region* R is a set of points including the point $(0, 0)$. The point (x', y') is immediately beyond the point (x, y) if either $x'=x+1$, $y'=y$ or $x'=x$, $y'=y+1$. A *path* in R from the point a_0 to the point a_n is a sequence of points $a_0, a_1, \cdots, a_n$ such that a_i $(i>0)$ is immediately beyond a_{i-1}. We can order the points in the XY-plane using the following convention—(x', y') occurs after (x, y) if either $x<x'$ or $x=x'$, $y<y'$. A *boundary point* (an element on the boundary B of R) is a point not in R which is the last point of a path from the origin. *Accessible points* are the points in R which can be reached by paths from the origin. The *index* of a *point* is the sum of its co-ordinates. A *finite* (or *bounded*) region is a region for which the indices of the

accessible points are less than some number n. The probability of an accessible point or a boundary point $(x, y)=r$ included in a path from the origin is $\mathrm{P}(r)=K(r)p^y q^x$, where $K(r)$ is the number of paths from the origin to the point r. A region for which $\sum_{r \in B} \mathrm{P}(r)=1$ will be called a *closed region*. The corresponding sampling plan is a closed plan. For any accessible point $t=(x, y)$ of a plan, $t(\alpha)$ will denote the total number of ways of passing from t to the boundary point α of the plan only through *its accessible points*. Throughout the paper, unless otherwise stated, every plan with origin at $(0, 0)$ will be assumed to be closed. An *estimator* f is defined only at the boundary points $r \in B$; it is defined to be an unbiased estimator (ue) of $g(p)$ if and only if $\sum_{r \in B} f(r) \cdot k(r)p^y q^x = g(p)$ identically in p, $0<p<1$. Also we should restrict our attention to ue's f of $1/p$ which are *proper* in the sense that $f(r) \geqq 1 \ \forall r \in B$. An estimator which is not proper is said to be *improper*.

3. Miscellaneous results

3.1. *Invalidity of Gupta's necessary conditions*

According to Gupta [4], a set of necessary conditions for unbiased estimation of $1/p$ is:

i) The sampling plan must be unbounded.

ii) $(0, c)=r_1$ must be on the boundary for some $0<c<\infty$, and the estimator takes the value 1 at r_1.

iii) Next boundary point r_2 must have its x-co-ordinate less than (index of r_1+2).

iv) In general if we have the first s boundary points then the next boundary point must have its x-co-ordinate less than (index of r_s $+2$).

It will now be demonstrated that condition ii) is unnecessary and hence that the conditions iii) and iv) do not bear any real essence. As a matter of fact, condition iii) will also be directly shown to be unnecessary even if condition ii) is assumed to be satisfied.

First consider the (closed) unbounded binomial sampling plan P1. It has no boundary point on the line $X=0$. In the following table are displayed the boundary points r and also $k(r)$ and $f(r)$ values from which it is not difficult to verify that $\mathrm{E}(f \mid p, \mathrm{P1})=1/p$.

r	(1, 1)	(2, 1)	(3, 1)	$\cdots$	$(n+1, 1)$	$\cdots$	(1, 2)	(1, 3)	$\cdots$	$(1, n+1)$	$\cdots$
$k(r)$	2	1	1	$\cdots$	1	$\cdots$	1	1	$\cdots$	1	$\cdots$
$f(r)$	$\frac{3}{2}$	3	4	$\cdots$	$n+2$	$\cdots$	1	1	$\cdots$	1	$\cdots$

Next consider the sampling plan P1′. The point $(0, 1)=r_1$ is on the boundary. According to condition iii) above, the next boundary point r_2 must have its x-co-ordinate <3 which is not the case with P1′. Still it provides an ue f of $1/p$ as the following table of values of f will demonstrate.

r	(0, 1)	(3, 1)	(3, 2)	$\cdots$	$(3, n+1)$	$\cdots$	(4, 1)	(5, 1)	$\cdots$	$(n+1, 1)$	$\cdots$
$k(r)$	1	3	3	$\cdots$	$n+2$	$\cdots$	1	1	$\cdots$	1	$\cdots$
$f(r)$	1	3	$\frac{7}{3}$	$\cdots$	$\frac{2n+5}{n+2}$	$\cdots$	5	6	$\cdots$	$n+2$	$\cdots$

Remark. It should be clear from P1 that in some cases the conditions iii) and iv) do not bear any meaning. P1′, of course, demonstrates that such conditions are really unnecessary.

In view of what have been presented here, only the unboundedness of the plan remains as a necessary condition and that, too, is more or less obvious.

3.2. *A simpler sufficient condition for ue of* $1/p$

In this sub-section we intend to present a simpler sufficient condition for ue of $1/p$. For completeness, we first state the sufficient condition of Gupta [4].

A sufficient condition for estimability of $1/p$ (*Gupta* [4])

If the closed plan with boundary $B=\{r_i=(x_i, y_i)\}$ be such that by changing its boundary points from r_i to $r_i'=(x_i, y_i+1)$ we get a closed plan $B'=\{r_i'=(x_i, y_i+1)\}$, then $1/p$ is estimable for the plan B.

The proposed sufficient condition is the following:

LEMMA 3.2. *If no point on the line '$Y=1$' is inaccessible,* $1/p$ *is unbiasedly estimable.*

PROOF. We make use of Wolfowitz's result (Theorem 2′ of [7]) here. With $t(\alpha)$ as defined in Section 2, we have, for every $x_0 \geqq 0$ and under the given condition, $\sum_{\alpha \in B} t(\alpha)p^y q^x = pq^{x_0}$ since $t=(x_0, 1)$ is an accessible or boundary point of the plan. Note that Theorem 2′ of [7] holds trivially for t any boundary point as well.

We now form the estimator $f(\alpha)=\sum_{x_0=0}^{\infty} (x_0+1)t(\alpha)/k(\alpha)$, $\forall \alpha \in B$. Then $\mathrm{E}[f(\alpha)]=\sum_{x_0=0}^{\infty} (x_0+1) \sum_{\alpha \in B} t(\alpha)p^y q^x = \sum_{0}^{\infty} (x_0+1)pq^{x_0}=1/p$ and hence $1/p$ becomes unbiasedly estimable.

Note 1. We first note that no point of '$Y=1$' is inaccessible only in the following situations: (i) both '$Y=0$' and '$Y=1$' do not possess boundary points; (ii) whenever '$Y=0$' has a boundary point $(x_{00}, 0)$, the point $(x_{00}-1, 1)$ is an accessible point. From this, it is not difficult to verify that whenever the proposed sufficient condition holds, the sufficient condition of Gupta also holds necessarily. But the converse is not true. Hence the stated condition has less scope but its simplicity is readily understandable.

Note 2. Whenever the stated sufficient condition holds, the ue $f(\alpha)$ is readily constructed. Again, Gupta's sufficient condition being also satisfied, another ue, namely, $f^*(\alpha)=k'(\alpha)/k(\alpha)$ may be constructed by Gupta's method [4]. How do f and f^* compare? The concept of sufficiency in the binomial sequential case, as developed by Blackwell [1], enables us to conclude that both f and f^* are based on the sufficient statistic. Both of them then turn out to be admissible. Of course, when the plan is complete [5], they are identical. Even if the plan is not complete, there are situations when f and f^* turn out to be identical. It can be easily verified that *an n.s. condition for* $f=f^*$ *is that* '$Y=0$' *has no boundary point*. When this is not so, f assumes the value zero at the only boundary point of '$Y=0$' and hence becomes *improper*. Therefore, in such a situation the *proper* estimator based on Gupta's method should always be used.

4. N.s. conditions for ue of $1/p$

4.1. *A necessary condition for ue of* $1/p$

Consider a closed sampling plan with the set B of boundary points $\alpha=(x, y)$.

DEFINITION. A sampling plan is defined to be bounded in the X-direction if for some finite x_0 and for all boundary points $\alpha=(x, y)$, we have the inequality $x \leqq x_0$.

In the sequel, by an X-bounded sampling plan we shall understand its boundedness in the X-direction. A sampling plan which is not X-bounded will be defined to be X-unbounded. Thus for an X-unbounded sampling plan, for every positive integral value of x_0, we have at least one and hence an infinite number of boundary points of the plan to the right of the line '$X=X_0$'. The Y-bounded sampling plans may also be analogously defined.

THEOREM 4.1.1. *A necessary condition for an unbounded sampling plan to provide an ue of* $1/p$ *is that the plan is* X-*unbounded.*

PROOF. We take the unbounded plan to be X-bounded and disprove the existence of any ue of $1/p$.

Since the plan is X-bounded, there is an x_0 such that no boundary point α lies to the right of '$X=x_0$'. It is easy to verify that this necessarily means that $(x, 0)$ is a boundary point for some x, $1\leqq x\leqq x_0$. Suppose $x=x_{00}\leqq x_0$. Since $k(\alpha)=1$ for $\alpha=(x_{00}, 0)$, we have

$$1=q^{x_{00}}+\sum_{\alpha} k(\alpha)p^y q^x\,, \qquad \text{identically in } p,\ 0<p<1 \tag{4.1.1}$$

where $\sum_{\alpha}$ is over all $\alpha\in B$, $\alpha=(x, y)$, $x\leqq x_0$, $y>0$. If $f(\alpha)$ is the value of an ue of $1/p$ at α, then

$$1/p=f(x_{00}, 0)q^{x_{00}}+\sum_{\alpha} k(\alpha)f(\alpha)p^y q^x\,, \qquad \text{identically in } p,\ 0<p<1\,. \tag{4.1.2}$$

From (4.1.2), we have $1/p\geqq f(x_{00}, 0)q^{x_{00}}\ \forall\, p$, $0<p<1$ implying, for $p=q=1/2$,

$$f(x_{00}, 0)\leqq 2^{x_{00}+1}\,. \tag{4.1.3}$$

Again we may write $1/p=f(x_{00}, 0)q^{x_{00}}+T$ where

$$T=T_1+T_2\,, \qquad T_i=\sum_{\alpha}{}_i\, k(\alpha)f(\alpha)p^y q^x\,, \qquad i=1, 2 \tag{4.1.4}$$

where $\sum_{\alpha 1}$ is over all $\alpha\in\sum_{\alpha}$, $\alpha=(x, y)$, $x+y\leqq x_0$ and $\sum_{\alpha 2}$ is over all $\alpha\in\sum_{\alpha}$, $\alpha=(x, y)$, $x+y>x_0$. From (4.1.2) and (4.1.4), using the same argument as in (4.1.3), we have

$$\begin{aligned} f(\alpha)k(\alpha)&\leqq 2^{x_0+1} && \forall\,\alpha\in\textstyle\sum_{\alpha 1} \\ &\leqq 2^{n+1} && \forall\,\alpha=(x, y)\in\textstyle\sum_{\alpha 2},\ x+y=n\ (>x_0) \\ &&& \text{i.e., } \forall\,\alpha\in\textstyle\sum_{\alpha 2}^{(n)}\ \text{(say)}\,. \end{aligned} \tag{4.1.5}$$

In (4.1.2), let us now take limit as $p\to 0$. We have L.H.S. $\to\infty$ and $\lim f(x_{00}, 0)q^{x_{00}}=f(x_{00}, 0)\leqq 2^{x_{00}+1}$. Now $\lim T_1\leqq 2^{x_0+1}[\lim \sum_{\alpha 1} p^y q^x]=0$. Again,

$$\lim T_2=\lim \sum_{n>x_0} \Big[\sum_{\alpha 2}^{(n)} f(\alpha)k(\alpha)p^y q^x\Big]\,. \tag{4.1.6}$$

Now for every $n>x_0$ and any boundary point $\alpha\in\sum_{\alpha 2}^{(n)}$, we have

$$\begin{aligned} \lim f(\alpha)k(\alpha)p^y q^x &\leqq 2^{n+1}\lim p^y q^x\leqq 2^{n+1}\lim p^{n-x_0}q^x \\ &=[\lim\,(2p)^{n-x_0}q^x]2^{x_0+1} \\ &=0 \qquad \text{uniformly in } n\,. \end{aligned}$$

Then the bracketted quantity [ ] in (4.1.6) tends to zero as $p \rightarrow 0$ uniformly in n. Therefore, in (4.1.2), R.H.S.$\rightarrow$a quantity $\leqq 2^{x_{00}+1}$ and L.H.S.$\rightarrow \infty$. This is a contradiction to the validity of (4.1.2) as an identity in p, $0<p<1$. This essentially implies non-existence of an ue of $1/p$. The theorem is thus proved.

Note. The arguments used above in establishing the uselessness of any X-bounded sampling plan to provide any ue of $1/p$ may also be used step by step without any change to establish the uselessness of such plans to provide ue's of any negative powers of p.

4.2. *Sufficient conditions for ue of* $1/p$

In this subsection we would like to formulate a sufficient condition (Theorem 4.2.2) for ue of $1/p$ from an X-unbounded sampling plan. This sufficient condition will also be shown to be equivalent to that of Gupta (Theorem 4.2.3). Towards this, we first take up a few preliminary results.

Consider an unbounded but not necessarily closed sampling plan with B as the set of boundary points α. Let $t=(x, y)$ be an accessible point of the plan. Define $t(\alpha)$ as in Section 2. Then we have the following result.

LEMMA 4.2.1. *A n.s. condition for the plan to be closed is that for every accessible point* $t=(x_0, 1)$ *of the plan on the line* '$Y=1$'

$$\sum_{\alpha \in B} t(\alpha) p^y q^x = p q^{x_0} \tag{4.2.1}$$

holds identically in p, $0<p<1$.

PROOF. Necessity of this result is due to Wolfowitz [7].

To establish sufficiency of this result, one may distinguish the following cases:

i) '$Y=0$' has *all* accessible points;

ii) '$Y=0$' has *one* (*and essentially one*) boundary point.

Now, first note that Theorem 2' of Wolfowitz [7] holds trivially for any boundary point of any plan. This means that in our case (4.2.1) holds for any boundary point $t=(x_0, 1)$ of a plan (not necessarily closed).

Next note that we may write, for any plan,

$$\begin{aligned} k(\alpha) &= k(0, \alpha)+k(1, \alpha)+\cdots+k(j, \alpha)+1 \qquad \textit{or} \\ &= k(0, \alpha)+k(1, \alpha)+\cdots+k(j, \alpha)+k(j+1, \alpha)+\cdots \end{aligned}$$

according as, for some j ($\geqq 0$), $(j+1, 0)$ is *or* is not a boundary point of the plan. Here $k(\alpha)$=total number of paths of reaching a boundary point α from (0, 0) and $k(j', \alpha)=t(\alpha)$ with $t=(j', 1)$, an accessible point;

$=1$ with $(j', 1)$ a boundary point. Also it may be noted that in either form of representation of $k(\alpha)$ as above, every component $k(j', \alpha)$ corresponds to a point $(j', 1)$ which is either accessible or boundary. Case (i) corresponds to the latter form of $k(\alpha)$ while case (ii) corresponds to the former form. In any case, use of the stated conditions in the lemma and of the above considerations immediately convinces one regarding the closure of the plan. The condition is therefore sufficient. This proves the lemma.

Note. Actually, for closedness, what we demand of the plan is the validity of (4.2.1) *only* for the accessible points of the set $\{(0, 1), (1, 1), \cdots, (j, 1)\}$ whenever $(j+1, 0)$ is a boundary point. Thus the stated sufficient condition may be strengthened under case (ii).

We will now go deep into the problem of providing an ue of $1/p$. Let us now examine the possibility of having an ue of $1/p$ based on a (closed) plan for which the stated sufficient condition of Lemma 3.2 does not hold. This then means that for such a plan we are having one and hence an infinite number of inaccessible points on the line '$Y=1$'. Let the first inaccessible point be $(x_0, 1)$ for some $x_0>1$. Certainly then every $(x, 1)$ is inaccessible, $x>x_0$.

In general terms, let us now introduce the notion of *transformed* plans. For a given plan P with the set B of boundary points α, let (x', y') be any point (may be inaccessible as well) in the XY-plane. Then the transformed plan $P^T(x', y')$, corresponding to (x', y'), with the set $B^T(x', y')$ of boundary points $\alpha^T(x', y')$ is defined as follows ((x', y') being understood, we shall simply write P^T, B^T and α^T):

i) Every α^T belonging to B^T also belongs to B necessarily.

ii) The points $\{(x, y);\ x \geqq x',\ y \geqq y'\}$ constitute the *totality* of all points (accessible, boundary and inaccessible) of P^T.

iii) Every boundary point $\alpha \in B$ is either a boundary point $\alpha^T \in B^T$ or an inaccessible point in P^T.

iv) The rules for obtaining the boundary points $\alpha^T \in B^T$ are as follows:

(a) If $(x', y') \in B$ i.e., if $\alpha=(x', y')$, then $\alpha^T=\alpha$ is the only boundary point of B^T.

(b) If $(x', y') \notin B$, then $\inf_{x>x'} \alpha=(x, y') \in B$ is the only point on '$Y=y'$' that belongs to B^T.

(c) If $(x', y') \notin B$, then $\inf_{y>y'} \alpha=(x', y) \in B$ is the only point on '$X=x'$' that belongs to B^T.

(d) If $(x', y') \notin B$, any boundary point $\alpha \in B$ also belongs to B^T if and only if it can be reached by a path from (x', y'). Otherwise, it is treated as an inaccessible point of P^T.

It may be noted that whenever the point (x', y') is an accessible point of P, we have, with $t=(x', y')$ and identically in p, $0<p<1$, $p^{y'}q^{x'} = \sum_{\alpha \in B^T(x',y')} t(\alpha)p^y q^x$ i.e., $1 = \sum_{\alpha \in B^T(x',y')} t(\alpha)p^{y-y'}q^{x-x'}$. Even when $t=(x', y')$ is an inaccessible point of P, we may define, for every $\alpha \in B^T(x', y')$, $t(\alpha)$=total number of ways of passing from t to α *only through the accessible points of* $P^T(x', y')$. The transformed plan $P^T(x', y')$ will be defined to be closed *only when* the above identity holds—no matter whether (x', y') is accessible or not.

In the setting of the previous paragraph, we then have the following result.

THEOREM 4.2.1. *A sufficient condition for an ue of $1/p$ to exist is that, for any $x \geqq x_0$, the transformed plan $P^T(x, 1)$ is closed.*

PROOF. The conditions of this theorem enable us to write

$$1 = \sum_{\alpha \in B^T(x',1)} t_{x'}(\alpha)p^{y-1}q^{x-x'} \qquad \text{for all } x' \geqq x_0 . \tag{4.2.2}$$

Here $t_{x'}(\alpha)$ is the total number of paths in $P^T(x', 1)$ from the point $(x', 1)$ to a boundary point $\alpha \in B^T(x', 1)$. Also note that by Lemma 4.2.1 (i.e., by Wolfowitz's result), (4.2.2) holds for all $x' < x_0$ as well.

Now form the estimator $f(\alpha) = \sum_{j=0}^{\infty} t_j(\alpha)(j+1)/k(\alpha)$. We have

$$\begin{aligned}
\mathrm{E}\,[f(\alpha)] &= \sum_{\alpha \in B} f(\alpha)k(\alpha)p^y q^x \\
&= \sum_{j=0}^{\infty} (j+1)[\sum_{\alpha \in B} t_j(\alpha)p^y q^x] \\
&= \sum_{j=0}^{x_0-1} (j+1)pq^j + \sum_{j=x_0}^{\infty} (j+1)pq^j[\sum_{\alpha \in B^T(j,1)} t_j(\alpha)p^{y-1}q^{x-j}] \\
&= p\sum_{j=0}^{\infty} (j+1)q^j = \frac{1}{p} .
\end{aligned}$$

Note. Implicit in this process of construction is the implication of the necessary condition (vide Theorem 4.1.1) for existence of an ue of $1/p$. Since necessarily the plan is X-unbounded, this in our present setting must mean that whatever $x \geqq x_0$, every $P^T(x, 1)$ is non-trivially defined and hence the requirement of (4.2.2) being satisfied has a rational basis.

The following result has also a long-term consequence.

LEMMA 4.2.2. *In the same setting as above, whenever* (4.2.2) *is satisfied for $x' = x_0$, it is satisfied for all $x' > x_0$.*

PROOF. Consider the given identity

$$1 = \sum_{\alpha \in B^T(x_0, 1)} t_{x_0}(\alpha) p^{y-1} q^{x-x_0} .$$

Now note that, for this plan $P^T(x_0, 1)$, all points $(x, 1)$, $x \geqq x_0$, are accessible points. Therefore, applying the result of Wolfowitz, we have, for any $x'' \geqq 0$ and identically in p, $0<p<1$,

$$q^{x''} = \sum_{\alpha \in B^T(x_0, 1)} t_{x_0+x''}(\alpha) p^{y-1} q^{x-x_0}$$

$$\text{i.e.,} \quad 1 = \sum_{\alpha \in B^T(x_0+x'', 1)} t_{x_0+x''}(\alpha) p^{y-1} q^{x-(x_0+x'')} .$$

This establishes the lemma.

Combining the contents of Theorem 4.2.1 with that of Lemma 4.2.2, we may finally conclude as follows:

THEOREM 4.2.2. (A sufficient condition equivalent (equivalence is proved below) to that of Gupta). *In the same setting as above, a sufficient condition for an ue of* $1/p$ *to exist is that the plan* $P^T(x_0, 1)$ *is closed.*

Our next result demonstrates equivalence of this sufficient condition to that of Gupta. Before observing this equivalence, let us first look at the plan. For the given plan P, the point $(x_0, 1)$ is the first inaccessible point on the line '$Y=1$'. This necessarily means that $(x_{00}, 0)$ is a boundary point on the line '$Y=0$' for some x_{00}, $1 \leqq x_{00} \leqq x_0$. Define the plan EP with the set of boundary points EB as an 'extension' of the given plan P in the sense that $\alpha^* = (x, y) \in EB$ if and only if $\alpha = (x, y-1) \in B$. Symbolically, we may write, $EB = \{\alpha_i^* = (x_i, y_i+1) / \alpha_i = (x_i, y_i) \in B\}$. Any $P^T(x, y)$ and the corresponding $EP^T(x, y+1)$ are related in the same way as $P^T(0, 0)$ $(\equiv P)$ and $EP^T(0, 1)$ and they have identical behaviour regarding closure. Gupta's sufficient condition then states that EB is closed. Sufficient condition stated here is that $P^T(x_0, 1)$ is closed. In the following theorem, we establish their equivalence.

THEOREM 4.2.3. *Whenever* $P^T(x_0, 1)$ *is closed,* EP *is closed and conversely.*

PROOF. First suppose $P^T(x_0, 1)$ is closed. By Lemma 4.2.2, we then conclude that $P^T(x', 1)$ is also closed, whatever $x' \geqq x_0$. Again, since for $x < x_0$, $(x, 1)$ is not inaccessible, we have the result that $P^T(x, 1)$ is closed for all $x \geqq x_{00}+1$. This, is view of Lemma 4.2.1, means that $P^T(x_{00}+1, 0)$ is closed and hence that $EP^T(x_{00}+1, 1)$ is closed. Again $P^T(x, 0)$ being closed for every x, $0 \leqq x \leqq x_{00}$ (with $P^T(0, 0) \equiv P$, by convention), we must have the corresponding $EP^T(x, 1)$ closed. It now suffices to note that EP has no boundary point on the line '$Y=0$' and only one boundary point $(x_{00}, 1)$ on the line '$Y=1$' for, an use of

Lemma 4.2.1 immediately describes the closure of EP.

Conversely, suppose that EP is closed. This necessarily means that $EP^T(x_{00}+1, 1)$ is closed which, in its turn, implies the closure of $P^T(x_{00}+1, 0)$. Obviously one is now convinced of having $P^T(x_0, 1)$ closed whenever $x_0 \geqq x_{00}+1$. The case of $x_0 = x_{00}$ is less obvious and it requires a critical close argument. This we develop below.

In the above setting with $x_0 = x_{00}$, we are to show that the closure of $P^T(x_0+1, 1)$ implies that of $P^T(x_0, 1)$. We may distinguish the two cases:

Case 1. On '$X=x_0$' there is at least one boundary point of P other than $(x_0, 0)$.

Suppose the first boundary point is (x_0, y_0) for some $y_0 \geqq 2$. Then, by Lemma 4.2.1 and the note following it, $P^T(x_0, 1)$ is closed if and only if for every accessible point (x_0+1, y) of $P^T(x_0, 1)$—belonging to the set of points $\{(x_0+1, y);\ 1 \leqq y \leqq y_0-1\}$—$P^T(x_0+1, y)$ is closed. It is not difficult to verify the validity of these latter conditions, with due considerations to the possibilities of having or not having boundary point(s) in between $(x_0+1, 1)$ and (x_0+1, y_0-1).

Case 2. No point on '$X=x_0$' (other than $(x_0, 0)$) is a boundary point of P.

Here also we may establish the closure of $P^T(x_0, 1)$, taking into account the possibilities of having or not having boundary point(s) on the line '$X=x_0+1$'.

The equivalence is thus established.

Note 1. This (equivalent) sufficient condition, though it may seem to be a bit artificial, is often simpler to check. We state below a set of simple rules for examining the closure or otherwise of $P^T(x_0, 1)$. The proofs are not difficult and hence omitted.

Rules for examining the closure of $P^T(x_0, 1)$

Rule 1. If, for some j $(\geqq 2)$, every point on the line '$Y=j$' of $P^T(x_0, 1)$ is an accessible point of P, then $P^T(x_0, 1)$ is closed.

Rule 2. Suppose on '$Y=2$' $(x_0^{(2)}, 2)$ is a boundary point of P, for some $x_0^{(2)} \geqq x_0$. Then $P^T(x_0, 1)$ is closed if and only if $P^T(x_0^{(2)}+1, 2)$ is closed and we may examine the closure of the latter following these rules.

Rule 3. Suppose every point on '$Y=2$' of $P^T(x_0, 1)$ is an inaccessible point of P. Then $P^T(x_0, 1)$ is closed if and only if $P^T(x_0, 2)$ is closed and we may examine the closure of the latter following these rules.

Rule 4. If the plan P is Y-bounded, $P^T(x_0, 1)$ is closed.

Rule 5. If the plan P is complete, $P^T(x_0, 1)$ is not closed.

By way of illustration we present below a sampling plan just to see how the rules work. Consider the following plan:

r	(3, 0)	(0, 3)	(1, 1)	(3, 1)	(1, 3)	$\{(2j+1, 2k+1);\ j, k \geqq 1\}$
$k(r)$	1	1	2	1	1	$\left\{\binom{j+k-2}{k-1};\ j, k \geqq 1\right\}$

Here $x_0=4$. An immediate application of rule 1 then reveals that $P^T(4, 1)$ is closed. It would be comparatively difficult to verify the closure of EP directly.

Note 2. A serious understanding of, and clear insight into, these rules will help one conclude as follows:

Whenever $P^T(x_0, 1)$ is not closed, on any line '$Y=j$' we may have at most a finite number of boundary points. More specifically, on any line '$Y=j$' it will be possible to obtain a point (j', j) such that no point (x, j), $x \geqq j'$, is a boundary point of P. Of course, as $j \to \infty$, $j' \to \infty$ (since the plan is X-unbounded).

Such a typical situation is reflected in the following plan.

r	(1, 0)	(1, 1)	(2, 2)	(3, 3)	$\cdot\cdot$	(n, n)	$\cdot\cdot$
$k(r)$	1	1	1	2	$\cdot\cdot$	2	$\cdot\cdot$

$\cdot\cdot$	(1, 4)	(1, 5)	$\cdot\cdot$	$(1, n)$	$\cdot\cdot$	(2, 5)	(3, 6)	$\cdot\cdot$	$(n, n+3)$	$\cdot\cdot$
$\cdot\cdot$	3	1	$\cdot\cdot$	1	$\cdot\cdot$	2	2	$\cdot\cdot$	2	$\cdot\cdot$

Here $x_0=2$ and $B^T(2, 1)=\{(2, 2), (3, 3), \cdots, (n, n), \cdots\}$.*

Note 3. These considerations encourage us to hold the opinion that a necessary condition for $1/p$ to be ue is that $P^T(x_0, 1)$ is closed (and hence that Gupta's condition is necessary as well). Indeed we need think more for supplying a rigorous proof of necessity of Gupta's condition for ue of $1/p$. The problem of characterization of the class of unbounded sampling plans providing ue's of $1/p$ will be successfully dealt with whenever this conjecture turns out to be true.

* Clearly the plan $P^T(2, 1)$ is not closed. Also it may be verified that $1/p$ is not ue from this plan.

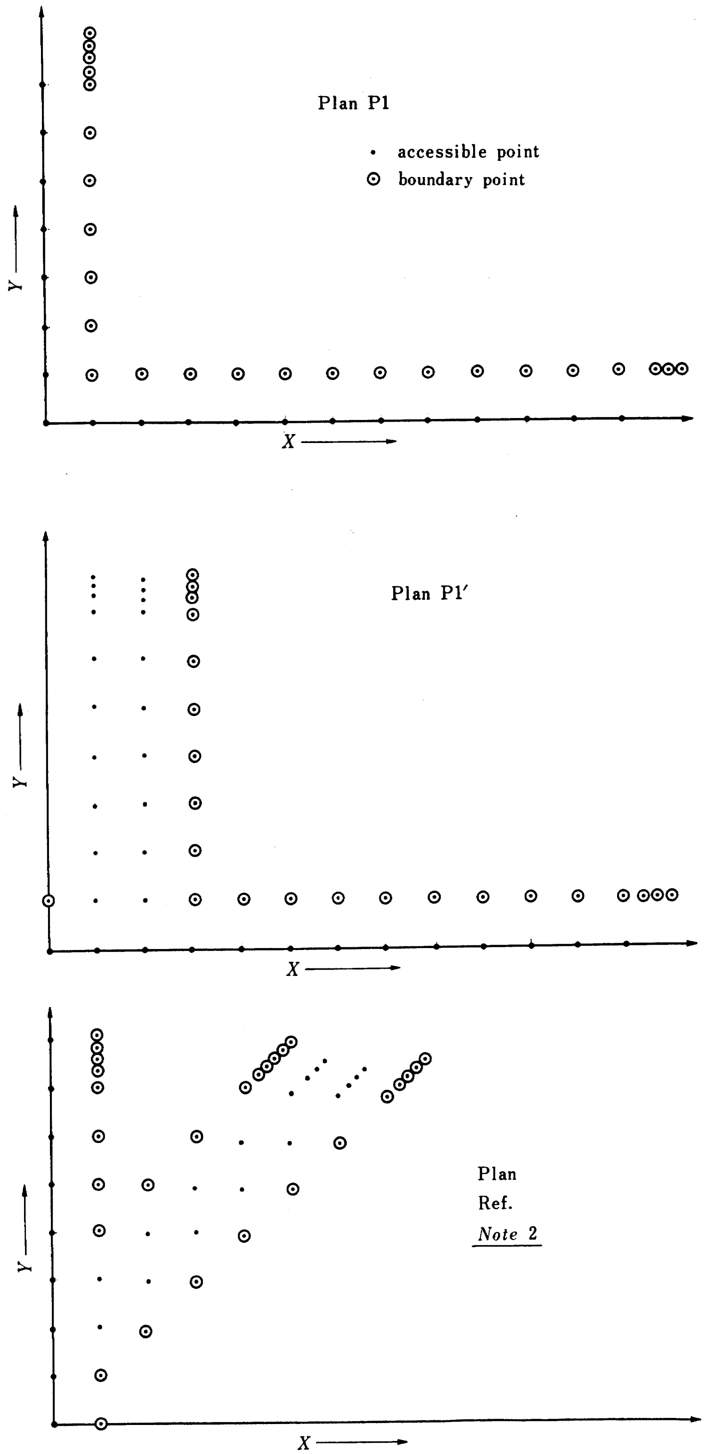
Plan P1
• accessible point
⊙ boundary point
Y
X
Plan P1′
Y
X
Plan
Ref.
Note 2
Y
X

Fig.

Acknowledgement

Most thankfully we acknowledge several helpful comments made by Dr. J. K. Ghosh of the Indian Statistical Institute, Calcutta.

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